9 Properties of Transformations

9.1 Translate Figures and Use Vectors
9.2 Use Properties of Matrices
9.3 Perform Reflections
9.4 Perform Rotations
9.5 Apply Compositions of Transformations
9.6 Identify Symmetry
9.7 Identify and Perform Dilations

Before

In previous chapters, you learned the following skills, which you'll use in Chapter 9: translating, reflecting, and rotating polygons, and using similar triangles.

Prerequisite Skills

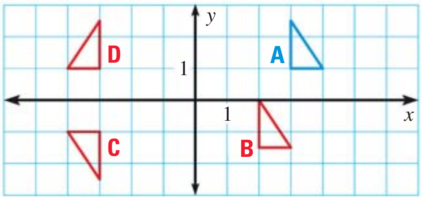
VOCABULARY CHECK

Match the transformation of Triangle A with its graph.

1. Translation of Triangle A

2. Reflection of Triangle A

3. Rotation of Triangle A



SKILLS AND ALGEBRA CHECK

The vertices of $JKLM$ are $J(-1, 6)$, $K(2, 5)$, $L(2, 2)$, and $M(-1, 1)$. Graph its image after the transformation described. *(Review p. 272 for 9.1, 9.3.)*

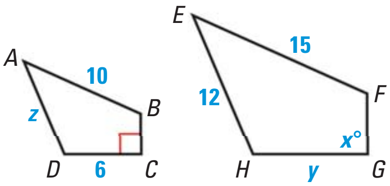
4. Translate 3 units left and 1 unit down.

5. Reflect in the y-axis.

In the diagram, $ABCD \sim EFGH$.
(Review p. 234 for 9.7.)

6. Find the scale factor of $ABCD$ to $EFGH$.

7. Find the values of x, y, and z.



@HomeTutor Prerequisite skills practice at classzone.com

Now

In Chapter 9, you will apply the big ideas listed below and reviewed in the Chapter Summary on page 635. You will also use the key vocabulary listed below.

Big Ideas

1. **Performing congruence and similarity transformations**
2. **Making real-world connections to symmetry and tessellations**
3. **Applying matrices and vectors in Geometry**

KEY VOCABULARY

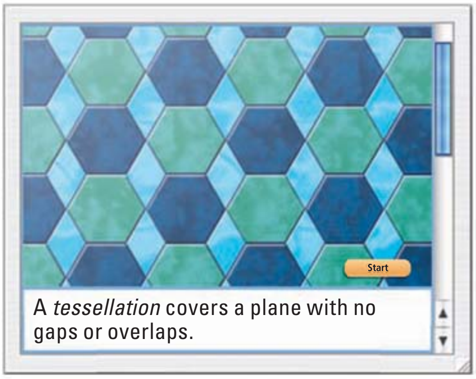
- image, *p. 572*
- preimage, *p. 572*
- isometry, *p. 573*
- vector, *p. 574*
- component form, *p. 574*
- matrix, *p. 580*
- element, *p. 580*
- dimensions, *p. 580*
- line of reflection, *p. 589*
- center of rotation, *p. 598*
- angle of rotation, *p. 598*
- glide reflection, *p. 608*
- composition of transformations, *p. 609*
- line symmetry, *p. 619*
- rotational symmetry, *p. 620*
- scalar multiplication, *p. 627*

Why?

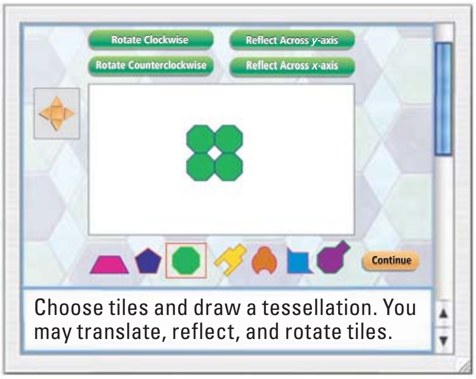
You can use properties of shapes to determine whether shapes tessellate. For example, you can use angle measurements to determine which shapes can be used to make a tessellation.

Animated Geometry

The animation illustrated below for Example 3 on page 617 helps you answer this question: How can you use tiles to tessellate a floor?



A *tessellation* covers a plane with no gaps or overlaps.



Choose tiles and draw a tessellation. You may translate, reflect, and rotate tiles.

Animated Geometry at classzone.com

Other animations for Chapter 9 : pages582, 590, 599, 602, 611, 619, and 626

9.1 Translate Figures and Use Vectors

Before	You used a coordinate rule to translate a figure.
Now	You will use a vector to translate a figure.
Why?	So you can find a distance covered on snowshoes, as in Exs. 35–37.

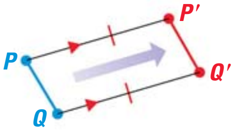
Key Vocabulary

- **image**
- **preimage**
- **isometry**
- **vector**
  initial point, terminal point, horizontal component, vertical component
- **component form**
- **translation,** *p. 272*

In Lesson 4.8, you learned that a *transformation* moves or changes a figure in some way to produce a new figure called an **image**. Another name for the original figure is the **preimage**.

Recall that a *translation* moves every point of a figure the same distance in the same direction. More specifically, a translation maps, or moves, the points P and Q of a plane figure to the points P' (read "*P prime*") and Q', so that one of the following statements is true:

- $PP' = QQ'$ and $\overline{PP'} \parallel \overline{QQ'}$, or
- $PP' = QQ'$ and $\overline{PP'}$ and $\overline{QQ'}$ are collinear.



EXAMPLE 1 Translate a figure in the coordinate plane

Graph quadrilateral $ABCD$ with vertices $A(-1, 2)$, $B(-1, 5)$, $C(4, 6)$, and $D(4, 2)$. Find the image of each vertex after the translation $(x, y) \to (x + 3, y - 1)$. Then graph the image using prime notation.

Solution

USE NOTATION
You can use *prime notation* to name an image. For example, if the preimage is $\triangle ABC$, then its image is $\triangle A'B'C'$, read as *"triangle A prime, B prime, C prime."*

First, draw $ABCD$. Find the translation of each vertex by adding 3 to its x-coordinate and subtracting 1 from its y-coordinate. Then graph the image.

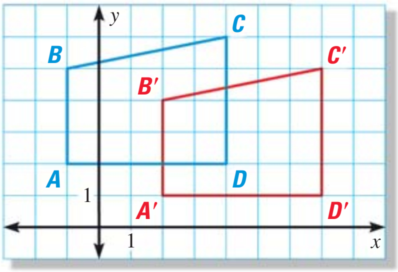
$$(x, y) \to (x + 3, y - 1)$$

$A(-1, 2) \to A'(2, 1)$

$B(-1, 5) \to B'(2, 4)$

$C(4, 6) \to C'(7, 5)$

$D(4, 2) \to D'(7, 1)$



✓ GUIDED PRACTICE for Example 1

1. Draw $\triangle RST$ with vertices $R(2, 2)$, $S(5, 2)$, and $T(3, 5)$. Find the image of each vertex after the translation $(x, y) \to (x + 1, y + 2)$. Graph the image using prime notation.

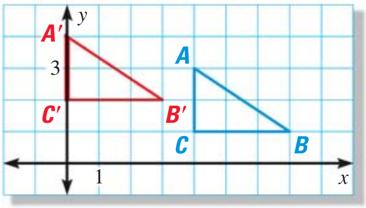
2. The image of $(x, y) \to (x + 4, y - 7)$ is $\overline{P'Q'}$ with endpoints $P'(-3, 4)$ and $Q'(2, 1)$. Find the coordinates of the endpoints of the preimage.

ISOMETRY An **isometry** is a transformation that preserves length and angle measure. Isometry is another word for congruence transformation (page 272).

EXAMPLE 2 Write a translation rule and verify congruence

READ DIAGRAMS
In this book, the preimage is always shown in blue, and the image is always shown in red.

Write a rule for the translation of $\triangle ABC$ to $\triangle A'B'C'$. Then verify that the transformation is an isometry.



Solution

To go from A to A', move 4 units left and 1 unit up. So, a rule for the translation is $(x, y) \rightarrow (x - 4, y + 1)$.

Use the SAS Congruence Postulate. Notice that $CB = C'B' = 3$, and $AC = A'C' = 2$. The slopes of $\overline{CB}$ and $\overline{C'B'}$ are 0, and the slopes of $\overline{CA}$ and $\overline{C'A'}$ are undefined, so the sides are perpendicular. Therefore, $\angle C$ and $\angle C'$ are congruent right angles. So, $\triangle ABC \cong \triangle A'B'C'$. The translation is an isometry.

✓ GUIDED PRACTICE for Example 2

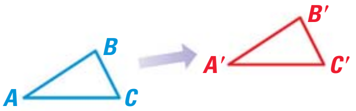
3. In Example 2, write a rule to translate $\triangle A'B'C'$ back to $\triangle ABC$.

THEOREM For Your Notebook

THEOREM 9.1 Translation Theorem

A translation is an isometry.

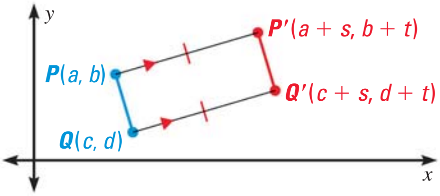
Proof: below; Ex. 46, p. 579



$\triangle ABC \cong \triangle A'B'C'$

PROOF Translation Theorem

A translation is an isometry.



GIVEN ▶ $P(a, b)$ and $Q(c, d)$ are two points on a figure translated by $(x, y) \rightarrow (x + s, y + t)$.

PROVE ▶ $PQ = P'Q'$

The translation maps $P(a, b)$ to $P'(a + s, b + t)$ and $Q(c, d)$ to $Q'(c + s, d + t)$.

Use the Distance Formula to find PQ and $P'Q'$. $PQ = \sqrt{(c - a)^2 + (d - b)^2}$.

$$P'Q' = \sqrt{[(c + s) - (a + s)]^2 + [(d + t) - (b + t)]^2}$$
$$= \sqrt{(c + s - a - s)^2 + (d + t - b - t)^2}$$
$$= \sqrt{(c - a)^2 + (d - b)^2}$$

Therefore, $PQ = P'Q'$ by the Transitive Property of Equality.

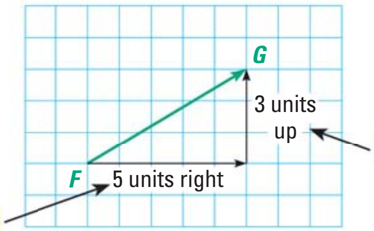
VECTORS Another way to describe a translation is by using a vector. A **vector** is a quantity that has both direction and *magnitude,* or size. A vector is represented in the coordinate plane by an arrow drawn from one point to another.

KEY CONCEPT *For Your Notebook*

Vectors

The diagram shows a vector named $\overrightarrow{FG}$, read as "vector *FG*."

The **initial point**, or starting point, of the vector is *F*.

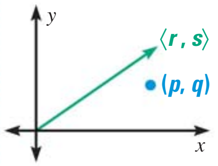


The **terminal point**, or ending point, of the vector is *G*.

The **component form** of a vector combines the horizontal and vertical components. So, the component form of $\overrightarrow{FG}$ is $\langle 5, 3\rangle$.

USE NOTATION

Use brackets to write the component form of the vector $\langle r, s\rangle$. Use parentheses to write the coordinates of the point (p, q).

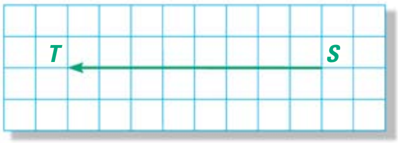


EXAMPLE 3 Identify vector components

Name the vector and write its component form.

a.

b.



Solution

a. The vector is $\overrightarrow{BC}$. From initial point *B* to terminal point *C*, you move 9 units right and 2 units down. So, the component form is $\langle 9, -2\rangle$.

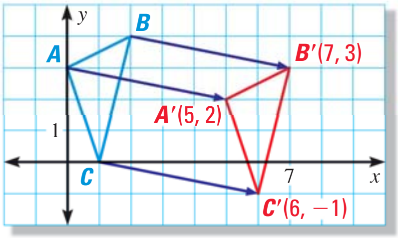
b. The vector is $\overrightarrow{ST}$. From initial point *S* to terminal point *T*, you move 8 units left and 0 units vertically. The component form is $\langle -8, 0\rangle$.

EXAMPLE 4 Use a vector to translate a figure

The vertices of $\triangle ABC$ are $A(0, 3)$, $B(2, 4)$, and $C(1, 0)$. Translate $\triangle ABC$ using the vector $\langle 5, -1\rangle$.

Solution

First, graph $\triangle ABC$. Use $\langle 5, -1\rangle$ to move each vertex 5 units to the right and 1 unit down. Label the image vertices. Draw $\triangle A'B'C'$. Notice that the vectors drawn from preimage to image vertices are parallel.



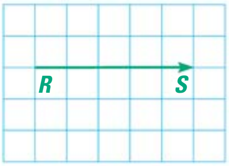
USE VECTORS

Notice that the vector can have different initial points. The vector describes only the direction and magnitude of the translation.

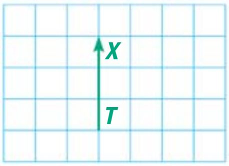
✓ GUIDED PRACTICE for Examples 3 and 4

Name the vector and write its component form.

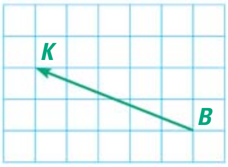
4.



5.



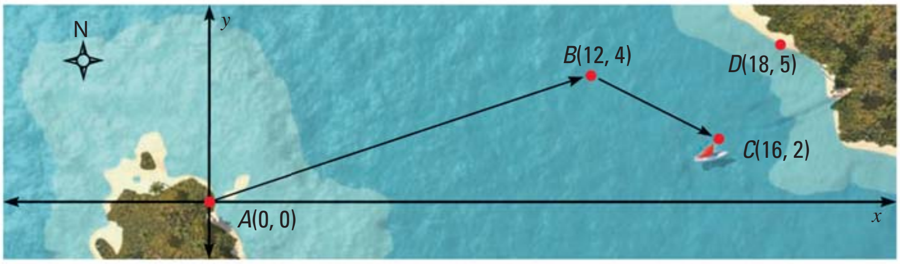
6.



7. The vertices of $\triangle LMN$ are $L(2, 2)$, $M(5, 3)$, and $N(9, 1)$. Translate $\triangle LMN$ using the vector $\langle -2, 6\rangle$.

EXAMPLE 5 Solve a multi-step problem

NAVIGATION A boat heads out from point A on one island toward point D on another. The boat encounters a storm at B, 12 miles east and 4 miles north of its starting point. The storm pushes the boat off course to point C, as shown.



a. Write the component form of $\overrightarrow{AB}$.

b. Write the component form of $\overrightarrow{BC}$.

c. Write the component form of the vector that describes the straight line path from the boat's current position C to its intended destination D.

Solution

a. The component form of the vector from $A(0, 0)$ to $B(12, 4)$ is

$$\overrightarrow{AB} = \langle 12 - 0, 4 - 0\rangle = \langle 12, 4\rangle.$$

b. The component form of the vector from $B(12, 4)$ to $C(16, 2)$ is

$$\overrightarrow{BC} = \langle 16 - 12, 2 - 4\rangle = \langle 4, -2\rangle.$$

c. The boat is currently at point C and needs to travel to D. The component form of the vector from $C(16, 2)$ to $D(18, 5)$ is

$$\overrightarrow{CD} = \langle 18 - 16, 5 - 2\rangle = \langle 2, 3\rangle.$$

✓ GUIDED PRACTICE for Example 5

8. **WHAT IF?** In Example 5, suppose there is no storm. Write the component form of the vector that describes the straight path from the boat's starting point A to its final destination D.

9.1 EXERCISES

HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 7, 11, and 35

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 14, and 42

SKILL PRACTICE

1. **VOCABULARY** Copy and complete: A __?__ is a quantity that has both __?__ and magnitude.

2. ★ **WRITING** *Describe* the difference between a vector and a ray.

EXAMPLE 1
on p. 572
for Exs. 3–10

IMAGE AND PREIMAGE **Use the translation $(x, y) \rightarrow (x - 8, y + 4)$.**

3. What is the image of $A(2, 6)$?

4. What is the image of $B(-1, 5)$?

5. What is the preimage of $C'(-3, -10)$?

6. What is the preimage of $D'(4, -3)$?

GRAPHING AN IMAGE **The vertices of $\triangle PQR$ are $P(-2, 3)$, $Q(1, 2)$, and $R(3, -1)$. Graph the image of the triangle using prime notation.**

7. $(x, y) \rightarrow (x + 4, y + 6)$

8. $(x, y) \rightarrow (x + 9, y - 2)$

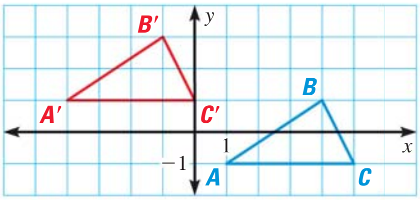
9. $(x, y) \rightarrow (x - 2, y - 5)$

10. $(x, y) \rightarrow (x - 1, y + 3)$

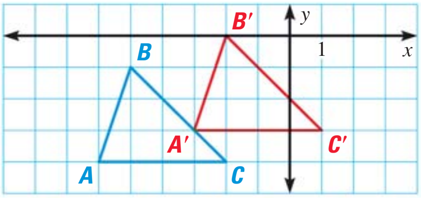
EXAMPLE 2
on p. 573
for Exs. 11–14

WRITING A RULE **$\triangle A'B'C'$ is the image of $\triangle ABC$ after a translation. Write a rule for the translation. Then *verify* that the translation is an isometry.**

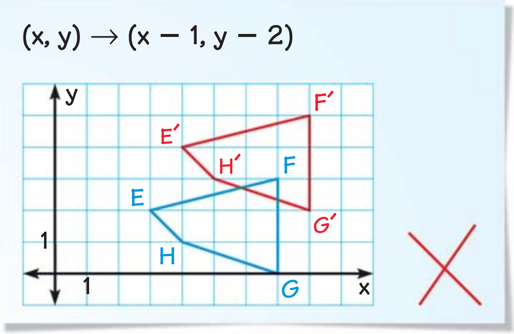
11.



12.



13. **ERROR ANALYSIS** *Describe* and correct the error in graphing the translation of quadrilateral *EFGH*.



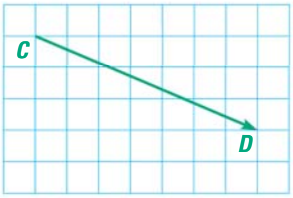
14. ★ **MULTIPLE CHOICE** Translate $Q(0, -8)$ using $(x, y) \rightarrow (x - 3, y + 2)$.

Ⓐ $Q'(-2, 5)$ Ⓑ $Q'(3, -10)$ Ⓒ $Q'(-3, -6)$ Ⓓ $Q'(2, -11)$

EXAMPLE 3
on p. 574
for Exs. 15–23

IDENTIFYING VECTORS **Name the vector and write its component form.**

15.



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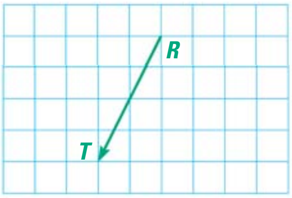
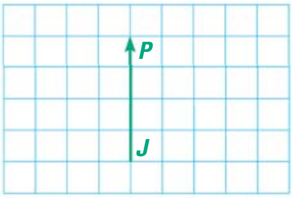


17.



VECTORS **Use the point $P(-3, 6)$. Find the component form of the vector that describes the translation to P'.**

18. $P'(0, 1)$ **19.** $P'(-4, 8)$ **20.** $P'(-2, 0)$ **21.** $P'(-3, -5)$

TRANSLATIONS **Think of each translation as a vector. *Describe* the vertical component of the vector. *Explain*.**

22.

23.

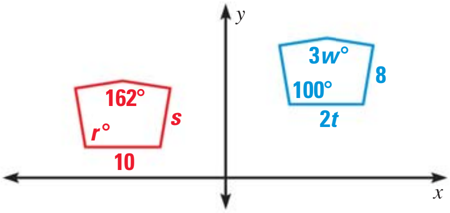
EXAMPLE 4
on p. 574
for Exs. 24–27

TRANSLATING A TRIANGLE **The vertices of $\triangle DEF$ are $D(2, 5)$, $E(6, 3)$, and $F(4, 0)$. Translate $\triangle DEF$ using the given vector. Graph $\triangle DEF$ and its image.**

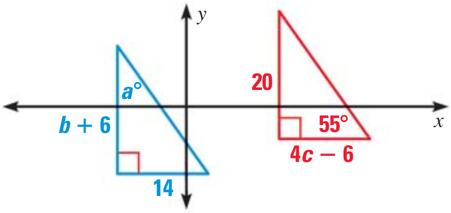
24. $\langle 6, 0\rangle$ **25.** $\langle 5, -1\rangle$ **26.** $\langle -3, -7\rangle$ **27.** $\langle -2, -4\rangle$

xy ALGEBRA **Find the value of each variable in the translation.**

28.



29.



30. **xy ALGEBRA** Translation A maps (x, y) to $(x + n, y + m)$. Translation B maps (x, y) to $(x + s, y + t)$.

a. Translate a point using Translation A, then Translation B. Write a rule for the final image of the point.

b. Translate a point using Translation B, then Translation A. Write a rule for the final image of the point.

c. *Compare* the rules you wrote in parts (a) and (b). Does it matter which translation you do first? *Explain*.

31. **MULTI-STEP PROBLEM** The vertices of a rectangle are $Q(2, -3)$, $R(2, 4)$, $S(5, 4)$, and $T(5, -3)$.

a. Translate $QRST$ 3 units left and 2 units down. Find the areas of $QRST$ $Q'R'S'T'$.

b. *Compare* the areas. Make a conjecture about the areas of a preimage and its image after a translation.

32. **CHALLENGE** The vertices of $\triangle ABC$ are $A(2, 2)$, $B(4, 2)$, and $C(3, 4)$.

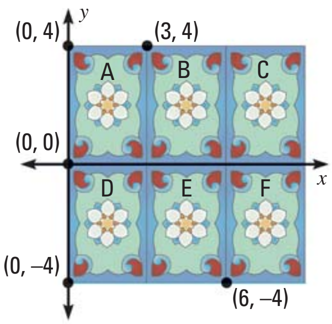
a. Graph the image of $\triangle ABC$ after the transformation $(x, y) \rightarrow (x + y, y)$. Is the transformation an isometry? *Explain*. Are the areas of $\triangle ABC$ and $\triangle A'B'C'$ the same?

b. Graph a new triangle, $\triangle DEF$, and its image after the transformation given in part (a). Are the areas of $\triangle DEF$ and $\triangle D'E'F'$ the same?

PROBLEM SOLVING

EXAMPLE 2
on p. 573
for Exs. 33–34

HOME DESIGN **Designers can use computers to make patterns in fabrics or floors. On the computer, a copy of the design in Rectangle A is used to cover an entire floor. The translation $(x, y) \rightarrow (x + 3, y)$ maps Rectangle A to Rectangle B.**



33. Use coordinate notation to describe the translations that map Rectangle A to Rectangles C, D, E, and F.

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34. Write a rule to translate Rectangle F back to Rectangle A.

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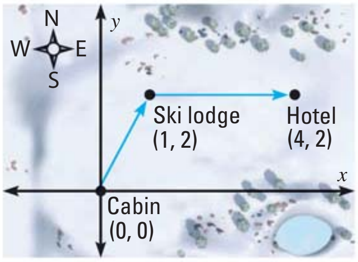
EXAMPLE 5
on p. 575
for Exs. 35–37

SNOWSHOEING **You are snowshoeing in the mountains. The distances in the diagram are in miles. Write the component form of the vector.**

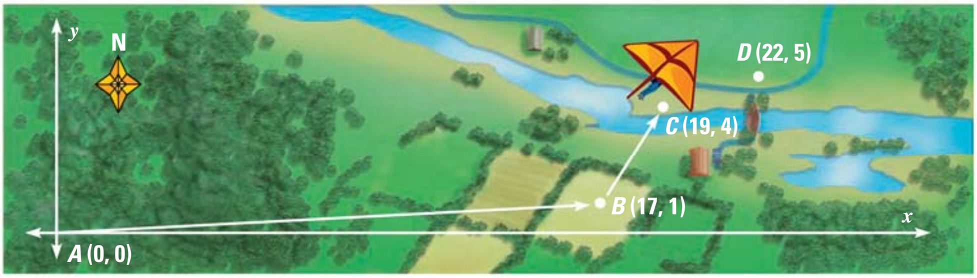


35. From the cabin to the ski lodge

36. From the ski lodge to the hotel

37. From the hotel back to your cabin

HANG GLIDING **A hang glider travels from point *A* to point *D*. At point *B*, the hang glider changes direction, as shown in the diagram. The distances in the diagram are in kilometers.**



38. Write the component form for $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

39. Write the component form of the vector that describes the path from the hang glider's current position *C* to its intended destination *D*.

40. What is the total distance the hang glider travels?

41. Suppose the hang glider went straight from *A* to *D*. Write the component form of the vector that describes this path. What is this distance?

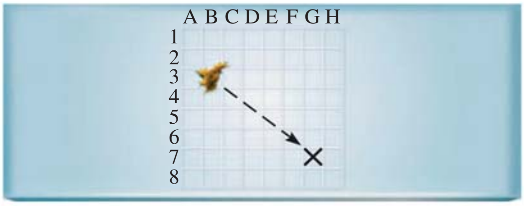
42. ★ **EXTENDED RESPONSE** Use the equation $2x + y = 4$.

a. Graph the line and its image after the translation $\langle -5, 4\rangle$. What is an equation of the image of the line?

b. *Compare* the line and its image. What are the slopes? the y-intercepts? the x-intercepts?

c. Write an equation of the image of $2x + y = 4$ after the translation $\langle 2, -6\rangle$ *without* using a graph. *Explain* your reasoning.

43. **SCIENCE** You are studying an amoeba through a microscope. Suppose the amoeba moves on a grid-indexed microscope slide in a straight line from square B3 to square G7.

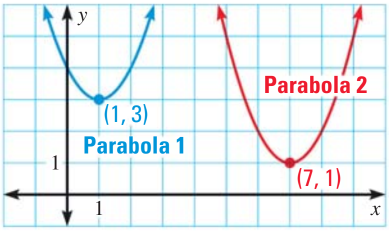


a. *Describe* the translation.

b. Each grid square is 2 millimeters on a side. How far does the amoeba travel?

c. Suppose the amoeba moves from B3 to G7 in 24.5 seconds. What is its speed in millimeters per second?

44. **MULTI-STEP PROBLEM** You can write the equation of a parabola in the form $y = (x - h)^2 + k$, where (h, k) is the *vertex* of the parabola. In the graph, an equation of Parabola 1 is $y = (x - 1)^2 + 3$, with vertex $(1, 3)$. Parabola 2 is the image of Parabola 1 after a translation.



a. Write a rule for the translation.

b. Write an equation of Parabola 2.

c. Suppose you translate Parabola 1 using the vector $\langle -4, 8\rangle$. Write an equation of the image.

d. An equation of Parabola 3 is $y = (x + 5)^2 - 3$. Write a rule for the translation of Parabola 1 to Parabola 3. *Explain* your reasoning.

45. **TECHNOLOGY** The standard form of an exponential equation is $y = a^x$, where $a > 0$ and $a \neq 1$. Use the equation $y = 2^x$.

a. Use a graphing calculator to graph $y = 2^x$ and $y = 2^x - 4$. *Describe* the translation from $y = 2^x$ to $y = 2^x - 4$.

b. Use a graphing calculator to graph $y = 2^x$ and $y = 2^{x-4}$. *Describe* the translation from $y = 2^x$ to $y = 2^{x-4}$.

46. **CHALLENGE** Use properties of congruent triangles to prove part of Theorem 9.1, that a translation preserves angle measure.

MIXED REVIEW

PREVIEW
Prepare for Lesson 9.2 in Exs. 47–50.

Find the sum, difference, product, or quotient. (p. 869)

47. $-16 - 7$

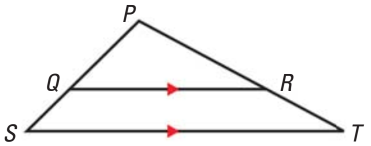
48. $6 + (-12)$

49. $(13)(-2)$

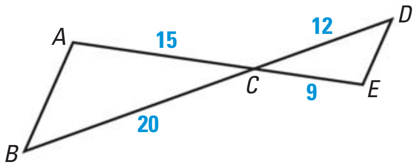
50. $16 \div (-4)$

Determine whether the two triangles are similar. If they are, write a similarity statement. (pp. 381, 388)

51.



52.



Points *A, B, C,* and *D* are the vertices of a quadrilateral. Give the most specific name for *ABCD*. *Justify* your answer. (p. 552)

53. $A(2, 0), B(7, 0), C(4, 4), D(2, 4)$

54. $A(3, 0), B(7, 2), C(3, 4), D(1, 2)$

9.2 Use Properties of Matrices

Before You performed translations using vectors.
Now You will perform translations using matrix operations.
Why So you can calculate the total cost of art supplies, as in Ex. 36.

Key Vocabulary
- matrix
- element
- dimensions

A **matrix** is a rectangular arrangement of numbers in rows and columns. (The plural of matrix is *matrices*.) Each number in a matrix is called an **element**.

$$\begin{array}{rcccc} & & & \text{column} & \\ & 5 & 4 & 4 & 9 \\ \text{row} & -3 & 5 & 2 & 6 \\ & 3 & -7 & 8 & 7 \end{array} \longleftarrow$$

The element in the second row and third column is 2.

READ VOCABULARY
An element of a matrix may also be called an *entry*.

The **dimensions** of a matrix are the numbers of rows and columns. The matrix above has three rows and four columns, so the dimensions of the matrix are 3 × 4 (read "3 by 4").

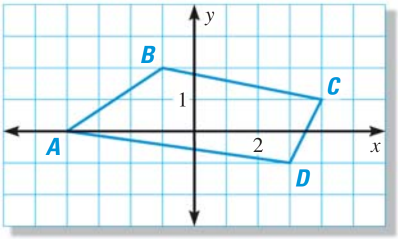
You can represent a figure in the coordinate plane using a matrix with two rows. The first row has the x-coordinate(s) of the vertices. The second row has the corresponding y-coordinate(s). Each column represents a vertex, so the number of columns depends on the number of vertices of the figure.

EXAMPLE 1 Represent figures using matrices

Write a matrix to represent the point or polygon.

a. Point A

b. Quadrilateral $ABCD$



Solution

AVOID ERRORS
The columns in a polygon matrix follow the consecutive order of the vertices of the polygon.

a. Point matrix for A

$$\begin{bmatrix} -4 \\ 0 \end{bmatrix} \begin{array}{l} \longleftarrow x\text{-coordinate} \\ \longleftarrow y\text{-coordinate} \end{array}$$

b. Polygon matrix for $ABCD$

$$\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \left[\begin{array}{cccc} -4 & -1 & 4 & 3 \\ 0 & 2 & 1 & -1 \end{array}\right] \end{array} \begin{array}{l} \longleftarrow x\text{-coordinates} \\ \longleftarrow y\text{-coordinates} \end{array}$$

✓ GUIDED PRACTICE for Example 1

1. Write a matrix to represent △ABC with vertices $A(3, 5)$, $B(6, 7)$ and $C(7, 3)$.
2. How many rows and columns are in a matrix for a hexagon?

ADDING AND SUBTRACTING To add or subtract matrices, you add or subtract corresponding elements. The matrices must have the same dimensions.

EXAMPLE 2 Add and subtract matrices

a. $\begin{bmatrix} 5 & -3 \\ 6 & -6 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 5+1 & -3+2 \\ 6+3 & -6+(-4) \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 9 & -10 \end{bmatrix}$

b. $\begin{bmatrix} 6 & 8 & 5 \\ 4 & 9 & -1 \end{bmatrix} - \begin{bmatrix} 1 & -7 & 0 \\ 4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 6-1 & 8-(-7) & 5-0 \\ 4-4 & 9-(-2) & -1-3 \end{bmatrix} = \begin{bmatrix} 5 & 15 & 5 \\ 0 & 11 & -4 \end{bmatrix}$

TRANSLATIONS You can use matrix addition to represent a translation in the coordinate plane. The image matrix for a translation is the sum of the translation matrix and the matrix that represents the preimage.

EXAMPLE 3 Represent a translation using matrices

The matrix $\begin{bmatrix} 1 & 5 & 3 \\ 1 & 0 & -1 \end{bmatrix}$ represents $\triangle ABC$. Find the image matrix that represents the translation of $\triangle ABC$ 1 unit left and 3 units up. Then graph $\triangle ABC$ and its image.

AVOID ERRORS

In order to add two matrices, they must have the same dimensions, so the translation matrix here must have three columns like the polygon matrix.

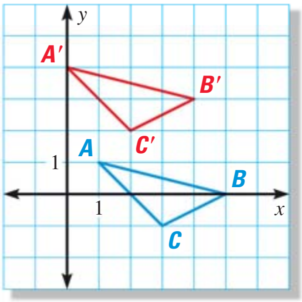
Solution

The translation matrix is $\begin{bmatrix} -1 & -1 & -1 \\ 3 & 3 & 3 \end{bmatrix}$.

Add this to the polygon matrix for the preimage to find the image matrix.

$$\begin{bmatrix} -1 & -1 & -1 \\ 3 & 3 & 3 \end{bmatrix} + \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1 & 5 & 3 \\ 1 & 0 & -1 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A' & B' & C' \end{array} \\ \begin{bmatrix} 0 & 4 & 2 \\ 4 & 3 & 2 \end{bmatrix} \end{array}$$

Translation matrix   **Polygon matrix**   **Image matrix**



✓ GUIDED PRACTICE for Examples 2 and 3

In Exercises 3 and 4, add or subtract.

3. $\begin{bmatrix} -3 & 7 \end{bmatrix} + \begin{bmatrix} 2 & -5 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -4 \\ 3 & -5 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix}$

5. The matrix $\begin{bmatrix} 1 & 2 & 6 & 7 \\ 2 & -1 & 1 & 3 \end{bmatrix}$ represents quadrilateral *JKLM*. Write the translation matrix and the image matrix that represents the translation of *JKLM* 4 units right and 2 units down. Then graph *JKLM* and its image.

MULTIPLYING MATRICES The product of two matrices A and B is defined only when the number of columns in A is equal to the number of rows in B. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product AB is an $m \times p$ matrix.

USE NOTATION

Recall that the dimensions of a matrix are always written as rows × columns.

$$\begin{array}{ccccc} A & \cdot & B & = & AB \\ (m \text{ by } n) & \cdot & (n \text{ by } p) & = & (m \text{ by } p) \\ & \nwarrow \text{equal} \nearrow & & & \text{dimensions of } AB \end{array}$$

You will use matrix multiplication in later lessons to represent transformations.

EXAMPLE 4 Multiply matrices

Multiply $\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 8 \end{bmatrix}$.

Solution

The matrices are both 2 × 2, so their product is defined. Use the following steps to find the elements of the product matrix.

STEP 1 **Multiply** the numbers in the first row of the first matrix by the numbers in the first column of the second matrix. Put the result in the first row, first column of the product matrix.

$$\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 1(2) + 0(-1) & ? \\ ? & ? \end{bmatrix}$$

STEP 2 **Multiply** the numbers in the first row of the first matrix by the numbers in the second column of the second matrix. Put the result in the first row, second column of the product matrix.

$$\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 1(2) + 0(-1) & 1(-3) + 0(8) \\ ? & ? \end{bmatrix}$$

STEP 3 **Multiply** the numbers in the second row of the first matrix by the numbers in the first column of the second matrix. Put the result in the second row, first column of the product matrix.

$$\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 1(2) + 0(-1) & 1(-3) + 0(8) \\ 4(2) + 5(-1) & ? \end{bmatrix}$$

STEP 4 **Multiply** the numbers in the second row of the first matrix by the numbers in the second column of the second matrix. Put the result in the second row, second column of the product matrix.

$$\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 1(2) + 0(-1) & 1(-3) + 0(8) \\ 4(2) + 5(-1) & 4(-3) + 5(8) \end{bmatrix}$$

STEP 5 **Simplify** the product matrix.

$$\begin{bmatrix} 1(2) + 0(-1) & 1(-3) + 0(8) \\ 4(2) + 5(-1) & 4(-3) + 5(8) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 3 & 28 \end{bmatrix}$$

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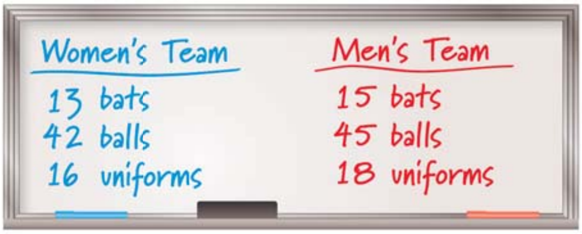
EXAMPLE 5 Solve a real-world problem

SOFTBALL Two softball teams submit equipment lists for the season. A bat costs \$20, a ball costs \$5, and a uniform costs \$40. Use matrix multiplication to find the total cost of equipment for each team.



Solution

First, write the equipment lists and the costs per item in matrix form. You will use matrix multiplication, so you need to set up the matrices so that the number of columns of the equipment matrix matches the number of rows of the cost per item matrix.

EQUIPMENT · **COST** = **TOTAL COST**

$$\begin{array}{r|ccc|} & \text{Bats} & \text{Balls} & \text{Uniforms} \\ \text{Women} & 13 & 42 & 16 \\ \text{Men} & 15 & 45 & 18 \end{array} \cdot \begin{array}{r|c|} & \text{Dollars} \\ \text{Bats} & 20 \\ \text{Balls} & 5 \\ \text{Uniforms} & 40 \end{array} = \begin{array}{r|c|} & \text{Dollars} \\ \text{Women} & ? \\ \text{Men} & ? \end{array}$$

You can find the total cost of equipment for each team by multiplying the equipment matrix by the cost per item matrix. The equipment matrix is 2 × 3 and the cost per item matrix is 3 × 1, so their product is a 2 × 1 matrix.

$$\begin{bmatrix} 13 & 42 & 16 \\ 15 & 45 & 18 \end{bmatrix}\begin{bmatrix} 20 \\ 5 \\ 40 \end{bmatrix} = \begin{bmatrix} 13(20) + 42(5) + 16(40) \\ 15(20) + 45(5) + 18(40) \end{bmatrix} = \begin{bmatrix} 1110 \\ 1245 \end{bmatrix}$$

▶ The total cost of equipment for the women's team is \$1110, and the total cost for the men's team is \$1245.

ANOTHER WAY

You could solve this problem arithmetically, multiplying the number of bats by the price of bats, and so on, then adding the costs for each team.

✓ GUIDED PRACTICE for Examples 4 and 5

Use the matrices below. Is the product defined? *Explain.*

$$A = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 6.7 & 0 \\ -9.3 & 5.2 \end{bmatrix}$$

6. AB **7.** BA **8.** AC

Multiply.

9. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 8 \\ -4 & 7 \end{bmatrix}$ **10.** $\begin{bmatrix} 5 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -2 \end{bmatrix}$ **11.** $\begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ 5 & 1 \end{bmatrix}$

12. WHAT IF? In Example 5, find the total cost if a bat costs \$25, a ball costs \$4, and a uniform costs \$35.

9.2 EXERCISES

HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 13, 19, and 31

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 17, 24, 25, and 35

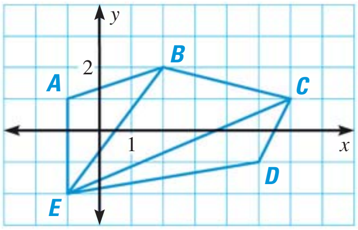
SKILL PRACTICE

1. **VOCABULARY** Copy and complete: To find the sum of two matrices, add corresponding __?__.

2. ★ **WRITING** How can you determine whether two matrices can be added? How can you determine whether two matrices can be multiplied?

EXAMPLE 1
on p. 580
for Exs. 3–6

USING A DIAGRAM **Use the diagram to write a matrix to represent the given polygon.**



3. $\triangle EBC$

4. $\triangle ECD$

5. Quadrilateral *BCDE*

6. Pentagon *ABCDE*

EXAMPLE 2
on p. 581
for Exs. 7–12

MATRIX OPERATIONS **Add or subtract.**

7. $\begin{bmatrix} 3 & 5 \end{bmatrix} + \begin{bmatrix} 9 & 2 \end{bmatrix}$

8. $\begin{bmatrix} -12 & 5 \\ 1 & -4 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 0 & 8 \end{bmatrix}$

9. $\begin{bmatrix} 9 & 8 \\ -2 & 3 \\ 0 & -4 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ 2 & -3 \\ -5 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 4.6 & 8.1 \end{bmatrix} - \begin{bmatrix} 3.8 & -2.1 \end{bmatrix}$

11. $\begin{bmatrix} -5 & 6 \\ -8 & 9 \end{bmatrix} - \begin{bmatrix} 8 & 10 \\ 4 & -7 \end{bmatrix}$

12. $\begin{bmatrix} 1.2 & 6 \\ 5.3 & 1.1 \end{bmatrix} - \begin{bmatrix} 2.5 & -3.3 \\ 7 & 4 \end{bmatrix}$

EXAMPLE 3
on p. 581
for Exs. 13–17

TRANSLATIONS **Find the image matrix that represents the translation of the polygon. Then graph the polygon and its image.**

13. $\begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} -2 & 2 & 1 \\ 4 & 1 & -3 \end{bmatrix} \end{array}$; 4 units up

14. $\begin{array}{c} \begin{matrix} F & G & H & J \end{matrix} \\ \begin{bmatrix} 2 & 5 & 8 & 5 \\ 2 & 3 & 1 & -1 \end{bmatrix} \end{array}$; 2 units left and 3 units down

15. $\begin{array}{c} \begin{matrix} L & M & N & P \end{matrix} \\ \begin{bmatrix} 3 & 0 & 2 & 2 \\ -1 & 3 & 3 & -1 \end{bmatrix} \end{array}$; 4 units right and 2 units up

16. $\begin{array}{c} \begin{matrix} Q & R & S \end{matrix} \\ \begin{bmatrix} -5 & 0 & 1 \\ 1 & 4 & 2 \end{bmatrix} \end{array}$; 3 units right and 1 unit down

17. ★ **MULTIPLE CHOICE** The matrix that represents quadrilateral *ABCD* is $\begin{bmatrix} 3 & 8 & 9 & 7 \\ 3 & 7 & 3 & 1 \end{bmatrix}$. Which matrix represents the image of the quadrilateral after translating it 3 units right and 5 units up?

Ⓐ $\begin{bmatrix} 6 & 11 & 12 & 10 \\ 8 & 12 & 8 & 6 \end{bmatrix}$

Ⓑ $\begin{bmatrix} 0 & 5 & 6 & 4 \\ 8 & 12 & 8 & 6 \end{bmatrix}$

Ⓒ $\begin{bmatrix} 6 & 11 & 12 & 10 \\ -2 & 2 & -2 & -4 \end{bmatrix}$

Ⓓ $\begin{bmatrix} 0 & 6 & 6 & 4 \\ -2 & 3 & -2 & -4 \end{bmatrix}$

EXAMPLE 4
on p. 582
for Exs. 18–26

MATRIX OPERATIONS **Multiply.**

18. $\begin{bmatrix} 5 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

19. $\begin{bmatrix} 1.2 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ -1.5 \end{bmatrix}$

20. $\begin{bmatrix} 6 & 7 \\ -5 & 8 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 9 & -3 \end{bmatrix}$

21. $\begin{bmatrix} 0.4 & 6 \\ -6 & 2.3 \end{bmatrix}\begin{bmatrix} 5 & 8 \\ -1 & 2 \end{bmatrix}$

22. $\begin{bmatrix} 4 & 8 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

23. $\begin{bmatrix} 9 & 1 & 2 \\ 8 & -1 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$

24. ★ **MULTIPLE CHOICE** Which product is not defined?

Ⓐ $\begin{bmatrix} 1 & 7 \\ 3 & 12 \end{bmatrix}\begin{bmatrix} 6 \\ 15 \end{bmatrix}$

Ⓑ $\begin{bmatrix} 3 & 20 \end{bmatrix}\begin{bmatrix} 9 \\ 30 \end{bmatrix}$

Ⓒ $\begin{bmatrix} 15 \\ -3 \end{bmatrix}\begin{bmatrix} 1 & 6 \\ 4 & 0 \end{bmatrix}$

Ⓓ $\begin{bmatrix} 30 \\ -7 \end{bmatrix}\begin{bmatrix} 5 & 5 \end{bmatrix}$

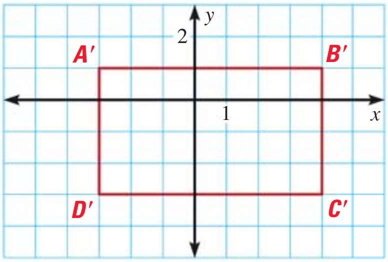
25. ★ **OPEN-ENDED MATH** Write two matrices that have a defined product. Then find the product.

26. **ERROR ANALYSIS** *Describe* and correct the error in the computation.

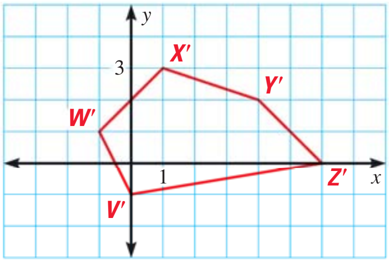
$$\begin{bmatrix} 9 & -2 \\ 4 & 10 \end{bmatrix}\begin{bmatrix} -6 & 12 \\ 3 & -6 \end{bmatrix} = \begin{bmatrix} 9(-6) & -2(12) \\ 4(3) & 10(-6) \end{bmatrix}$$

TRANSLATIONS **Use the described translation and the graph of the image to find the matrix that represents the preimage.**

27. 4 units right and 2 units down



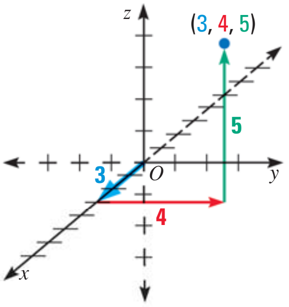
28. 6 units left and 5 units up



29. **MATRIX EQUATION** Use the description of a translation of a triangle to find the value of each variable. *Explain* your reasoning. What are the coordinates of the vertices of the image triangle?

$$\begin{bmatrix} 12 & 12 & w \\ -7 & v & -7 \end{bmatrix} + \begin{bmatrix} 9 & a & b \\ 6 & -2 & c \end{bmatrix} = \begin{bmatrix} m & 20 & -8 \\ n & -9 & 13 \end{bmatrix}$$

30. **CHALLENGE** A point in space has three coordinates (x, y, z), as shown at the right. From the origin, a point can be forward or back on the x-axis, left or right on the y-axis, and up or down on the z-axis.



a. You translate a point three units forward, four units right, and five units up. Write a translation matrix for the point.

b. You translate a figure that has five vertices. Write a translation matrix to move the figure five units back, ten units left, and six units down.

PROBLEM SOLVING

EXAMPLE 5
on p. 583
for Ex. 31

31. **COMPUTERS** Two computer labs submit equipment lists. A mouse costs \$10, a package of CDs costs \$32, and a keyboard costs \$15. Use matrix multiplication to find the total cost of equipment for each lab.

Lab 1	Lab 2
25 Mice	15 Mice
10 CDs	20 CDs
18 Keyboards	12 Keyboards

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32. **SWIMMING** Two swim teams submit equipment lists. The women's team needs 30 caps and 26 goggles. The men's team needs 15 caps and 25 goggles. A cap costs \$10 and goggles cost \$15.

a. Use matrix addition to find the total number of caps and the total number of goggles for each team.

b. Use matrix multiplication to find the total equipment cost for each team.

c. Find the total cost for both teams.

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MATRIX PROPERTIES **In Exercises 33–35, use matrices *A*, *B*, and *C*.**

$$A = \begin{bmatrix} 5 & 1 \\ 10 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 4 \\ -5 & 1 \end{bmatrix}$$

33. **MULTI-STEP PROBLEM** Use the 2×2 matrices above to explore the Commutative Property of Multiplication.

a. What does it mean that multiplication is *commutative*?

b. Find and *compare* AB and BA.

c. Based on part (b), make a conjecture about whether matrix multiplication is commutative.

34. **MULTI-STEP PROBLEM** Use the 2×2 matrices above to explore the Associative Property of Multiplication.

a. What does it mean that multiplication is *associative*?

b. Find and *compare* $A(BC)$ and $(AB)C$.

c. Based on part (b), make a conjecture about whether matrix multiplication is associative.

35. ★ **SHORT RESPONSE** Find and *compare* $A(B + C)$ and $AB + AC$. Make a conjecture about matrices and the Distributive Property.

36. **ART** Two art classes are buying supplies. A brush is \$4 and a paint set is \$10. Each class has only \$225 to spend. Use matrix multiplication to find the maximum number of brushes Class A can buy and the maximum number of paint sets Class B can buy. *Explain.*

Class A	Class B
x brushes	18 brushes
12 paint sets	y paint sets

37. CHALLENGE The total United States production of corn was 8,967 million bushels in 2002, and 10,114 million bushels in 2003. The table shows the percents of the total grown by four states.

a. Use matrix multiplication to find the number of bushels (in millions) harvested in each state each year.

b. How many bushels (in millions) were harvested in these two years in Iowa?

c. The price for a bushel of corn in Nebraska was \$2.32 in 2002, and \$2.45 in 2003. Use matrix multiplication to find the total value of corn harvested in Nebraska in these two years.

	2002	2003
Iowa	21.5%	18.6%
Illinois	16.4%	17.9%
Nebraska	10.5%	11.1%
Minnesota	11.7%	9.6%

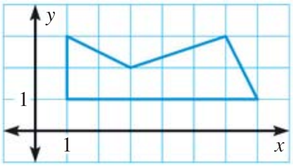
MIXED REVIEW

PREVIEW
Prepare for Lesson 9.3 in Exs. 38–39.

Copy the figure and draw its image after the reflection. *(p. 272)*

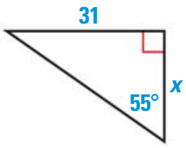
38. Reflect the figure in the x-axis.

39. Reflect the figure in the y-axis.

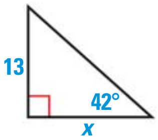


Find the value of x to the nearest tenth. *(p. 466)*

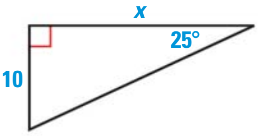
40.



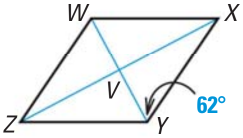
41.



42.



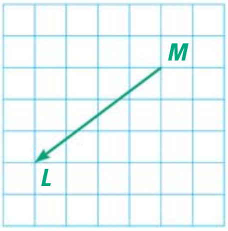
The diagonals of rhombus $WXYZ$ intersect at V. Given that $m\angle XYW = 62°$, find the indicated measure. *(p. 533)*



43. $m\angle ZYW =$ _?_ **44.** $m\angle WXY =$ _?_ **45.** $m\angle XVY =$ _?_

QUIZ *for Lessons 9.1–9.2*

1. In the diagram shown, name the vector and write its component form. *(p. 572)*



Use the translation $(x, y) \to (x + 3, y - 2)$. *(p. 572)*

2. What is the image of (−1, 5)?

3. What is the image of (6, 3)?

4. What is the preimage of (−4, −1)?

Add, subtract, or multiply. *(p. 580)*

5. $\begin{bmatrix} 5 & -3 \\ 8 & -2 \end{bmatrix} + \begin{bmatrix} -9 & 6 \\ 4 & -7 \end{bmatrix}$

6. $\begin{bmatrix} -6 & 1 \\ 3 & 12 \end{bmatrix} - \begin{bmatrix} 4 & 15 \\ -7 & 8 \end{bmatrix}$

7. $\begin{bmatrix} 7 & -6 & 2 \\ 8 & 3 & 5 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ -9 & 0 \\ 3 & -7 \end{bmatrix}$

Investigating Geometry ACTIVITY *Use before Lesson 9.3*

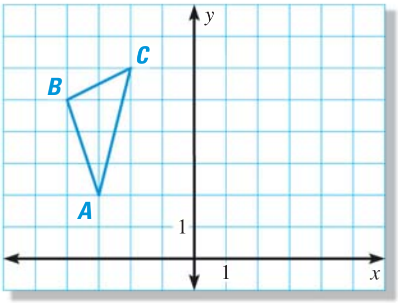
9.3 Reflections in the Plane

MATERIALS • graph paper • straightedge

QUESTION **What is the relationship between the line of reflection and the segment connecting a point and its image?**

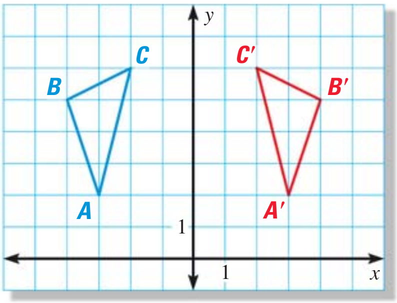
EXPLORE **Graph a reflection of a triangle**

STEP 1



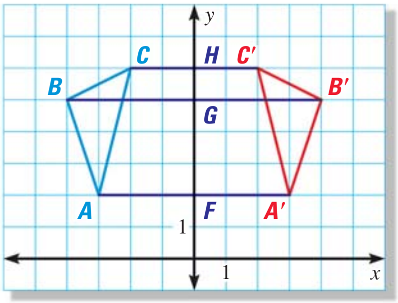
Draw a triangle Graph $A(-3, 2)$, $B(-4, 5)$, and $C(-2, 6)$. Connect the points to form $\triangle ABC$.

STEP 2



Graph a reflection Reflect $\triangle ABC$ in the y-axis. Label points A', B', and C' appropriately.

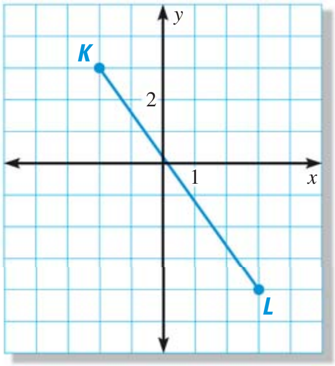
STEP 3



Draw segments Draw $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$. Label the points where these segments intersect the y-axis as F, G, and H, respectively.

DRAW CONCLUSIONS **Use your observations to complete these exercises**

1. Find the lengths of $\overline{CH}$ and $\overline{HC'}$, $\overline{BG}$ and $\overline{GB'}$, and $\overline{AF}$ and $\overline{FA'}$. *Compare* the lengths of each pair of segments.
2. Find the measures of $\angle CHG$, $\angle BGF$, and $\angle AFG$. *Compare* the angle measures.
3. How is the y-axis related to $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$?
4. Use the graph at the right.
   - **a.** $\overline{K'L'}$ is the reflection of $\overline{KL}$ in the x-axis. Copy the diagram and draw $\overline{K'L'}$.
   - **b.** Draw $\overline{KK'}$ and $\overline{LL'}$. Label the points where the segments intersect the x-axis as J and M.
   - **c.** How is the x-axis related to $\overline{KK'}$ and $\overline{LL'}$?
5. How is the line of reflection related to the segment connecting a point and its image?



9.3 Perform Reflections

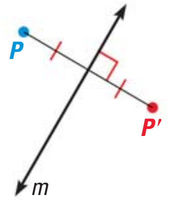
Before	You reflected a figure in the *x*- or *y*-axis.
Now	You will reflect a figure in any given line.
Why?	So you can identify reflections, as in Exs. 31–33.

Key Vocabulary
- **line of reflection**
- **reflection,** *p. 272*

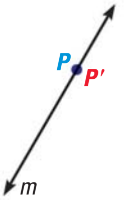
In Lesson 4.8, you learned that a *reflection* is a transformation that uses a line like a mirror to reflect an image. The mirror line is called the **line of reflection**.

A reflection in a line m maps every point P in the plane to a point P', so that for each point one of the following properties is true:

- If P is not on m, then m is the perpendicular bisector of $\overline{PP'}$, or
- If P is on m, then $P = P'$.



Point *P* not on *m*



Point *P* on *m*

EXAMPLE 1 Graph reflections in horizontal and vertical lines

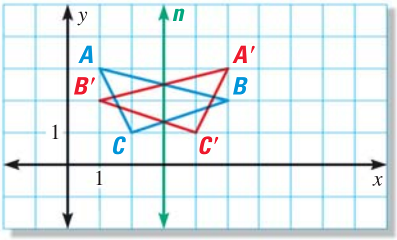
The vertices of $\triangle ABC$ are $A(1, 3)$, $B(5, 2)$, and $C(2, 1)$. Graph the reflection of $\triangle ABC$ described.

a. In the line n: $x = 3$

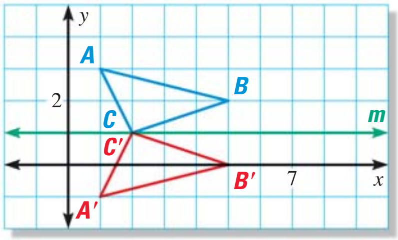
b. In the line m: $y = 1$

Solution

a. Point A is 2 units left of n, so its reflection A' is 2 units right of n at $(5, 3)$. Also, B' is 2 units left of n at $(1, 2)$, and C' is 1 unit right of n at $(4, 1)$.



b. Point A is 2 units above m, so A' is 2 units below m at $(1, -1)$. Also, B' is 1 unit below m at $(5, 0)$. Because point C is on line m, you know that $C = C'$.



✓ GUIDED PRACTICE for Example 1

Graph a reflection of $\triangle ABC$ from Example 1 in the given line.

1. $y = 4$

2. $x = -3$

3. $y = 2$

EXAMPLE 2 Graph a reflection in $y = x$

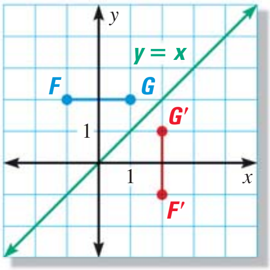
The endpoints of $\overline{FG}$ are $F(-1, 2)$ and $G(1, 2)$. Reflect the segment in the line $y = x$. Graph the segment and its image.

REVIEW SLOPE

The product of the slopes of perpendicular lines is -1.

Solution

The slope of $y = x$ is 1. The segment from F to its image, $\overline{FF'}$, is perpendicular to the line of reflection $y = x$, so the slope of $\overline{FF'}$ will be -1 (because $1(-1) = -1$). From F, move 1.5 units right and 1.5 units down to $y = x$. From that point, move 1.5 units right and 1.5 units down to locate $F'(3, -1)$.



The slope of $\overline{GG'}$ will also be -1. From G, move 0.5 units right and 0.5 units down to $y = x$. Then move 0.5 units right and 0.5 units down to locate $G'(2, 1)$.

COORDINATE RULES You can use coordinate rules to find the images of points reflected in four special lines.

KEY CONCEPT *For Your Notebook*

Coordinate Rules for Reflections

- If (a, b) is reflected in the x-axis, its image is the point $(a, -b)$.
- If (a, b) is reflected in the y-axis, its image is the point $(-a, b)$.
- If (a, b) is reflected in the line $y = x$, its image is the point (b, a).
- If (a, b) is reflected in the line $y = -x$, its image is the point $(-b, -a)$.

EXAMPLE 3 Graph a reflection in $y = -x$

Reflect $\overline{FG}$ from Example 2 in the line $y = -x$. Graph $\overline{FG}$ and its image.

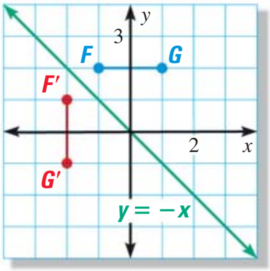
Solution

Use the coordinate rule for reflecting in $y = -x$.

$$(a, b) \rightarrow (-b, -a)$$

$F(-1, 2) \rightarrow F'(-2, 1)$

$G(1, 2) \rightarrow G'(-2, -1)$



Animated Geometry at classzone.com

✓ GUIDED PRACTICE for Examples 2 and 3

4. Graph $\triangle ABC$ with vertices $A(1, 3)$, $B(4, 4)$, and $C(3, 1)$. Reflect $\triangle ABC$ in the lines $y = -x$ and $y = x$. Graph each image.

5. In Example 3, *verify* that $\overline{FF'}$ is perpendicular to $y = -x$.

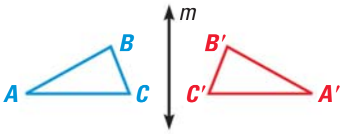
REFLECTION THEOREM You saw in Lesson 9.1 that the image of a translation is congruent to the original figure. The same is true for a reflection.

THEOREM

For Your Notebook

THEOREM 9.2 Reflection Theorem

A reflection is an isometry.



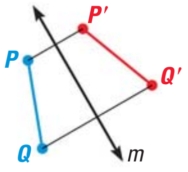
$\triangle ABC \cong \triangle A'B'C'$

Proof: Exs. 35–38, p. 595

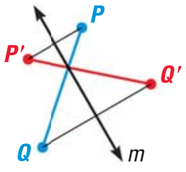
WRITE PROOFS

Some theorems, such as the Reflection Theorem, have more than one case. To prove this type of theorem, each case must be proven.

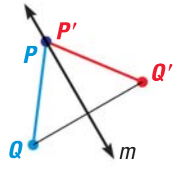
PROVING THE THEOREM To prove the Reflection Theorem, you need to show that a reflection preserves the length of a segment. Consider a segment $\overline{PQ}$ that is reflected in a line m to produce $\overline{P'Q'}$. There are four cases to prove:



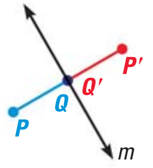
Case 1 P and Q are on the same side of m.



Case 2 P and Q are on opposite sides of m.



Case 3 P lies on m, and $\overline{PQ}$ is not $\perp$ to m.



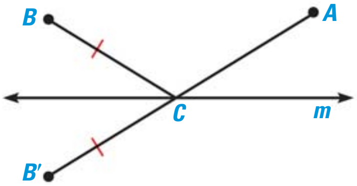
Case 4 Q lies on m, and $\overline{PQ} \perp m$.

EXAMPLE 4 Find a minimum distance

PARKING You are going to buy books. Your friend is going to buy CDs. Where should you park to minimize the distance you both will walk?

Solution

Reflect B in line m to obtain B'. Then draw $\overline{AB'}$. Label the intersection of $\overline{AB'}$ and m as C. Because AB' is the shortest distance between A and B' and $BC = B'C$, park at point C to minimize the combined distance, $AC + BC$, you both have to walk.



✓ GUIDED PRACTICE for Example 4

6. Look back at Example 4. Answer the question by using a reflection of point A instead of point B.

REFLECTION MATRIX You can find the image of a polygon reflected in the x-axis or y-axis using matrix multiplication. Write the reflection matrix to the *left* of the polygon matrix, then multiply.

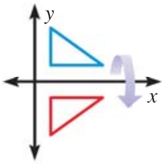
Notice that because matrix multiplication is not commutative, the order of the matrices in your product is important. The reflection matrix must be first followed by the polygon matrix.

KEY CONCEPT *For Your Notebook*

Reflection Matrices

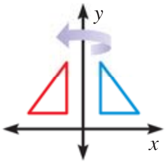
Reflection in the x-axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



Reflection in the y-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



EXAMPLE 5 Use matrix multiplication to reflect a polygon

The vertices of $\triangle DEF$ are $D(1, 2)$, $E(3, 3)$, and $F(4, 0)$. Find the reflection of $\triangle DEF$ in the y-axis using matrix multiplication. Graph $\triangle DEF$ and its image.

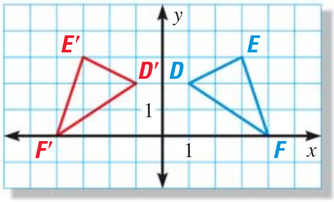
Solution

STEP 1 **Multiply** the polygon matrix by the matrix for a reflection in the y-axis.

$$\underset{\text{Reflection matrix}}{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}} \underset{\text{Polygon matrix}}{\begin{array}{c} \begin{array}{ccc} D & E & F \end{array} \\ \begin{bmatrix} 1 & 3 & 4 \\ 2 & 3 & 0 \end{bmatrix} \end{array}} = \begin{bmatrix} -1(1) + 0(2) & -1(3) + 0(3) & -1(4) + 0(0) \\ 0(1) + 1(2) & 0(3) + 1(3) & 0(4) + 1(0) \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{ccc} D' & E' & F' \end{array} \\ \begin{bmatrix} -1 & -3 & -4 \\ 2 & 3 & 0 \end{bmatrix} \end{array} \quad \text{Image matrix}$$

STEP 2 **Graph** $\triangle DEF$ and $\triangle D'E'F'$.



✓ GUIDED PRACTICE for Example 5

The vertices of $\triangle LMN$ are $L(-3, 3)$, $M(1, 2)$, and $N(-2, 1)$. Find the described reflection using matrix multiplication.

7. Reflect $\triangle LMN$ in the x-axis.

8. Reflect $\triangle LMN$ in the y-axis.

9.3 EXERCISES

HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 5, 13, and 33

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 12, 25, and 40

SKILL PRACTICE

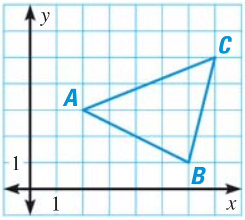
1. **VOCABULARY** What is a *line of reflection*?

2. ★ **WRITING** *Explain* how to find the distance from a point to its image if you know the distance from the point to the line of reflection.

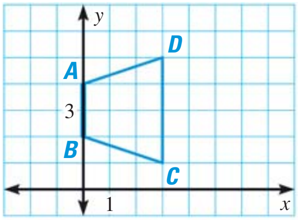
EXAMPLE 1
on p. 589
for Exs. 3–8

REFLECTIONS **Graph the reflection of the polygon in the given line.**

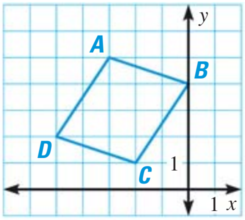
3. x-axis



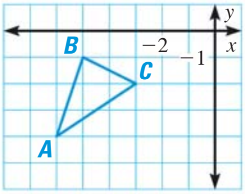
4. y-axis



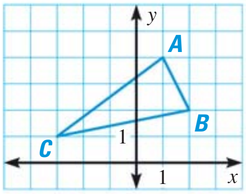
5. $y = 2$



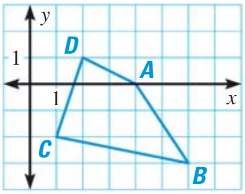
6. $x = -1$



7. y-axis

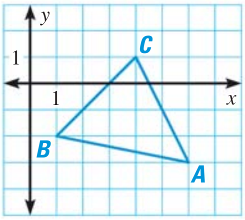


8. $y = -3$

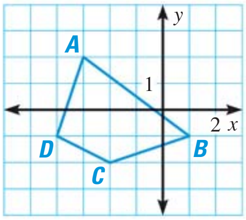


EXAMPLES 2 and 3
on p. 590
for Exs. 9–12

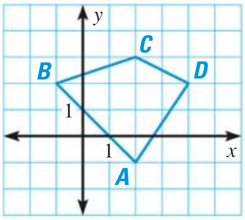
9. $y = x$



10. $y = -x$



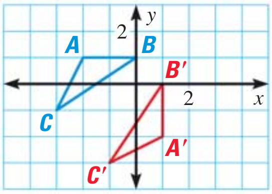
11. $y = x$



12. ★ **MULTIPLE CHOICE** What is the line of reflection for $\triangle ABC$ and its image?

Ⓐ $y = 0$ (the x-axis)　　Ⓑ $y = -x$

Ⓒ $x = 1$　　Ⓓ $y = x$



EXAMPLE 5
on p. 592
for Exs. 13–17

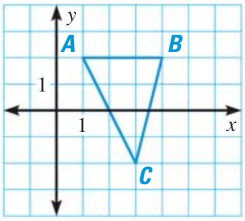
USING MATRIX MULTIPLICATION **Use matrix multiplication to find the image. Graph the polygon and its image.**

13. Reflect $\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} -2 & 3 & 4 \\ 5 & -3 & 6 \end{bmatrix} \end{array}$ in the x-axis.

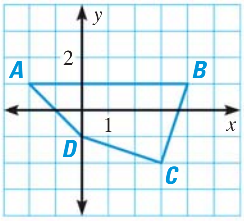
14. Reflect $\begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{bmatrix} 2 & 6 & 5 & 2 \\ -2 & -3 & -8 & -5 \end{bmatrix} \end{array}$ in the y-axis.

FINDING IMAGE MATRICES **Write a matrix for the polygon. Then find the image matrix that represents the polygon after a reflection in the given line.**

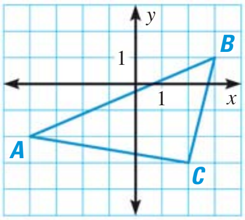
15. y-axis



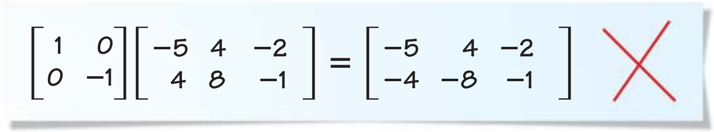
16. x-axis



17. y-axis



18. **ERROR ANALYSIS** *Describe* and correct the error in finding the image matrix of $\triangle PQR$ reflected in the y-axis.



$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -5 & 4 & -2 \\ 4 & 8 & -1 \end{bmatrix} = \begin{bmatrix} -5 & 4 & -2 \\ -4 & -8 & -1 \end{bmatrix}$$

MINIMUM DISTANCE **Find point C on the x-axis so $AC + BC$ is a minimum.**

19. $A(1, 4)$, $B(6, 1)$

20. $A(4, -3)$, $B(12, -5)$

21. $A(-8, 4)$, $B(-1, 3)$

TWO REFLECTIONS **The vertices of $\triangle FGH$ are $F(3, 2)$, $G(1, 5)$, and $H(-1, 2)$. Reflect $\triangle FGH$ in the first line. Then reflect $\triangle F'G'H'$ in the second line. Graph $\triangle F'G'H'$ and $\triangle F''G''H''$.**

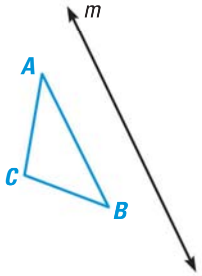
22. In $y = 2$, then in $y = -1$

23. In $y = -1$, then in $x = 2$

24. In $y = x$, then in $x = -3$

25. ★ **SHORT RESPONSE** Use your graphs from Exercises 22–24. What do you notice about the order of vertices in the preimages and images?

26. **CONSTRUCTION** Use these steps to construct a reflection of $\triangle ABC$ in line m using a straightedge and a compass.



STEP 1 **Draw** $\triangle ABC$ and line m.

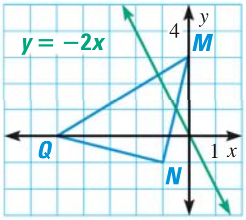
STEP 2 **Use** one compass setting to find two points that are equidistant from A on line m. Use the same compass setting to find a point on the other side of m that is the same distance from line m. Label that point A'.

STEP 3 **Repeat** Step 2 to find points B' and C'. Draw $\triangle A'B'C'$.

27. xy **ALGEBRA** The line $y = 3x + 2$ is reflected in the line $y = -1$. What is the equation of the image?

28. xy **ALGEBRA** Reflect the graph of the quadratic equation $y = 2x^2 - 5$ in the x-axis. What is the equation of the image?

29. **REFLECTING A TRIANGLE** Reflect $\triangle MNQ$ in the line $y = -2x$.



30. **CHALLENGE** Point $B'(1, 4)$ is the image of $B(3, 2)$ after a reflection in line c. Write an equation of line c.

PROBLEM SOLVING

REFLECTIONS **Identify the case of the Reflection Theorem represented.**

31.

32.

33.

EXAMPLE 4
on p. 591
for Ex. 34

34. DELIVERING PIZZA You park at some point K on line n. You deliver a pizza to house H, go back to your car, and deliver a pizza to house J. Assuming that you can cut across both lawns, how can you determine the parking location K that minimizes the total walking distance?

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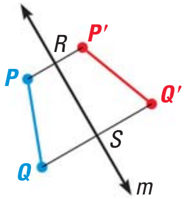
35. PROVING THEOREM 9.2 Prove Case 1 of the Reflection Theorem.



Case 1 The segment does not intersect the line of reflection.

GIVEN ▶ A reflection in m maps P to P' and Q to Q'.

PROVE ▶ $PQ = P'Q'$

Plan for Proof

a. Draw $\overline{PP'}$, $\overline{QQ'}$, $\overline{RQ}$, and $\overline{RQ'}$. Prove that $\triangle RSQ \cong \triangle RSQ'$.

b. Use the properties of congruent triangles and perpendicular bisectors to prove that $PQ = P'Q'$.

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PROVING THEOREM 9.2 **In Exercises 36–38, write a proof for the given case of the Reflection Theorem. (Refer to the diagrams on page 591.)**

36. Case 2 The segment intersects the line of reflection.

GIVEN ▶ A reflection in m maps P to P' and Q to Q'.
Also, $\overline{PQ}$ intersects m at point R.

PROVE ▶ $PQ = P'Q'$

37. Case 3 One endpoint is on the line of reflection, and the segment is not perpendicular to the line of reflection.

GIVEN ▶ A reflection in m maps P to P' and Q to Q'.
Also, P lies on line m, and $\overline{PQ}$ is not perpendicular to m.

PROVE ▶ $PQ = P'Q'$

38. Case 4 One endpoint is on the line of reflection, and the segment is perpendicular to the line of reflection.

GIVEN ▶ A reflection in m maps P to P' and Q to Q'.
Also, Q lies on line m, and $\overline{PQ}$ is perpendicular to line m.

PROVE ▶ $PQ = P'Q'$

39. REFLECTING POINTS Use $C(1, 3)$.

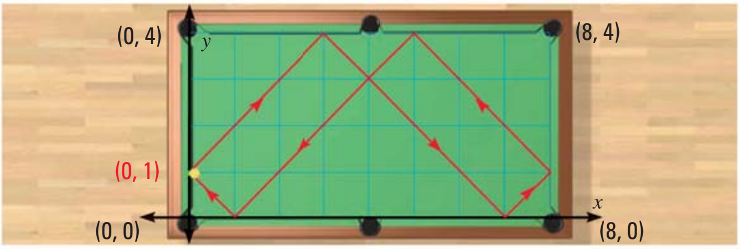
a. Point A has coordinates $(-1, 1)$. Find point B on $\overrightarrow{AC}$ so $AC = CB$.

b. The endpoints of $\overline{FG}$ are $F(2, 0)$ and $G(3, 2)$. Find point H on $\overrightarrow{FC}$ so $FC = CH$. Find point J on $\overrightarrow{GC}$ so $GC = CJ$.

c. Explain why parts (a) and (b) can be called *reflection in a point*.

PHYSICS The Law of Reflection states that the angle of incidence is congruent to the angle of reflection. Use this information in Exercises 40 and 41.

40. ★ SHORT RESPONSE Suppose a billiard table has a coordinate grid on it. If a ball starts at the point $(0, 1)$ and rolls at a 45° angle, it will eventually return to its starting point. Would this happen if the ball started from other points on the y-axis between $(0, 0)$ and $(0, 4)$? *Explain.*

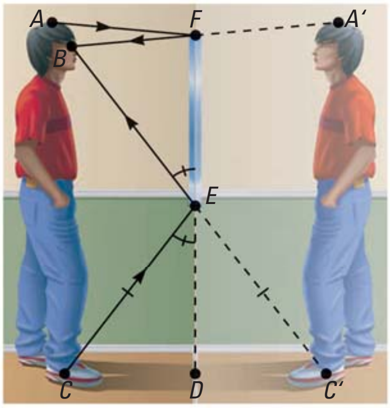


41. CHALLENGE Use the diagram to prove that you can see your full self in a mirror that is only half of your height. Assume that you and the mirror are both perpendicular to the floor.

a. Think of a light ray starting at your foot and reflected in a mirror. Where does it have to hit the mirror in order to reflect to your eye?

b. Think of a light ray starting at the top of your head and reflected in a mirror. Where does it have to hit the mirror in order to reflect to your eye?

c. Show that the distance between the points you found in parts (a) and (b) is half your height.



MIXED REVIEW

PREVIEW
Prepare for Lesson 9.4 in Exs. 42–43.

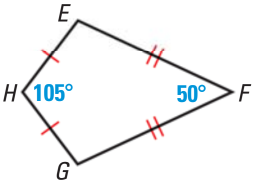
Tell whether the lines through the given points are *parallel*, *perpendicular*, or *neither*. *Justify* your answer. *(p. 171)*

42. Line 1: $(3, 7)$ and $(9, 7)$
Line 2: $(-2, 8)$ and $(-2, 1)$

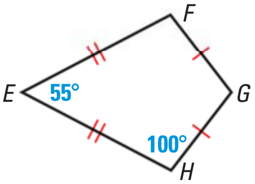
43. Line 1: $(-4, -1)$ and $(-8, -4)$
Line 2: $(1, -3)$ and $(5, 0)$

Quadrilateral *EFGH* is a kite. Find $m\angle G$. *(p. 542)*

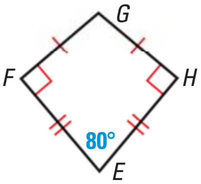
44.



45.



46.



MIXED REVIEW *of Problem Solving*

STATE TEST PRACTICE
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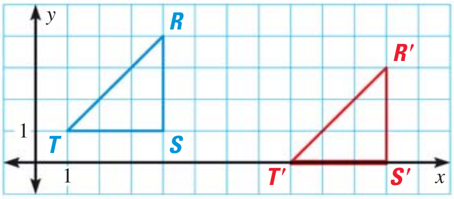
Lessons 9.1–9.3

1. **MULTI-STEP PROBLEM** $\triangle R'S'T'$ is the image of $\triangle RST$ after a translation.

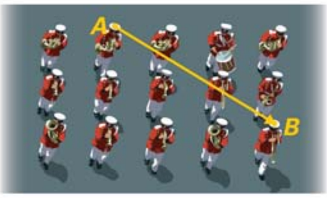


   a. Write a rule for the translation.

   b. *Verify* that the transformation is an isometry.

   c. Suppose $\triangle R'S'T'$ is translated using the rule $(x, y) \rightarrow (x + 4, y - 2)$. What are the coordinates of the vertices of $\triangle R''S''T''$?

2. **SHORT RESPONSE** During a marching band routine, a band member moves directly from point A to point B. Write the component form of the vector $\overrightarrow{AB}$. *Explain* your answer.



3. **SHORT RESPONSE** Trace the picture below. Reflect the image in line m. How is the distance from X to line m related to the distance from X' to line m? Write the property that makes this true.

4. **SHORT RESPONSE** The endpoints of $\overline{AB}$ are $A(2, 4)$ and $B(4, 0)$. The endpoints of $\overline{CD}$ are $C(3, 3)$ and $D(7, -1)$. Is the transformation from $\overline{AB}$ to $\overline{CD}$ an isometry? *Explain.*

5. **GRIDDED ANSWER** The vertices of $\triangle FGH$ are $F(-4, 3)$, $G(3, -1)$, and $H(1, -2)$. The coordinates of F' are $(-1, 4)$ after a translation. What is the x-coordinate of G'?

6. **OPEN-ENDED** Draw a triangle in a coordinate plane. Reflect the triangle in an axis. Write the reflection matrix that would yield the same result.

7. **EXTENDED RESPONSE** Two cross-country teams submit equipment lists for a season. A pair of running shoes costs \$60, a pair of shorts costs \$18, and a shirt costs \$15.

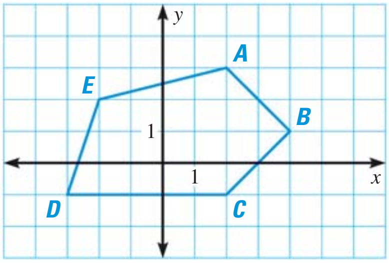
Women's Team	Men's Team
14 pairs of shoes	10 pairs of shoes
16 pairs of shorts	13 pairs of shorts
16 shirts	13 shirts

   a. Use matrix multiplication to find the total cost of equipment for each team.

   b. How much money will the teams need to raise if the school gives each team \$200?

   c. Repeat parts (a) and (b) if a pair of shoes costs \$65 and a shirt costs \$10. Does the change in prices change which team needs to raise more money? *Explain.*

8. **MULTI-STEP PROBLEM** Use the polygon as the preimage.



   a. Reflect the preimage in the y-axis.

   b. Reflect the preimage in the x-axis.

   c. *Compare* the order of vertices in the preimage with the order in each image.

9.4 Perform Rotations

Before You rotated figures about the origin.

Now You will rotate figures about a point.

Why? So you can classify transformations, as in Exs. 3–5.

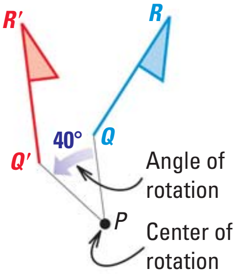
Key Vocabulary
- **center of rotation**
- **angle of rotation**
- **rotation,** *p. 272*

Recall from Lesson 4.8 that a *rotation* is a transformation in which a figure is turned about a fixed point called the **center of rotation**. Rays drawn from the center of rotation to a point and its image form the **angle of rotation**.

A rotation about a point P through an angle of $x°$ maps every point Q in the plane to a point Q' so that one of the following properties is true:

- If Q is not the center of rotation P, then $QP = Q'P$ and $m\angle QPQ' = x°$, or
- If Q is the center of rotation P, then the image of Q is Q.

A 40° counterclockwise rotation is shown at the right. Rotations can be *clockwise* or *counterclockwise*. In this chapter, all rotations are counterclockwise.



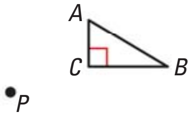
DIRECTION OF ROTATION

clockwise

counterclockwise

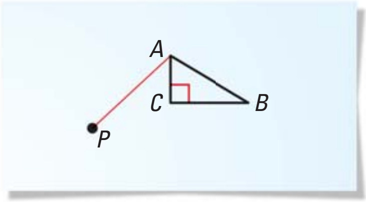
EXAMPLE 1 Draw a rotation

Draw a 120° rotation of $\triangle ABC$ about P.

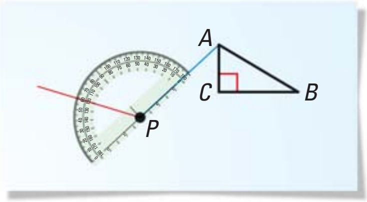


Solution

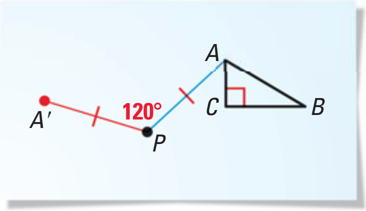
STEP 1 **Draw** a segment from A to P.



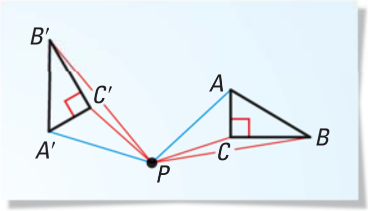
STEP 2 **Draw** a ray to form a 120° angle with $\overline{PA}$.



STEP 3 **Draw** A' so that $PA' = PA$.

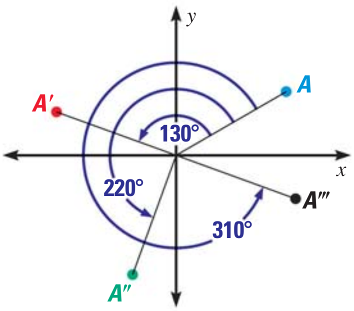


STEP 4 **Repeat** Steps 1–3 for each vertex. Draw $\triangle A'B'C'$.



ROTATIONS ABOUT THE ORIGIN You can rotate a figure more than 180°. The diagram shows rotations of point A 130°, 220°, and 310° about the origin. A rotation of 360° returns a figure to its original coordinates.

There are coordinate rules that can be used to find the coordinates of a point after rotations of 90°, 180°, or 270° about the origin.



USE ROTATIONS

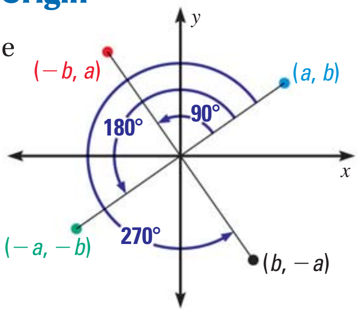
You can rotate a figure more than 360°. However, the effect is the same as rotating the figure by the angle minus 360°.

KEY CONCEPT *For Your Notebook*

Coordinate Rules for Rotations about the Origin

When a point (a, b) is rotated counterclockwise about the origin, the following are true:

1. For a rotation of 90°, $(a, b) \rightarrow (-b, a)$.
2. For a rotation of 180°, $(a, b) \rightarrow (-a, -b)$.
3. For a rotation of 270°, $(a, b) \rightarrow (b, -a)$.



EXAMPLE 2 Rotate a figure using the coordinate rules

Graph quadrilateral $RSTU$ with vertices $R(3, 1)$, $S(5, 1)$, $T(5, -3)$, and $U(2, -1)$. Then rotate the quadrilateral 270° about the origin.

Solution

Graph $RSTU$. Use the coordinate rule for a 270° rotation to find the images of the vertices.

$$(a, b) \rightarrow (b, -a)$$

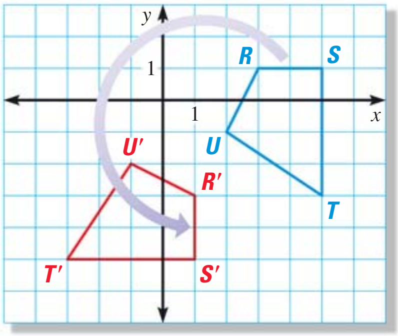
$R(3, 1) \rightarrow R'(1, -3)$

$S(5, 1) \rightarrow S'(1, -5)$

$T(5, -3) \rightarrow T'(-3, -5)$

$U(2, -1) \rightarrow U'(-1, -2)$

Graph the image $R'S'T'U'$.



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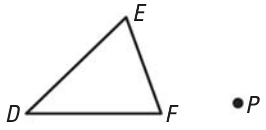
ANOTHER WAY

For an alternative method for solving the problem in Example 2, turn to page 606 for the **Problem Solving Workshop**.

✓ **GUIDED PRACTICE** for Examples 1 and 2

1. Trace $\triangle DEF$ and P. Then draw a 50° rotation of $\triangle DEF$ about P.

2. Graph $\triangle JKL$ with vertices $J(3, 0)$, $K(4, 3)$, and $L(6, 0)$. Rotate the triangle 90° about the origin.



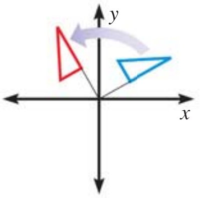
USING MATRICES You can find certain images of a polygon rotated about the origin using matrix multiplication. Write the rotation matrix to the left of the polygon matrix, then multiply.

KEY CONCEPT *For Your Notebook*

Rotation Matrices (Counterclockwise)

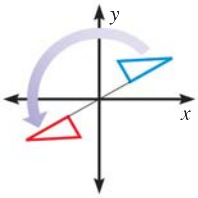
90° rotation

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



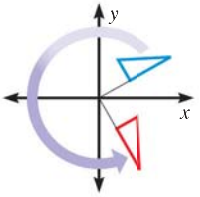
180° rotation

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$



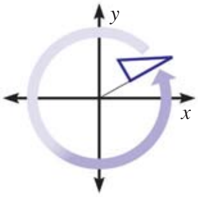
270° rotation

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$



360° rotation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



READ VOCABULARY

Notice that a 360° rotation returns the figure to its original position. Multiplying by the matrix that represents this rotation gives you the polygon matrix you started with, which is why it is also called the *identity matrix*.

EXAMPLE 3 Use matrices to rotate a figure

Trapezoid *EFGH* has vertices *E*(−3, 2), *F*(−3, 4), *G*(1, 4), and *H*(2, 2). Find the image matrix for a 180° rotation of *EFGH* about the origin. Graph *EFGH* and its image.

Solution

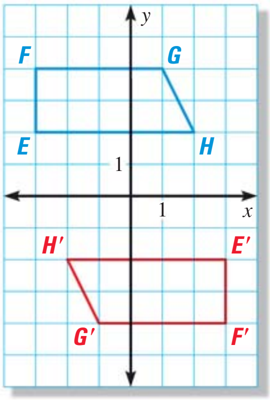
STEP 1 **Write** the polygon matrix: $\begin{array}{c} \begin{array}{cccc} E & F & G & H \end{array} \\ \begin{bmatrix} -3 & -3 & 1 & 2 \\ 2 & 4 & 4 & 2 \end{bmatrix} \end{array}$

STEP 2 **Multiply** by the matrix for a 180° rotation.

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{array}{c} \begin{array}{cccc} E & F & G & H \end{array} \\ \begin{bmatrix} -3 & -3 & 1 & 2 \\ 2 & 4 & 4 & 2 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} E' & F' & G' & H' \end{array} \\ \begin{bmatrix} 3 & 3 & -1 & -2 \\ -2 & -4 & -4 & -2 \end{bmatrix} \end{array}$$

Rotation matrix **Polygon matrix** **Image matrix**

STEP 3 **Graph** the preimage *EFGH*.
Graph the image *E′F′G′H′*.



AVOID ERRORS

Because matrix multiplication is not commutative, you should always write the rotation matrix first, then the polygon matrix.

✓ GUIDED PRACTICE for Example 3

Use the quadrilateral *EFGH* in Example 3. Find the image matrix after the rotation about the origin. Graph the image.

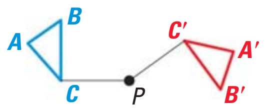
3. 90° **4.** 270° **5.** 360°

THEOREM — For Your Notebook

THEOREM 9.3 Rotation Theorem

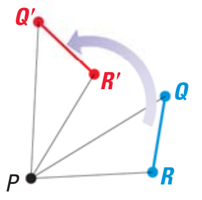
A rotation is an isometry.

Proof: Exs. 33–35, p. 604

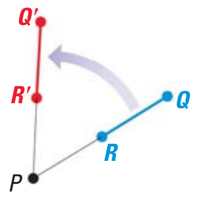


$\triangle ABC \cong \triangle A'B'C'$

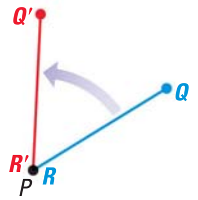
CASES OF THEOREM 9.3 To prove the Rotation Theorem, you need to show that a rotation preserves the length of a segment. Consider a segment $\overline{QR}$ rotated about point P to produce $\overline{Q'R'}$. There are three cases to prove:



Case 1 R, Q, and P are noncollinear.



Case 2 R, Q, and P are collinear.



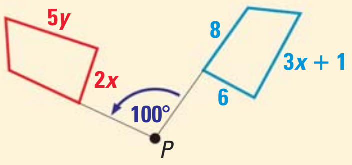
Case 3 P and R are the same point.

EXAMPLE 4 Standardized Test Practice

The quadrilateral is rotated about P. What is the value of y?

Ⓐ $\frac{8}{5}$ Ⓑ 2

Ⓒ 3   Ⓓ 10



Solution

By Theorem 9.3, the rotation is an isometry, so corresponding side lengths are equal. Then $2x = 6$, so $x = 3$. Now set up an equation to solve for y.

$5y = 3x + 1$	Corresponding lengths in an isometry are equal.
$5y = 3(3) + 1$	Substitute 3 for x.
$y = 2$	Solve for y.

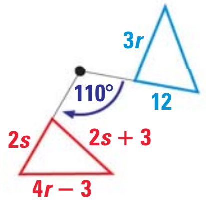
▶ The correct answer is B. Ⓐ Ⓑ Ⓒ Ⓓ

GUIDED PRACTICE for Example 4

6. Find the value of r in the rotation of the triangle.

Ⓐ 3   Ⓑ 5

Ⓒ 6   Ⓓ 15



9.4 EXERCISES

HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 13, 15, and 29

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 20, 21, 23, 24, and 37

SKILL PRACTICE

1. **VOCABULARY** What is a *center of rotation*?

2. ★ **WRITING** *Compare* the coordinate rules and the rotation matrices for a rotation of 90°.

EXAMPLE 1
on p. 598
for Exs. 3–11

IDENTIFYING TRANSFORMATIONS **Identify the type of transformation, *translation*, *reflection*, or *rotation*, in the photo. *Explain* your reasoning.**

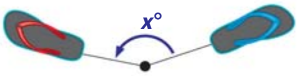
3.

4.

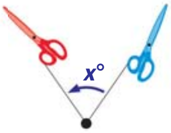
5.

ANGLE OF ROTATION **Match the diagram with the angle of rotation.**

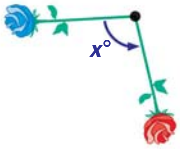
6.



7.



8.



A. 70°

B. 100°

C. 150°

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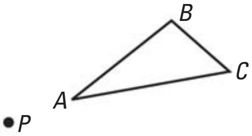
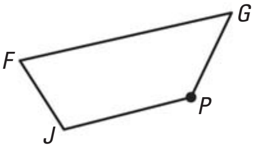
ROTATING A FIGURE **Trace the polygon and point *P* on paper. Then draw a rotation of the polygon the given number of degrees about *P*.**

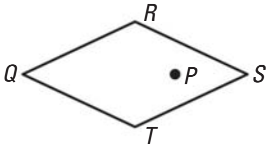
9. 30°



10. 150°



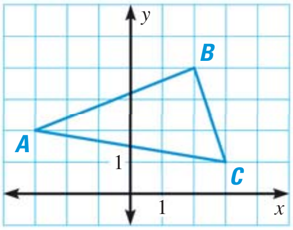
11. 130°



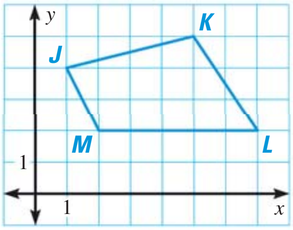
EXAMPLE 2
on p. 599
for Exs. 12–14

USING COORDINATE RULES **Rotate the figure the given number of degrees about the origin. List the coordinates of the vertices of the image.**

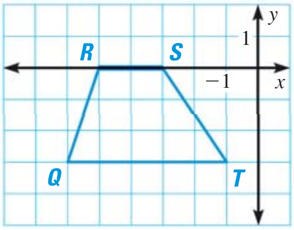
12. 90°



13. 180°



14. 270°



EXAMPLE 3
on p. 600
for Exs. 15–19

USING MATRICES Find the image matrix that represents the rotation of the polygon about the origin. Then graph the polygon and its image.

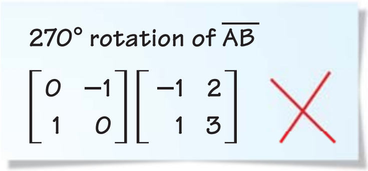
15. $\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1 & 5 & 4 \\ 4 & 6 & 3 \end{bmatrix} \end{array}$; 90°

16. $\begin{array}{c} \begin{array}{ccc} J & K & L \end{array} \\ \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & -3 \end{bmatrix} \end{array}$; 180°

17. $\begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{bmatrix} -4 & 2 & 2 & -4 \\ -4 & -2 & -5 & -7 \end{bmatrix} \end{array}$; 270°

ERROR ANALYSIS The endpoints of $\overline{AB}$ are $A(-1, 1)$ and $B(2, 3)$. *Describe* and correct the error in setting up the matrix multiplication for a 270° rotation about the origin.

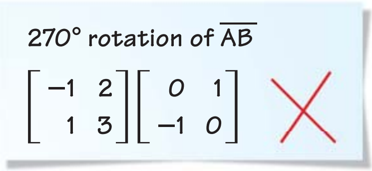
18.



270° rotation of $\overline{AB}$

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$ ✗

19.



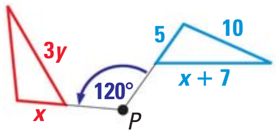
270° rotation of $\overline{AB}$

$\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ✗

EXAMPLE 4
on p. 601
for Exs. 20–21

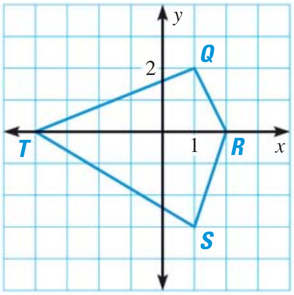
20. ★ **MULTIPLE CHOICE** What is the value of y in the rotation of the triangle about P?

Ⓐ 4   Ⓑ 5   Ⓒ $\frac{17}{3}$ Ⓓ 10



21. ★ **MULTIPLE CHOICE** Suppose quadrilateral $QRST$ is rotated 180° about the origin. In which quadrant is Q'?

Ⓐ I   Ⓑ II   Ⓒ III   Ⓓ IV



22. **FINDING A PATTERN** The vertices of $\triangle ABC$ are $A(2, 0)$, $B(3, 4)$, and $C(5, 2)$. Make a table to show the vertices of each image after a 90°, 180°, 270°, 360°, 450°, 540°, 630°, and 720° rotation. What would be the coordinates of A' after a rotation of 1890°? *Explain*.

23. ★ **MULTIPLE CHOICE** A rectangle has vertices at (4, 0), (4, 2), (7, 0), and (7, 2). Which image has a vertex at the origin?

Ⓐ Translation right 4 units and down 2 units

Ⓑ Rotation of 180° about the origin

Ⓒ Reflection in the line $x = 4$

Ⓓ Rotation of 180° about the point (2, 0)

24. ★ **SHORT RESPONSE** Rotate the triangle in Exercise 12 90° about the origin. Show that corresponding sides of the preimage and image are perpendicular. *Explain*.

25. **VISUAL REASONING** A point in space has three coordinates (x, y, z). What is the image of point (3, 2, 0) rotated 180° about the origin in the xz-plane? (*See* Exercise 30, page 585.)

CHALLENGE Rotate the line the given number of degrees (a) about the x-intercept and (b) about the y-intercept. Write the equation of each image.

26. $y = 2x - 3$; 90°

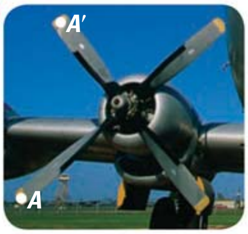
27. $y = -x + 8$; 180°

28. $y = \frac{1}{2}x + 5$; 270°

PROBLEM SOLVING

ANGLE OF ROTATION **Use the photo to find the angle of rotation that maps A onto A'. *Explain* your reasoning.**

29.



30.

31.

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32. **REVOLVING DOOR** You enter a revolving door and rotate the door 180°. What does this mean in the context of the situation? Now, suppose you enter a revolving door and rotate the door 360°. What does this mean in the context of the situation? *Explain*.

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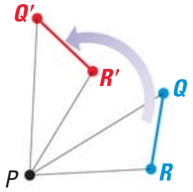
33. **PROVING THEOREM 9.3** Copy and complete the proof of Case 1.

Case 1 The segment is noncollinear with the center of rotation.

GIVEN ▶ A rotation about P maps Q to Q' and R to R'.

PROVE ▶ $QR = Q'R'$



STATEMENTS	REASONS
1. $PQ = PQ'$, $PR = PR'$, $m\angle QPQ' = m\angle RPR'$	1. Definition of _?_
2. $m\angle QPQ' = m\angle QPR' + m\angle R'PQ'$ $m\angle RPR' = m\angle RPQ + m\angle QPR'$	2. _?_
3. $m\angle QPR' + m\angle R'PQ' =$ $m\angle RPQ + m\angle QPR'$	3. _?_ Property of Equality
4. $m\angle QPR = m\angle Q'PR'$	4. _?_ Property of Equality
5. _?_ ≅ _?_	5. SAS Congruence Postulate
6. $\overline{QR} \cong \overline{Q'R'}$	6. _?_
7. $QR = Q'R'$	7. _?_

PROVING THEOREM 9.3 **Write a proof for Case 2 and Case 3. (Refer to the diagrams on page 601.)**

34. **Case 2** The segment is collinear with the center of rotation.

GIVEN ▶ A rotation about P maps Q to Q' and R to R'.
P, Q, and R are collinear.

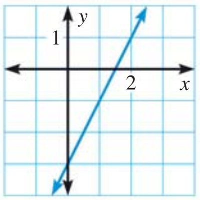
PROVE ▶ $QR = Q'R'$

35. **Case 3** The center of rotation is one endpoint of the segment.

GIVEN ▶ A rotation about P maps Q to Q' and R to R'.
P and R are the same point.

PROVE ▶ $QR = Q'R'$

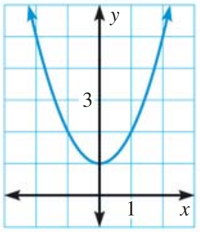
36. MULTI-STEP PROBLEM Use the graph of $y = 2x - 3$.



a. Rotate the line 90°, 180°, 270°, and 360° about the origin. *Describe* the relationship between the equation of the preimage and each image.

b. Do you think that the relationships you described in part (a) are true for *any* line? *Explain* your reasoning.

37. ★ EXTENDED RESPONSE Use the graph of the quadratic equation $y = x^2 + 1$ at the right.



a. Rotate the *parabola* by replacing y with x and x with y in the original equation, then graph this new equation.

b. What is the angle of rotation?

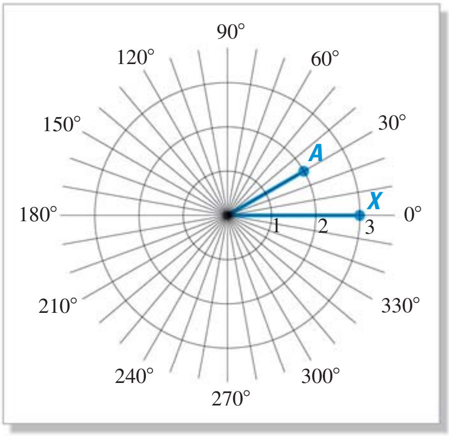
c. Are the image and the preimage both functions? *Explain.*

TWO ROTATIONS The endpoints of $\overline{FG}$ are $F(1, 2)$ and $G(3, 4)$. Graph $\overline{F'G'}$ and $\overline{F''G''}$ after the given rotations.

38. Rotation: 90° about the origin
Rotation: 180° about (0, 4)

39. Rotation: 270° about the origin
Rotation: 90° about (−2, 0)

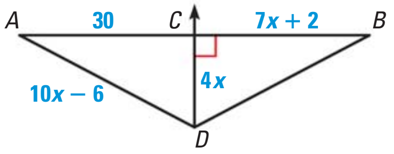
40. CHALLENGE A polar coordinate system locates a point in a plane by its distance from the origin O and by the measure of an angle with its vertex at the origin. For example, the point $A(2, 30°)$ at the right is 2 units from the origin and $m\angle XOA = 30°$. What are the polar coordinates of the image of point A after a 90° rotation? 180° rotation? 270° rotation? *Explain.*



MIXED REVIEW

PREVIEW
Prepare for Lesson 9.5 in Exs. 41–43.

In the diagram, $\overrightarrow{DC}$ is the perpendicular bisector of $\overline{AB}$. *(p. 303)*



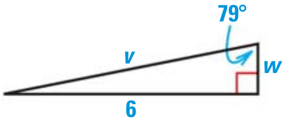
41. What segment lengths are equal?

42. What is the value of x?

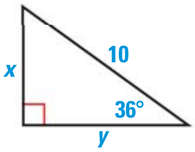
43. Find BD. *(p. 433)*

Use a sine or cosine ratio to find the value of each variable. Round decimals to the nearest tenth. *(p. 473)*

44.



45.



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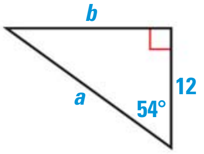


PROBLEM SOLVING WORKSHOP
LESSON 9.4

Using ALTERNATIVE METHODS

Another Way to Solve Example 2, page 599

MULTIPLE REPRESENTATIONS In Example 2 on page 599, you saw how to use a coordinate rule to rotate a figure. You can also *use tracing paper* and move a copy of the figure around the coordinate plane.

PROBLEM

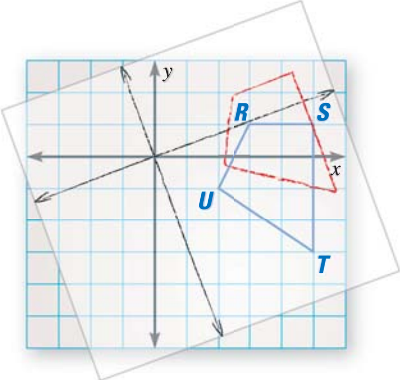
Graph quadrilateral *RSTU* with vertices $R(3, 1)$, $S(5, 1)$, $T(5, -3)$, and $U(2, -1)$. Then rotate the quadrilateral 270° about the origin.

METHOD

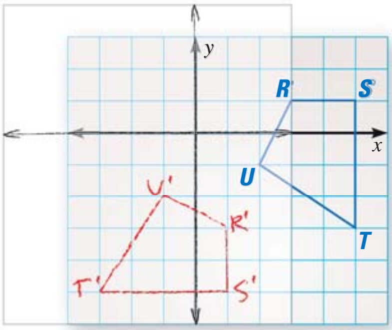
Using Tracing Paper You can use tracing paper to rotate a figure.

STEP 1 **Graph** the original figure in the coordinate plane.

STEP 2 **Trace** the quadrilateral and the axes on tracing paper.



STEP 3 **Rotate** the tracing paper 270°. Then transfer the resulting image onto the graph paper.



PRACTICE

1. **GRAPH** Graph quadrilateral *ABCD* with vertices $A(2, -2)$, $B(5, -3)$, $C(4, -5)$, and $D(2, -4)$. Then rotate the quadrilateral 180° about the origin using tracing paper.

2. **GRAPH** Graph $\triangle RST$ with vertices $R(0, 6)$, $S(1, 4)$, and $T(-2, 3)$. Then rotate the triangle 270° about the origin using tracing paper.

3. **SHORT RESPONSE** *Explain* why rotating a figure 90° clockwise is the same as rotating the figure 270° counterclockwise.

4. **SHORT RESPONSE** *Explain* how you could use tracing paper to do a reflection.

5. **REASONING** If you rotate the point (3, 4) 90° about the origin, what happens to the *x*-coordinate? What happens to the *y*-coordinate?

6. **GRAPH** Graph $\triangle JKL$ with vertices $J(4, 8)$, $K(4, 6)$, and $L(2, 6)$. Then rotate the triangle 90° about the point $(-1, 4)$ using tracing paper.

Investigating Geometry ACTIVITY *Use before Lesson 9.5*

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9.5 Double Reflections

MATERIALS • graphing calculator or computer

QUESTION What happens when you reflect a figure in two lines in a plane?

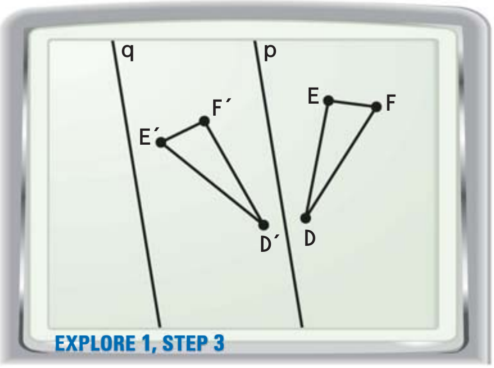
EXPLORE 1 Double reflection in parallel lines

STEP 1 ***Draw a scalene triangle*** Construct a scalene triangle like the one at the right. Label the vertices D, E, and F.

STEP 2 ***Draw parallel lines*** Construct two parallel lines p and q on one side of the triangle. Make sure that the lines do not intersect the triangle. Save as "EXPLORE1".

STEP 3 ***Reflect triangle*** Reflect $\triangle DEF$ in line p. Reflect $\triangle D'E'F'$ in line q. How is $\triangle D''E''F''$ related to $\triangle DEF$?

STEP 4 ***Make conclusion*** Drag line q. Does the relationship appear to be true if p and q are not on the same side of the figure?



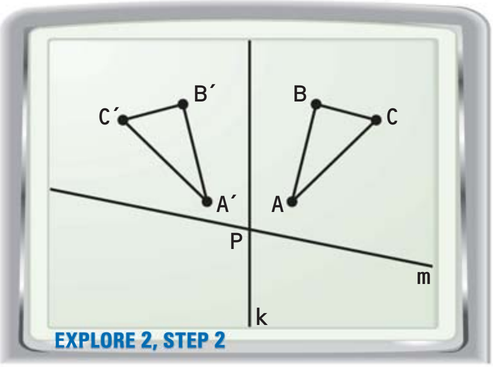
EXPLORE 1, STEP 3

EXPLORE 2 Double reflection in intersecting lines

STEP 1 ***Draw intersecting lines*** Follow Step 1 in Explore 1 for $\triangle ABC$. Change Step 2 from parallel lines to intersecting lines k and m. Make sure that the lines do not intersect the triangle. Label the point of intersection of lines k and m as P. Save as "EXPLORE2".

STEP 2 ***Reflect triangle*** Reflect $\triangle ABC$ in line k. Reflect $\triangle A'B'C'$ in line m. How is $\triangle A''B''C''$ related to $\triangle ABC$?

STEP 3 ***Measure angles*** Measure $\angle APA''$ and the acute angle formed by lines k and m. What is the relationship between these two angles? Does this relationship remain true when you move lines k and m?



EXPLORE 2, STEP 2

DRAW CONCLUSIONS Use your observations to complete these exercises

1. What other transformation maps a figure onto the same image as a reflection in two parallel lines?
2. What other transformation maps a figure onto the same image as a reflection in two intersecting lines?

9.5 Apply Compositions of Transformations

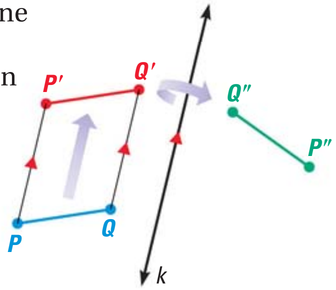
Before	You performed rotations, reflections, or translations.
Now	You will perform combinations of two or more transformations.
Why?	So you can describe the transformations that represent a rowing crew, as in Ex. 30.

Key Vocabulary
- **glide reflection**
- **composition of transformations**

A translation followed by a reflection can be performed one after the other to produce a *glide reflection*. A translation can be called a glide. A **glide reflection** is a transformation in which every point P is mapped to a point P'' by the following steps.

STEP 1 **First,** a translation maps P to P'.

STEP 2 **Then,** a reflection in a line k parallel to the direction of the translation maps P' to P''.



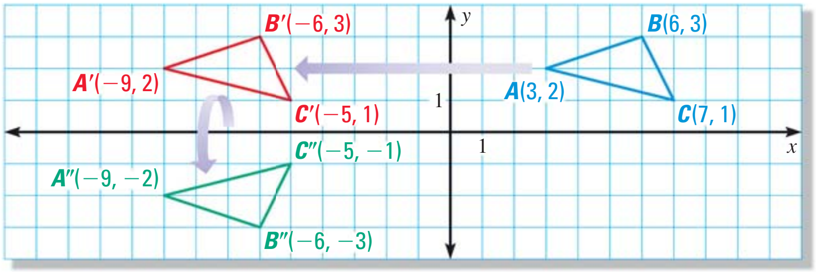
EXAMPLE 1 Find the image of a glide reflection

The vertices of $\triangle ABC$ are $A(3, 2)$, $B(6, 3)$, and $C(7, 1)$. Find the image of $\triangle ABC$ after the glide reflection.

Translation: $(x, y) \to (x - 12, y)$
Reflection: in the x-axis

Solution

Begin by graphing $\triangle ABC$. Then graph $\triangle A'B'C'$ after a translation 12 units left. Finally, graph $\triangle A''B''C''$ after a reflection in the x-axis.



AVOID ERRORS

The line of reflection must be parallel to the direction of the translation to be a glide reflection.

✓ GUIDED PRACTICE for Example 1

1. Suppose $\triangle ABC$ in Example 1 is translated 4 units down, then reflected in the y-axis. What are the coordinates of the vertices of the image?

2. In Example 1, *describe* a glide reflection from $\triangle A''B''C''$ to $\triangle ABC$.

COMPOSITIONS When two or more transformations are combined to form a single transformation, the result is a **composition of transformations**. A glide reflection is an example of a composition of transformations.

In this lesson, a composition of transformations uses isometries, so the final image is congruent to the preimage. This suggests the Composition Theorem.

THEOREM *For Your Notebook*

THEOREM 9.4 Composition Theorem

The composition of two (or more) isometries is an isometry.

Proof: Exs. 35–36, p. 614

EXAMPLE 2 Find the image of a composition

The endpoints of $\overline{RS}$ are $R(1, -3)$ and $S(2, -6)$. Graph the image of $\overline{RS}$ after the composition.

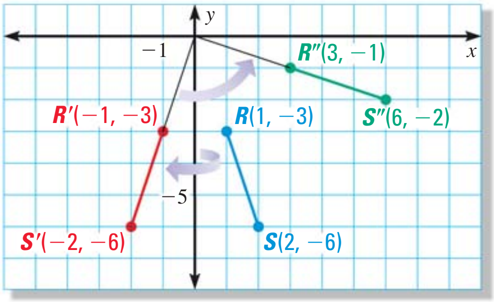
Reflection: in the y-axis
Rotation: 90° about the origin

Solution

STEP 1 **Graph** $\overline{RS}$.

STEP 2 **Reflect** $\overline{RS}$ in the y-axis. $\overline{R'S'}$ has endpoints $R'(-1, -3)$ and $S'(-2, -6)$.

STEP 3 **Rotate** $\overline{R'S'}$ 90° about the origin. $\overline{R''S''}$ has endpoints $R''(3, -1)$ and $S''(6, -2)$.



AVOID ERRORS
Unless you are told otherwise, do the transformations in the order given.

TWO REFLECTIONS Compositions of two reflections result in either a translation or a rotation, as described in Theorems 9.5 and 9.6.

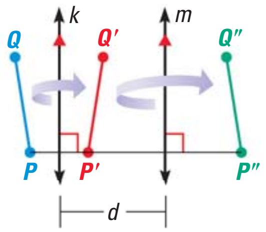
THEOREM *For Your Notebook*

THEOREM 9.5 Reflections in Parallel Lines Theorem

If lines k and m are parallel, then a reflection in line k followed by a reflection in line m is the same as a translation.

If P'' is the image of P, then:

1. $\overline{PP''}$ is perpendicular to k and m, and
2. $PP'' = 2d$, where d is the distance between k and m.



Proof: Ex. 37, p. 614

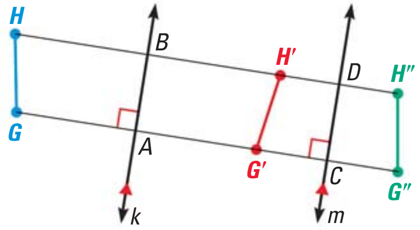
EXAMPLE 3 Use Theorem 9.5

In the diagram, a reflection in line k maps $\overline{GH}$ to $\overline{G'H'}$. A reflection in line m maps $\overline{G'H'}$ to $\overline{G''H''}$. Also, $HB = 9$ and $DH'' = 4$.

a. Name any segments congruent to each segment: $\overline{HG}$, $\overline{HB}$, and $\overline{GA}$.

b. Does $AC = BD$? Explain.

c. What is the length of $\overline{GG''}$?



Solution

a. $\overline{HG} \cong \overline{H'G'}$, and $\overline{HG} \cong \overline{H''G''}$. $\overline{HB} \cong \overline{H'B}$. $\overline{GA} \cong \overline{G'A}$.

b. Yes, $AC = BD$ because $\overline{GG''}$ and $\overline{HH''}$ are perpendicular to both k and m, so $\overline{BD}$ and $\overline{AC}$ are opposite sides of a rectangle.

c. By the properties of reflections, $H'B = 9$ and $H'D = 4$. Theorem 9.5 implies that $GG'' = HH'' = 2 \cdot BD$, so the length of $\overline{GG''}$ is $2(9 + 4)$, or 26 units.

✓ GUIDED PRACTICE for Examples 2 and 3

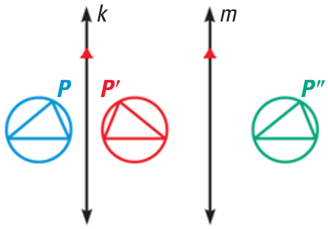
3. Graph $\overline{RS}$ from Example 2. Do the rotation first, followed by the reflection. Does the order of the transformations matter? *Explain.*

4. In Example 3, part (c), *explain* how you know that $GG'' = HH''$.

Use the figure below for Exercises 5 and 6. The distance between line k and line m is 1.6 centimeters.

5. The preimage is reflected in line k, then in line m. *Describe* a single transformation that maps the blue figure to the green figure.

6. What is the distance between P and P''? If you draw $\overline{PP'}$, what is its relationship with line k? *Explain.*



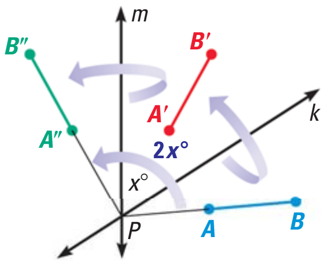
THEOREM For Your Notebook

THEOREM 9.6 Reflections in Intersecting Lines Theorem

If lines k and m intersect at point P, then a reflection in k followed by a reflection in m is the same as a rotation about point P.

The angle of rotation is $2x°$, where $x°$ is the measure of the acute or right angle formed by k and m.

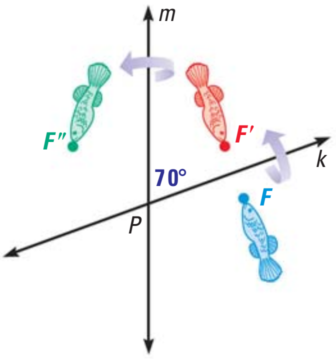
Proof: Ex. 38, p. 614



$m\angle BPB'' = 2x°$

EXAMPLE 4 Use Theorem 9.6

In the diagram, the figure is reflected in line *k*. The image is then reflected in line *m*. Describe a single transformation that maps *F* to *F*″.



Solution

The measure of the acute angle formed between lines *k* and *m* is 70°. So, by Theorem 9.6, a single transformation that maps *F* to *F*″ is a 140° rotation about point *P*.

You can check that this is correct by tracing lines *k* and *m* and point *F*, then rotating the point 140°.

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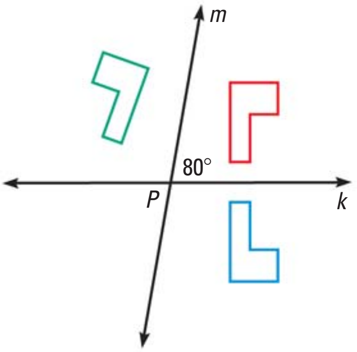
✓ GUIDED PRACTICE for Example 4



7. In the diagram at the right, the preimage is reflected in line *k*, then in line *m*. *Describe* a single transformation that maps the blue figure onto the green figure.

8. A rotation of 76° maps *C* to *C*′. To map *C* to *C*′ using two reflections, what is the angle formed by the intersecting lines of reflection?

9.5 EXERCISES

HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 7, 17, and 27

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 25, 29, and 34

SKILL PRACTICE

1. **VOCABULARY** Copy and complete: In a glide reflection, the direction of the translation must be ___?___ to the line of reflection.

2. ★ **WRITING** *Explain* why a glide reflection is an isometry.

EXAMPLE 1 on p. 608 for Exs. 3–6

GLIDE REFLECTION The endpoints of $\overline{CD}$ are $C(2, -5)$ and $D(4, 0)$. Graph the image of $\overline{CD}$ after the glide reflection.

3. **Translation:** $(x, y) \to (x, y - 1)$
Reflection: in the y-axis

4. **Translation:** $(x, y) \to (x - 3, y)$
Reflection: in $y = -1$

5. **Translation:** $(x, y) \to (x, y + 4)$
Reflection: in $x = 3$

6. **Translation:** $(x, y) \to (x + 2, y + 2)$
Reflection: in $y = x$

EXAMPLE 2
on p. 609
for Exs. 7–14

GRAPHING COMPOSITIONS **The vertices of $\triangle PQR$ are $P(2, 4)$, $Q(6, 0)$, and $R(7, 2)$. Graph the image of $\triangle PQR$ after a composition of the transformations in the order they are listed.**

7. **Translation:** $(x, y) \rightarrow (x, y - 5)$
Reflection: in the y-axis

8. **Translation:** $(x, y) \rightarrow (x - 3, y + 2)$
Rotation: 90° about the origin

9. **Translation:** $(x, y) \rightarrow (x + 12, y + 4)$
Translation: $(x, y) \rightarrow (x - 5, y - 9)$

10. **Reflection:** in the x-axis
Rotation: 90° about the origin

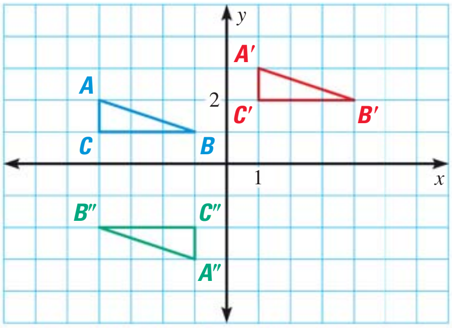
REVERSING ORDERS **Graph $\overline{F''G''}$ after a composition of the transformations in the order they are listed. Then perform the transformations in reverse order. Does the order affect the final image $\overline{F''G''}$?**

11. $F(-5, 2)$, $G(-2, 4)$
Translation: $(x, y) \rightarrow (x + 3, y - 8)$
Reflection: in the x-axis

12. $F(-1, -8)$, $G(-6, -3)$
Reflection: in the line $y = 2$
Rotation: 90° about the origin

DESCRIBING COMPOSITIONS ***Describe*** **the composition of transformations.**

13.



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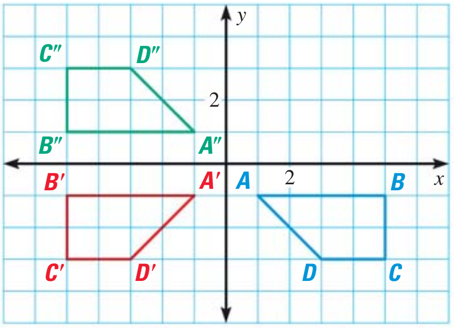


EXAMPLE 3
on p. 610
for Exs. 15–19

USING THEOREM 9.5 **In the diagram, $k \parallel m$, $\triangle ABC$ is reflected in line k, and $\triangle A'B'C'$ is reflected in line m.**

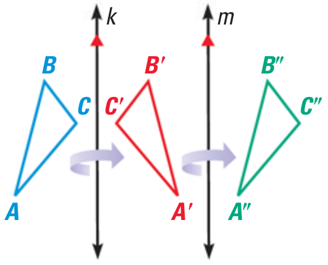
15. A translation maps $\triangle ABC$ onto which triangle?

16. Which lines are perpendicular to $\overleftrightarrow{AA''}$?

17. Name two segments parallel to $\overleftrightarrow{BB''}$.

18. If the distance between k and m is 2.6 inches, what is the length of $\overleftrightarrow{CC''}$?

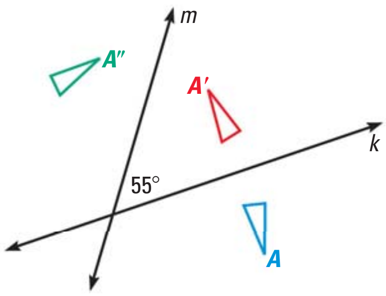
19. Is the distance from B' to m the same as the distance from B'' to m? *Explain.*



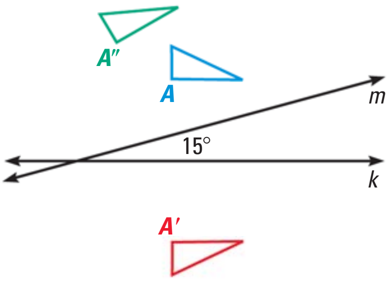
EXAMPLE 4
on p. 611
for Exs. 20–21

USING THEOREM 9.6 **Find the angle of rotation that maps A onto A''.**

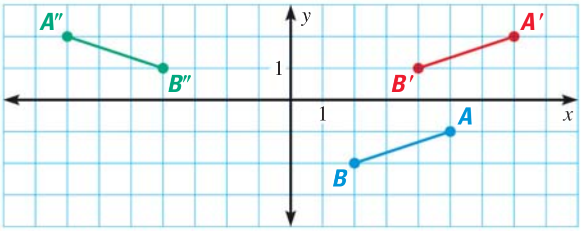
20.



21.



22. **ERROR ANALYSIS** A student described the translation of $\overline{AB}$ to $\overline{A'B'}$ followed by the reflection of $\overline{A'B'}$ to $\overline{A''B''}$ in the y-axis as a glide reflection. *Describe* and correct the student's error.



USING MATRICES **The vertices of $\triangle PQR$ are $P(1, 4)$, $Q(3, -2)$, and $R(7, 1)$. Use matrix operations to find the image matrix that represents the composition of the given transformations. Then graph $\triangle PQR$ and its image.**

23. **Translation:** $(x, y) \rightarrow (x, y + 5)$
Reflection: in the y-axis

24. **Reflection:** in the x-axis
Translation: $(x, y) \rightarrow (x - 9, y - 4)$

25. ★ **OPEN-ENDED MATH** Sketch a polygon. Apply three transformations of your choice on the polygon. What can you say about the congruence of the preimage and final image after multiple transformations? *Explain.*

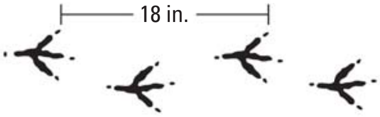
26. **CHALLENGE** The vertices of $\triangle JKL$ are $J(1, -3)$, $K(2, 2)$, and $L(3, 0)$. Find the image of the triangle after a 180° rotation about the point $(-2, 2)$, followed by a reflection in the line $y = -x$.

PROBLEM SOLVING

EXAMPLE 1
on p. 608
for Exs. 27–30

ANIMAL TRACKS **The left and right prints in the set of animal tracks can be related by a glide reflection. Copy the tracks and *describe* a translation and reflection that combine to create the glide reflection.**

27. bald eagle (2 legs)



28. armadillo (4 legs)

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29. ★ **MULTIPLE CHOICE** Which is *not* a glide reflection?

Ⓐ The teeth of a closed zipper

Ⓑ The tracks of a walking duck

Ⓒ The keys on a computer keyboard

Ⓓ The red squares on two adjacent rows of a checkerboard

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30. **ROWING** *Describe* the transformations that are combined to represent an eight-person rowing shell.

SWEATER PATTERNS **In Exercises 31–33, *describe* the transformations that are combined to make each sweater pattern.**

31.

32.

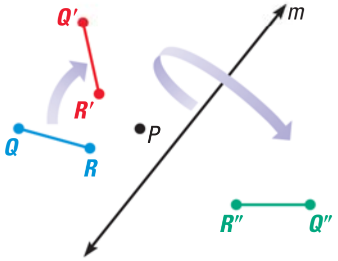
33.

34. ★ **SHORT RESPONSE** Use Theorem 9.5 to *explain* how you can make a glide reflection using three reflections. How are the lines of reflection related?

35. **PROVING THEOREM 9.4** Write a plan for proof for one case of the Composition Theorem.

GIVEN ▶ A rotation about P maps Q to Q' and R to R'. A reflection in m maps Q' to Q'' and R' to R''.

PROVE ▶ $QR = Q''R''$



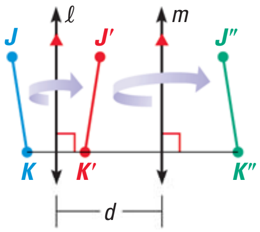
36. **PROVING THEOREM 9.4** A composition of a rotation and a reflection, as in Exercise 35, is one case of the Composition Theorem. List all possible cases, and prove the theorem for another pair of compositions.

37. **PROVING THEOREM 9.5** Prove the Reflection in Parallel Lines Theorem.

GIVEN ▶ A reflection in line ℓ maps $\overline{JK}$ to $\overline{J'K'}$, a reflection in line m maps $\overline{J'K'}$ to $\overline{J''K''}$, and $\ell \parallel m$.

PROVE ▶ **a.** $\overleftrightarrow{KK''}$ is perpendicular to ℓ and m.

b. $KK'' = 2d$, where d is the distance between ℓ and m.

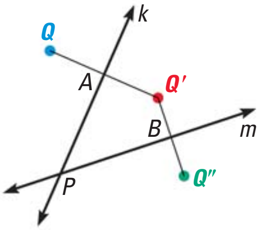


38. **PROVING THEOREM 9.6** Prove the Reflection in Intersecting Lines Theorem.

GIVEN ▶ Lines k and m intersect at point P. Q is any point not on k or m.

PROVE ▶ **a.** If you reflect point Q in k, and then reflect its image Q' in m, Q'' is the image of Q after a rotation about point P.

b. $m\angle QPQ'' = 2(m\angle APB)$



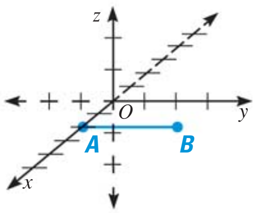
Plan for Proof First show $k \perp \overline{QQ'}$ and $\overline{QA} \cong \overline{Q'A}$. Then show $\triangle QAP \cong \triangle Q'AP$. In the same way, show $\triangle Q'BP \cong \triangle Q''BP$. Use congruent triangles and substitution to show that $\overline{QP} \cong \overline{Q''P}$. That proves part (a) by the definition of a rotation. Then use congruent triangles to prove part (b).

39. **VISUAL REASONING** You are riding a bicycle along a flat street.

a. What two transformations does the wheel's motion use?

b. *Explain* why this is not a composition of transformations.

40. MULTI-STEP PROBLEM A point in space has three coordinates (x, y, z). From the origin, a point can be forward or back on the x-axis, left or right on the y-axis, and up or down on the z-axis. The endpoints of segment $\overline{AB}$ in space are $A(2, 0, 0)$ and $B(2, 3, 0)$, as shown at the right.



a. Rotate $\overline{AB}$ 90° about the x-axis with center of rotation A. What are the coordinates of $\overline{A'B'}$?

b. Translate $\overline{A'B'}$ using the vector $\langle 4, 0, -1 \rangle$. What are the coordinates of $\overline{A''B''}$?

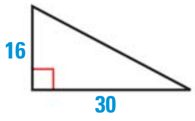
41. CHALLENGE *Justify* the following conjecture or provide a counterexample.

Conjecture When performing a composition of two transformations of the *same type*, order does not matter.

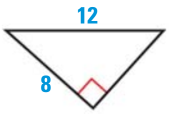
MIXED REVIEW

Find the unknown side length. Write your answer in simplest radical form. *(p. 433)*

42.



43.



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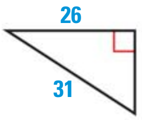


PREVIEW
Prepare for Lesson 9.6 in Exs. 45–48.

The coordinates of $\triangle PQR$ are $P(3, 1)$, $Q(3, 3)$, and $R(6, 1)$. Graph the image of the triangle after the translation. *(p. 572)*

45. $(x, y) \rightarrow (x + 3, y)$

46. $(x, y) \rightarrow (x - 3, y)$

47. $(x, y) \rightarrow (x, y + 2)$

48. $(x, y) \rightarrow (x + 3, y + 2)$

QUIZ *for Lessons 9.3–9.5*

The vertices of $\triangle ABC$ are $A(7, 1)$, $B(3, 5)$, and $C(10, 7)$. Graph the reflection in the line. *(p. 589)*

1. y-axis

2. $x = -4$

3. $y = -x$

Find the coordinates of the image of $P(2, -3)$ after the rotation about the origin. *(p. 598)*

4. 180° rotation

5. 90° rotation

6. 270° rotation

The vertices of $\triangle PQR$ are $P(-8, 8)$, $Q(-5, 0)$, and $R(-1, 3)$. Graph the image of $\triangle PQR$ after a composition of the transformations in the order they are listed. *(p. 608)*

7. **Translation:** $(x, y) \rightarrow (x + 6, y)$
Reflection: in the y-axis

8. **Reflection:** in the line $y = -2$
Rotation: 90° about the origin

9. **Translation:** $(x, y) \rightarrow (x - 5, y)$
Translation: $(x, y) \rightarrow (x + 2, y + 7)$

10. **Rotation:** 180° about the origin
Translation: $(x, y) \rightarrow (x + 4, y - 3)$

Extension

Use after Lesson 9.5

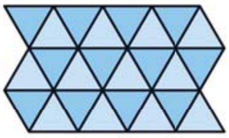
Tessellations

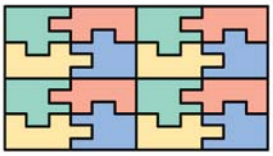
GOAL Make tessellations and discover their properties.

Key Vocabulary

- **tessellation**

A **tessellation** is a collection of figures that cover a plane with no gaps or overlaps. You can use transformations to make tessellations.



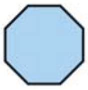


A *regular tessellation* is a tessellation of congruent regular polygons. In the figures above, the tessellation of equilateral triangles is a regular tessellation.

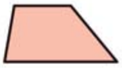
EXAMPLE 1 Determine whether shapes tessellate

Does the shape tessellate? If so, tell whether the tessellation is regular.

a. Regular octagon



b. Trapezoid



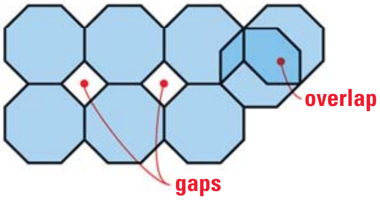
c. Regular hexagon

Solution

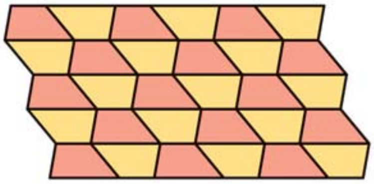
AVOID ERRORS

The sum of the angles surrounding every vertex of a tessellation is 360°. This means that no regular polygon with more than six sides can be used in a *regular* tesssellation.

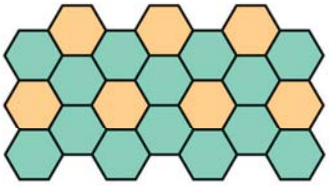
a. A regular octagon does not tessellate.



b. The trapezoid tessellates. The tessellation is not regular because the trapezoid is not a regular polygon.



c. A regular hexagon tessellates using translations. The tessellation is regular because it is made of congruent regular hexagons.



EXAMPLE 2 Draw a tessellation using one shape

Change a triangle to make a tessellation.

Solution

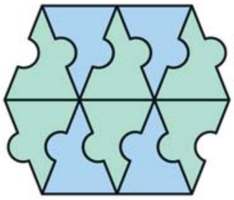
STEP 1

Cut a piece from the triangle.

STEP 2

Slide the piece to another side.

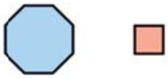
STEP 3



Translate and reflect the figure to make a tessellation.

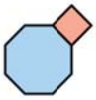
EXAMPLE 3 Draw a tessellation using two shapes

Draw a tessellation using the given floor tiles.



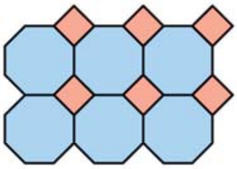
Solution

STEP 1



Combine one octagon and one square by connecting sides of the same length.

STEP 2



Translate the pair of polygons to make a tessellation

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READ VOCABULARY

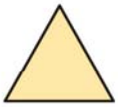
Notice that in the tessellation in Example 3, the same combination of regular polygons meet at each vertex. This type of tessellation is called *semi-regular*.

PRACTICE

EXAMPLE 1
on p. 616
for Exs. 1–4

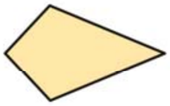
REGULAR TESSELLATIONS **Does the shape tessellate? If so, tell whether the tessellation is regular.**

1. Equilateral triangle



2. Circle

3. Kite



4. ★ **OPEN-ENDED MATH** Draw a rectangle. Use the rectangle to make two different tessellations.

5. **MULTI-STEP PROBLEM** Choose a tessellation and measure the angles at three vertices.

   a. What is the sum of the measures of the angles? What can you conclude?

   b. *Explain* how you know that any *quadrilateral* will tessellate.

EXAMPLE 2
on p. 617
for Exs. 6–9

DRAWING TESSELLATIONS In Exercises 6–8, use the steps in Example 2 to make a figure that will tessellate.

6. Make a tessellation using a triangle as the base figure.

7. Make a tessellation using a square as the base figure. Change both pairs of opposite sides.

8. Make a tessellation using a hexagon as the base figure. Change all three pairs of opposite sides.

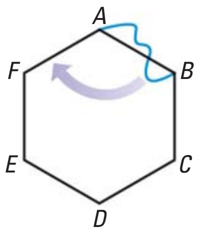
9. **ROTATION TESSELLATION** Use these steps to make another tessellation based on a regular hexagon *ABCDEF*.

   a. Connect points *A* and *B* with a curve. Rotate the curve 120° about *A* so that *B* coincides with *F*.

   b. Connect points *E* and *F* with a curve. Rotate the curve 120° about *E* so that *F* coincides with *D*.

   c. Connect points *C* and *D* with a curve. Rotate the curve 120° about *C* so that *D* coincides with *B*.

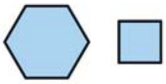
   d. Use this figure to draw a tessellation.



EXAMPLE 3
on p. 617
for Exs. 10–12

USING TWO POLYGONS Draw a tessellation using the given polygons.

10.



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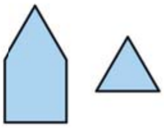

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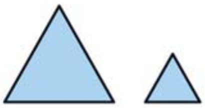
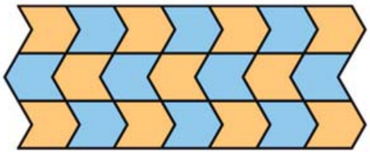


13. ★ **OPEN-ENDED MATH** Draw a tessellation using three different polygons.

TRANSFORMATIONS *Describe* the transformation(s) used to make the tessellation.

14.



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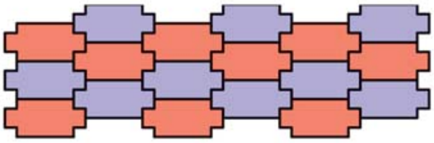

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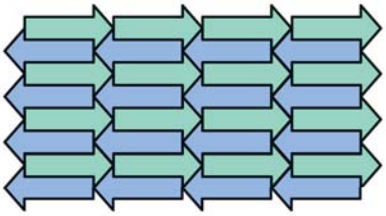

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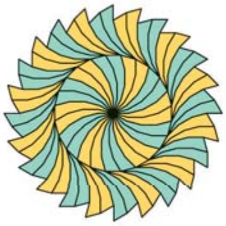


18. **USING SHAPES** On graph paper, outline a capital H. Use this shape to make a tessellation. What transformations did you use?

9.6 Identify Symmetry

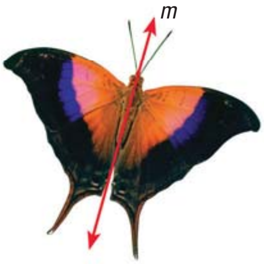
Before You reflected or rotated figures.

Now You will identify line and rotational symmetries of a figure.

Why? So you can identify the symmetry in a bowl, as in Ex. 11.

Key Vocabulary
- **line symmetry**
- **line of symmetry**
- **rotational symmetry**
- **center of symmetry**

A figure in the plane has **line symmetry** if the figure can be mapped onto itself by a reflection in a line. This line of reflection is a **line of symmetry**, such as line m at the right. A figure can have more than one line of symmetry.



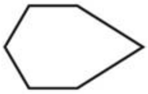
EXAMPLE 1 Identify lines of symmetry

How many lines of symmetry does the hexagon have?

a.

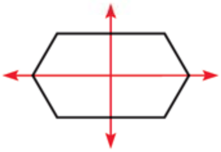
b.

c.

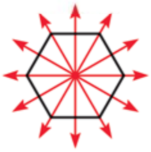


Solution

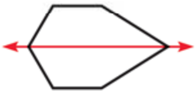
a. Two lines of symmetry



b. Six lines of symmetry



c. One line of symmetry



REVIEW REFLECTION
Notice that the lines of symmetry are also lines of reflection.

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✓ GUIDED PRACTICE for Example 1

How many lines of symmetry does the object appear to have?

1.

2.

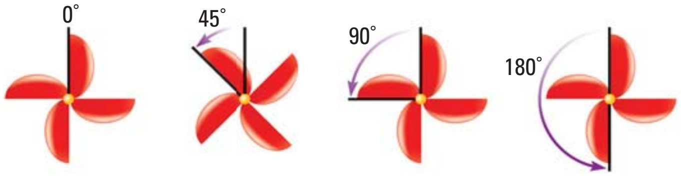
3.

4. Draw a hexagon with no lines of symmetry.

ROTATIONAL SYMMETRY A figure in a plane has **rotational symmetry** if the figure can be mapped onto itself by a rotation of 180° or less about the center of the figure. This point is the **center of symmetry**. Note that the rotation can be either clockwise or counterclockwise.

REVIEW ROTATION
For a figure with rotational symmetry, the *angle of rotation* is the smallest angle that maps the figure onto itself.

For example, the figure below has rotational symmetry, because a rotation of either 90° or 180° maps the figure onto itself (although a rotation of 45° does not).



The figure above also has *point symmetry*, which is 180° rotational symmetry.

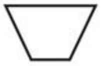
EXAMPLE 2 Identify rotational symmetry

Does the figure have rotational symmetry? If so, describe any rotations that map the figure onto itself.

a. Parallelogram

b. Regular octagon

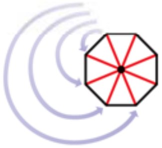
c. Trapezoid



Solution

a. The parallelogram has rotational symmetry. The center is the intersection of the diagonals. A 180° rotation about the center maps the parallelogram onto itself.

b. The regular octagon has rotational symmetry. The center is the intersection of the diagonals. Rotations of 45°, 90°, 135°, or 180° about the center all map the octagon onto itself.

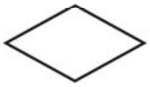


c. The trapezoid does not have rotational symmetry because no rotation of 180° or less maps the trapezoid onto itself.

✓ GUIDED PRACTICE for Example 2

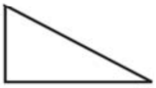
Does the figure have rotational symmetry? If so, *describe* any rotations that map the figure onto itself.

5. Rhombus



6. Octagon

7. Right triangle



★ EXAMPLE 3 Standardized Test Practice

Identify the line symmetry and rotational symmetry of the equilateral triangle at the right.

Ⓐ 3 lines of symmetry, 60° rotational symmetry

Ⓑ 3 lines of symmetry, 120° rotational symmetry

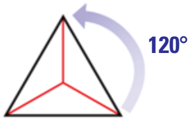
Ⓒ 1 line of symmetry, 180° rotational symmetry

Ⓓ 1 line of symmetry, no rotational symmetry

Solution

The triangle has line symmetry. Three lines of symmetry can be drawn for the figure.

For a figure with s lines of symmetry, the smallest rotation that maps the figure onto itself has the measure $\frac{360^\circ}{s}$. So, the equilateral triangle has $\frac{360^\circ}{3}$, or 120° rotational symmetry.



▸ The correct answer is B. Ⓐ ● Ⓒ Ⓓ

ELIMINATE CHOICES
An equilateral triangle can be mapped onto itself by reflecting over any of three different lines. So, you can eliminate choices C and D.

✓ GUIDED PRACTICE for Example 3

8. *Describe* the lines of symmetry and rotational symmetry of a non-equilateral isosceles triangle.

9.6 EXERCISES

HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 7, 13, and 31

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 13, 14, 21, and 23

SKILL PRACTICE

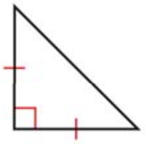
1. VOCABULARY What is a *center of symmetry*?

2. ★ WRITING Draw a figure that has one line of symmetry and does not have rotational symmetry. Can a figure have two lines of symmetry and no rotational symmetry?

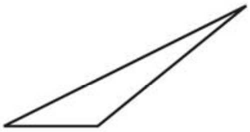
EXAMPLE 1
on p. 619
for Exs. 3–5

LINE SYMMETRY **How many lines of symmetry does the triangle have?**

3.



4.

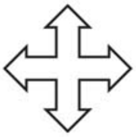


5.

EXAMPLE 2
on p. 620
for Exs. 6–9

ROTATIONAL SYMMETRY **Does the figure have rotational symmetry? If so, *describe* any rotations that map the figure onto itself.**

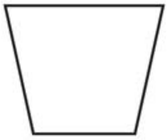
6.



7.

8.

9.



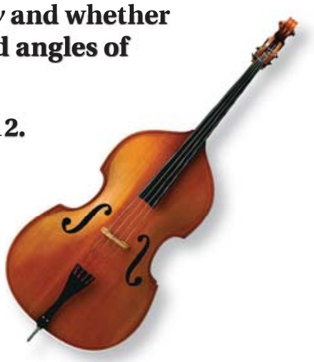
EXAMPLE 3
on p. 621
for Exs. 10–16

SYMMETRY **Determine whether the figure has *line symmetry* and whether it has *rotational symmetry*. Identify all lines of symmetry and angles of rotation that map the figure onto itself.**

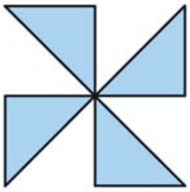
10.

11.

12.



13. ★ **MULTIPLE CHOICE** Identify the line symmetry and rotational symmetry of the figure at the right.



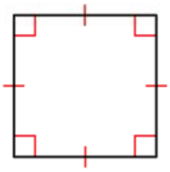
Ⓐ 1 line of symmetry, no rotational symmetry

Ⓑ 1 line of symmetry, 180° rotational symmetry

Ⓒ No lines of symmetry, 90° rotational symmetry

Ⓓ No lines of symmetry, 180° rotational symmetry

14. ★ **MULTIPLE CHOICE** Which statement best describes the rotational symmetry of a square?



Ⓐ The square has no rotational symmetry.

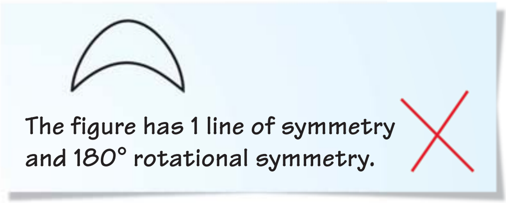
Ⓑ The square has 90° rotational symmetry.

Ⓒ The square has point symmetry.

Ⓓ Both B and C are correct.

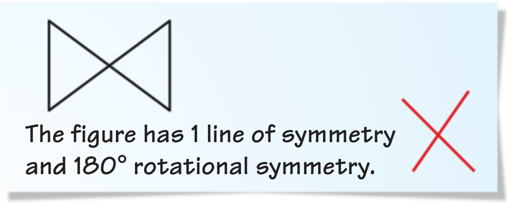
ERROR ANALYSIS ***Describe*** **and correct the error made in describing the symmetry of the figure.**

15.



The figure has 1 line of symmetry and 180° rotational symmetry.

16.



The figure has 1 line of symmetry and 180° rotational symmetry.

DRAWING FIGURES **In Exercises 17–20, use the description to draw a figure. If not possible, write *not possible*.**

17. A quadrilateral with no line of symmetry

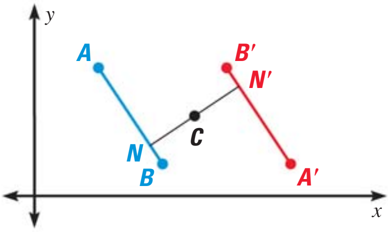
18. An octagon with exactly two lines of symmetry

19. A hexagon with no point symmetry

20. A trapezoid with rotational symmetry

21. ★ **OPEN-ENDED MATH** Draw a polygon with 180° rotational symmetry and with exactly two lines of symmetry.

22. **POINT SYMMETRY** In the graph, $\overline{AB}$ is reflected in the point C to produce the image $\overline{A'B'}$. To make a reflection in a point C for each point N on the preimage, locate N' so that $N'C = NC$ and N' is on $\overleftrightarrow{NC}$. *Explain* what kind of rotation would produce the same image. What kind of symmetry does quadrilateral $AB'A'B$ have?



23. ★ **SHORT RESPONSE** A figure has more than one line of symmetry. Can two of the lines of symmetry be parallel? *Explain*.

24. **REASONING** How many lines of symmetry does a circle have? How many angles of rotational symmetry does a circle have? *Explain*.

25. **VISUAL REASONING** How many planes of symmetry does a cube have?

26. **CHALLENGE** What can you say about the rotational symmetry of a regular polygon with n sides? *Explain*.

PROBLEM SOLVING

EXAMPLES 1 and 2 on pp. 619–620 for Exs. 27–30

WORDS Identify the line symmetry and rotational symmetry (if any) of each word.

27. MOW **28.** RADAR **29.** OHIO **30.** pod

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KALEIDOSCOPES In Exercises 31–33, use the following information about kaleidoscopes.

Inside a kaleidoscope, two mirrors are placed next to each other to form a V, as shown at the right. The angle between the mirrors determines the number of lines of symmetry in the image. Use the formula $n(m\angle 1) = 180^\circ$ to find the measure of $\angle 1$ between the mirrors or the number n of lines of symmetry in the image.

Calculate the angle at which the mirrors must be placed for the image of a kaleidoscope to make the design shown.

31.

32.

33.

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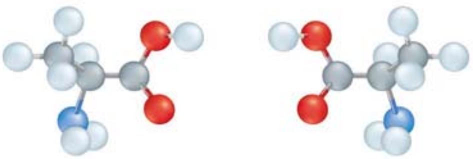
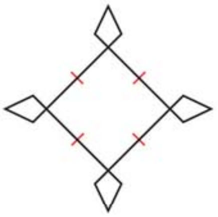
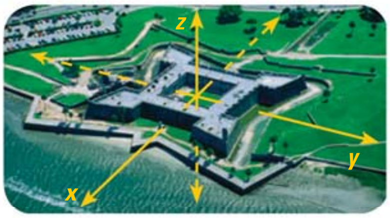
34. CHEMISTRY The diagram at the right shows two forms of the amino acid *alanine*. One form is laevo-alanine and the other is dextro-alanine. How are the structures of these two molecules related? *Explain*.



35. MULTI-STEP PROBLEM The *Castillo de San Marcos* in St. Augustine, Florida, has the shape shown.

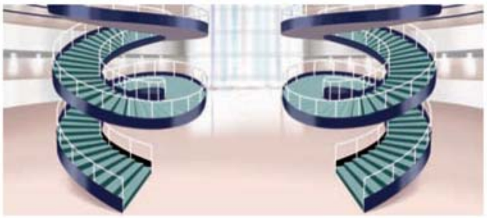




a. What kind(s) of symmetry does the shape of the building show?

b. Imagine the building on a three-dimensional coordinate system. Copy and complete the following statement: The lines of symmetry in part (a) are now described as __?__ of symmetry and the rotational symmetry about the center is now described as rotational symmetry about the __?__.

36. CHALLENGE Spirals have a type of symmetry called spiral, or helical, symmetry. *Describe* the two transformations involved in a spiral staircase. Then *explain* the difference in transformations between the two staircases at the right.



MIXED REVIEW

PREVIEW

Prepare for Lesson 9.7 in Exs. 37–39.

Solve the proportion. *(p. 356)*

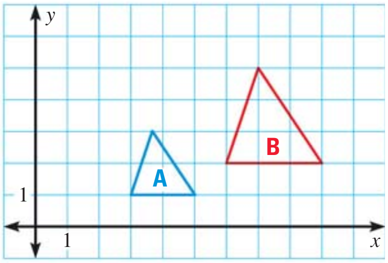
37. $\frac{5}{x} = \frac{15}{27}$

38. $\frac{a + 4}{7} = \frac{49}{56}$

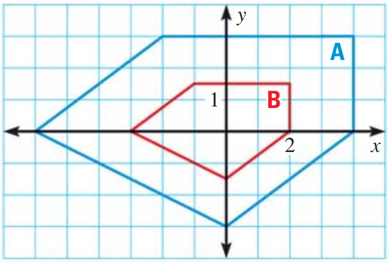
39. $\frac{5}{2b - 3} = \frac{1}{3b + 1}$

Determine whether the dilation from Figure A to Figure B is a *reduction* or an *enlargement*. Then find its scale factor. *(p. 409)*

40.



41.



Write a matrix to represent the given polygon. *(p. 580)*

42. Triangle A in Exercise 40

43. Triangle B in Exercise 40

44. Pentagon A in Exercise 41

45. Pentagon B in Exercise 41

Investigating Geometry CONSTRUCTION *Use before Lesson 9.7*

9.7 Investigate Dilations

MATERIALS • straightedge • compass • ruler

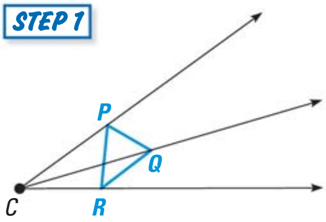
QUESTION How do you construct a dilation of a figure?

Recall from Lesson 6.7 that a dilation enlarges or reduces a figure to make a similar figure. You can use construction tools to make enlargement dilations.

EXPLORE Construct an enlargement dilation

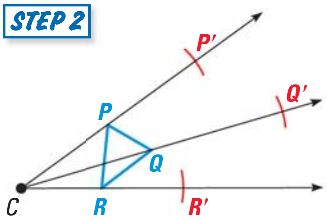
Use a compass and straightedge to construct a dilation of $\triangle PQR$ with a scale factor of 2, using a point C outside the triangle as the center of dilation.

STEP 1



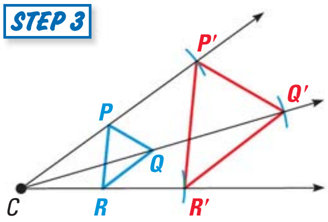
Draw a triangle Draw $\triangle PQR$ and choose the center of the dilation C outside the triangle. Draw lines from C through the vertices of the triangle.

STEP 2



Use a compass Use a compass to locate P' on $\overrightarrow{CP}$ so that $CP' = 2(CP)$. Locate Q' and R' in the same way.

STEP 3



Connect points Connect points P', Q', and R' to form $\triangle P'Q'R'$.

DRAW CONCLUSIONS Use your observations to complete these exercises

1. Find the ratios of corresponding side lengths of $\triangle PQR$ and $\triangle P'Q'R'$. Are the triangles similar? *Explain.*
2. Draw $\triangle DEF$. Use a compass and straightedge to construct a dilation with a scale factor of 3, using point D on the triangle as the center of dilation.
3. Find the ratios of corresponding side lengths of $\triangle DEF$ and $\triangle D'E'F'$. Are the triangles similar? *Explain.*
4. Draw $\triangle JKL$. Use a compass and straightedge to construct a dilation with a scale factor of 2, using a point A inside the triangle as the center of dilation.
5. Find the ratios of corresponding side lengths of $\triangle JKL$ and $\triangle J'K'L'$. Are the triangles similar? *Explain.*
6. What can you conclude about the corresponding angles measures of a triangle and an enlargement dilation of the triangle?

9.7 Identify and Perform Dilations

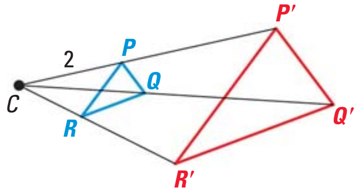
Before	You used a coordinate rule to draw a dilation.
Now	You will use drawing tools and matrices to draw dilations.
Why?	So you can determine the scale factor of a photo, as in Ex. 37.

Key Vocabulary
- **scalar multiplication**
- **dilation,** *p. 409*
- **reduction,** *p. 409*
- **enlargement,** *p. 409*

Recall from Lesson 6.7 that a dilation is a transformation in which the original figure and its image are similar.

A dilation with center C and scale factor k maps every point P in a figure to a point P' so that one of the following statements is true:

- If P is not the center point C, then the image point P' lies on $\overrightarrow{CP}$. The scale factor k is a positive number such that $k = \frac{CP'}{CP}$ and $k \neq 1$, or
- If P is the center point C, then $P = P'$.

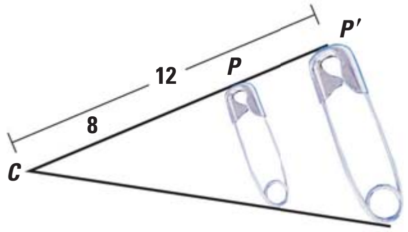


As you learned in Lesson 6.7, the dilation is a *reduction* if $0 < k < 1$ and it is an *enlargement* if $k > 1$.

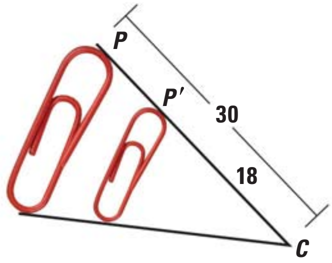
EXAMPLE 1 Identify dilations

Find the scale factor of the dilation. Then tell whether the dilation is a *reduction* or an *enlargement*.

a.



b.



Solution

a. Because $\frac{CP'}{CP} = \frac{12}{8}$, the scale factor is $k = \frac{3}{2}$. The image P' is an enlargement.

b. Because $\frac{CP'}{CP} = \frac{18}{30}$, the scale factor is $k = \frac{3}{5}$. The image P' is a reduction.

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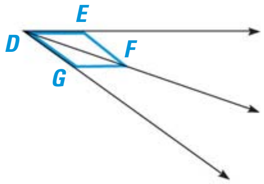
EXAMPLE 2 Draw a dilation

Draw and label ▱*DEFG*. Then construct a dilation of ▱*DEFG* with point *D* as the center of dilation and a scale factor of 2.

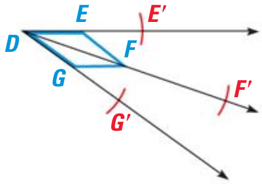
Solution

STEP 1



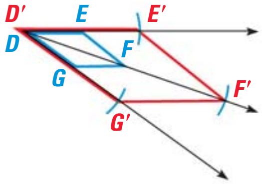
Draw *DEFG*. Draw rays from *D* through vertices *E*, *F*, and *G*.

STEP 2



Open the compass to the length of $\overline{DE}$. Locate *E*′ on $\overrightarrow{DE}$ so $DE' = 2(DE)$. Locate *F*′ and *G*′ the same way.

STEP 3



Add a second label *D*′ to point *D*. Draw the sides of *D*′*E*′*F*′*G*′.

✓ GUIDED PRACTICE for Examples 1 and 2

1. In a dilation, $CP' = 3$ and $CP = 12$. Tell whether the dilation is a *reduction* or an *enlargement* and find its scale factor.

2. Draw and label △*RST*. Then construct a dilation of △*RST* with *R* as the center of dilation and a scale factor of 3.

MATRICES **Scalar multiplication** is the process of multiplying each element of a matrix by a real number or *scalar*.

EXAMPLE 3 Scalar multiplication

Simplify the product: $4\begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & -3 \end{bmatrix}$.

Solution

$$4\begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 4(3) & 4(0) & 4(1) \\ 4(2) & 4(-1) & 4(-3) \end{bmatrix}$$ Multiply each element in the matrix by 4.

$$= \begin{bmatrix} 12 & 0 & 4 \\ 8 & -4 & -12 \end{bmatrix}$$ Simplify.

✓ GUIDED PRACTICE for Example 3

Simplify the product.

3. $5\begin{bmatrix} 2 & 1 & -10 \\ 3 & -4 & 7 \end{bmatrix}$

4. $-2\begin{bmatrix} -4 & 1 & 0 \\ 9 & -5 & -7 \end{bmatrix}$

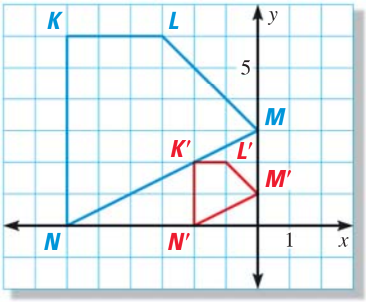
DILATIONS USING MATRICES You can use scalar multiplication to represent a dilation centered at the origin in the coordinate plane. To find the image matrix for a dilation centered at the origin, use the scale factor as the scalar.

EXAMPLE 4 Use scalar multiplication in a dilation

The vertices of quadrilateral *KLMN* are *K*(−6, 6), *L*(−3, 6), *M*(0, 3), and *N*(−6, 0). Use scalar multiplication to find the image of *KLMN* after a dilation with its center at the origin and a scale factor of $\frac{1}{3}$. Graph *KLMN* and its image.

Solution

$$\underset{\text{Scale factor}}{\frac{1}{3}}\ \underset{\text{Polygon matrix}}{\begin{array}{c}\begin{array}{cccc} K & L & M & N \end{array}\\ \begin{bmatrix} -6 & -3 & 0 & -6 \\ 6 & 6 & 3 & 0 \end{bmatrix}\end{array}} = \underset{\text{Image matrix}}{\begin{array}{c}\begin{array}{cccc} K' & L' & M' & N' \end{array}\\ \begin{bmatrix} -2 & -1 & 0 & -2 \\ 2 & 2 & 1 & 0 \end{bmatrix}\end{array}}$$



EXAMPLE 5 Find the image of a composition

The vertices of △*ABC* are *A*(−4, 1), *B*(−2, 2), and *C*(−2, 1). Find the image of △*ABC* after the given composition.

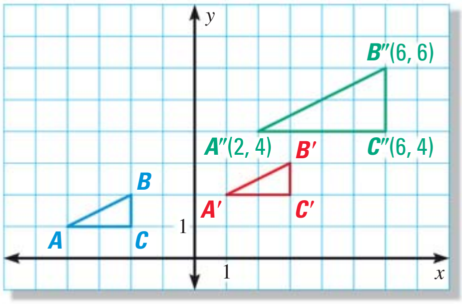
Translation: (*x*, *y*) → (*x* + 5, *y* + 1)
Dilation: centered at the origin with a scale factor of 2

Solution

STEP 1 **Graph** the preimage △*ABC* on the coordinate plane.

STEP 2 **Translate** △*ABC* 5 units to the right and 1 unit up. Label it △*A′B′C′*.

STEP 3 **Dilate** △*A′B′C′* using the origin as the center and a scale factor of 2 to find △*A″B″C″*.



✓ GUIDED PRACTICE for Examples 4 and 5

5. The vertices of △*RST* are *R*(1, 2), *S*(2, 1), and *T*(2, 2). Use scalar multiplication to find the vertices of △*R′S′T′* after a dilation with its center at the origin and a scale factor of 2.

6. A segment has the endpoints *C*(−1, 1) and *D*(1, 1). Find the image of $\overline{CD}$ after a 90° rotation about the origin followed by a dilation with its center at the origin and a scale factor of 2.

9.7 EXERCISES

HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 7, 19, and 35

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 24, 25, 27, 29, and 38

SKILL PRACTICE

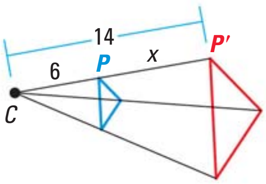
1. **VOCABULARY** What is a *scalar*?

2. ★ **WRITING** If you know the scale factor, *explain* how to determine if an image is larger or smaller than the preimage.

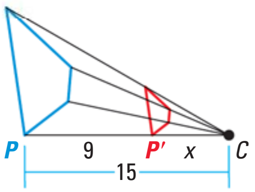
EXAMPLE 1 on p. 626 for Exs. 3–6

IDENTIFYING DILATIONS **Find the scale factor. Tell whether the dilation is a *reduction* or an *enlargement*. Find the value of x.**

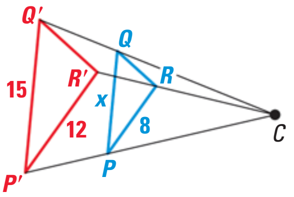
3.



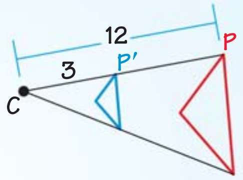
4.



5.



6. **ERROR ANALYSIS** *Describe* and correct the error in finding the scale factor k of the dilation.



$$k = \frac{CP}{CP'}$$

$$k = \frac{12}{3} = 4$$

EXAMPLE 2 on p. 627 for Exs. 7–14

CONSTRUCTION **Copy the diagram. Then draw the given dilation.**

7. Center H; $k = 2$

8. Center H; $k = 3$

9. Center J; $k = 2$

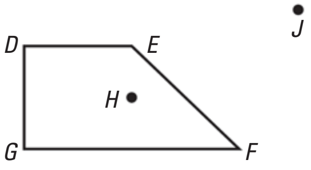
10. Center F; $k = 2$

11. Center J; $k = \frac{1}{2}$

12. Center F; $k = \frac{3}{2}$

13. Center D; $k = \frac{3}{2}$

14. Center G; $k = \frac{1}{2}$



EXAMPLE 3 on p. 627 for Exs. 15–17

SCALAR MULTIPLICATION **Simplify the product.**

15. $4\begin{bmatrix} 3 & 7 & 4 \\ 0 & 9 & -1 \end{bmatrix}$

16. $-5\begin{bmatrix} -2 & -5 & 7 & 3 \\ 1 & 4 & 0 & -1 \end{bmatrix}$

17. $9\begin{bmatrix} 0 & 3 & 2 \\ -1 & 7 & 0 \end{bmatrix}$

EXAMPLE 4 on p. 628 for Exs. 18–20

DILATIONS WITH MATRICES **Find the image matrix that represents a dilation of the polygon centered at the origin with the given scale factor. Then graph the polygon and its image.**

18. $\begin{array}{c} \begin{matrix} D & E & F \end{matrix} \\ \begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{bmatrix} \end{array}; k = 2$

19. $\begin{array}{c} \begin{matrix} G & H & J \end{matrix} \\ \begin{bmatrix} -2 & 0 & 6 \\ -4 & 2 & -2 \end{bmatrix} \end{array}; k = \frac{1}{2}$

20. $\begin{array}{c} \begin{matrix} J & L & M & N \end{matrix} \\ \begin{bmatrix} -6 & -3 & 3 & 3 \\ 0 & 3 & 0 & -3 \end{bmatrix} \end{array}; k = \frac{2}{3}$

EXAMPLE 5
on p. 628
for Exs. 21–23

COMPOSING TRANSFORMATIONS **The vertices of △*FGH* are *F*(−2, −2), *G*(−2, −4), and *H*(−4, −4). Graph the image of the triangle after a composition of the transformations in the order they are listed.**

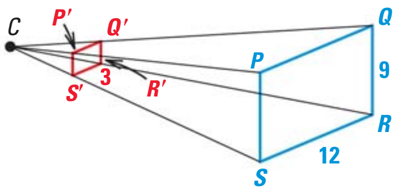
21. **Translation:** $(x, y) \rightarrow (x + 3, y + 1)$
Dilation: centered at the origin with a scale factor of 2

22. **Dilation:** centered at the origin with a scale factor of $\frac{1}{2}$
Reflection: in the y-axis

23. **Rotation:** 90° about the origin
Dilation: centered at the origin with a scale factor of 3

24. ★ **WRITING** Is a composition of transformations that includes a dilation ever an isometry? *Explain*.

25. ★ **MULTIPLE CHOICE** In the diagram, the center of the dilation of ▱*PQRS* is point *C*. The length of a side of ▱*P′Q′R′S′* is what percent of the length of the corresponding side of ▱*PQRS*?



Ⓐ 25% Ⓑ 33% Ⓒ 300% Ⓓ 400%

26. **REASONING** The distance from the center of dilation to the image of a point is shorter than the distance from the center of dilation to the preimage. Is the dilation a *reduction* or an *enlargement*? *Explain*.

27. ★ **SHORT RESPONSE** Graph a triangle in the coordinate plane. Rotate the triangle, then dilate it. Then do the same dilation first, followed by the rotation. In this composition of transformations, does it matter in which order the triangle is dilated and rotated? *Explain* your answer.

28. **REASONING** A dilation maps $A(5, 1)$ to $A'(2, 1)$ and $B(7, 4)$ to $B'(6, 7)$.

a. Find the scale factor of the dilation.

b. Find the center of the dilation.

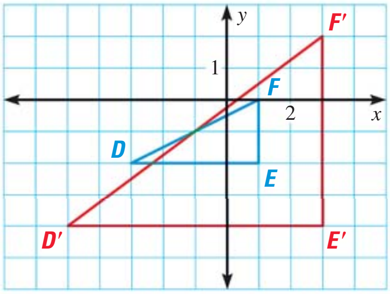
29. ★ **MULTIPLE CHOICE** Which transformation of (x, y) is a dilation?

Ⓐ $(3x, y)$ Ⓑ $(-x, 3y)$ Ⓒ $(3x, 3y)$ Ⓓ $(x + 3, y + 3)$

30. *xy* **ALGEBRA** Graph parabolas of the form $y = ax^2$ using three different values of a. Describe the effect of changing the value of a. Is this a dilation? *Explain*.

31. **REASONING** In the graph at the right, determine whether △*D′E′F′* is a dilation of △*DEF*. *Explain*.

32. **CHALLENGE** △*ABC* has vertices $A(4, 2)$, $B(4, 6)$, and $C(7, 2)$. Find the vertices that represent a dilation of △*ABC* centered at (4, 0) with a scale factor of 2.

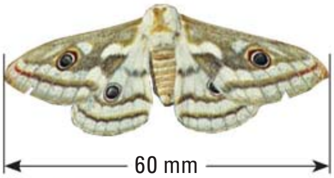


PROBLEM SOLVING

EXAMPLE 1
on p. 626
for Exs. 33–35

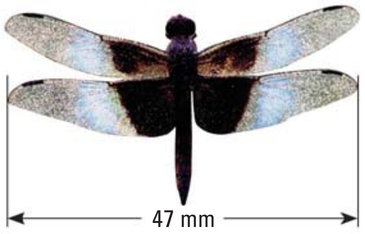
SCIENCE **You are using magnifying glasses. Use the length of the insect and the magnification level to determine the length of the image seen through the magnifying glass.**

33. Emperor moth
magnification 5x



34. Ladybug
magnification 10x

35. Dragonfly
magnification 20x



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36. MURALS A painter sketches plans for a mural. The plans are 2 feet by 4 feet. The actual mural will be 25 feet by 50 feet. What is the scale factor? Is this a dilation? *Explain*.

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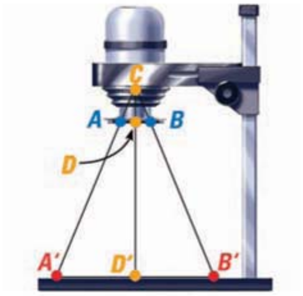
37. PHOTOGRAPHY By adjusting the distance between the negative and the enlarged print in a photographic enlarger, you can make prints of different sizes. In the diagram shown, you want the enlarged print to be 9 inches wide ($A'B'$). The negative is 1.5 inches wide (AB), and the distance between the light source and the negative is 1.75 inches (CD).

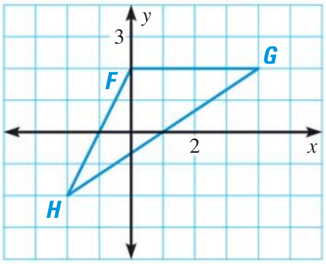


a. What is the scale factor of the enlargement?

b. What is the distance between the negative and the enlarged print?

38. ★ **OPEN-ENDED MATH** Graph a polygon in a coordinate plane. Draw a figure that is similar but not congruent to the polygon. What is the scale factor of the dilation you drew? What is the center of the dilation?

39. MULTI-STEP PROBLEM Use the figure at the right.



a. Write a polygon matrix for the figure. Multiply the matrix by the scalar -2.

b. Graph the polygon represented by the new matrix.

c. Repeat parts (a) and (b) using the scalar $-\frac{1}{2}$.

d. Make a conjecture about the effect of multiplying a polygon matrix by a negative scale factor.

40. AREA You have an 8 inch by 10 inch photo.

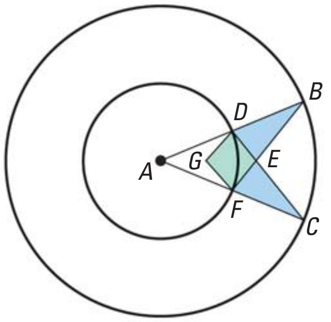
a. What is the area of the photo?

b. You photocopy the photo at 50%. What are the dimensions of the image? What is the area of the image?

c. How many images of this size would you need to cover the original photo?

41. REASONING You put a reduction of a page on the original page. *Explain* why there is a point that is in the same place on both pages.

42. CHALLENGE Draw two concentric circles with center A. Draw $\overline{AB}$ and $\overline{AC}$ to the larger circle to form a 45° angle. Label points D and F, where $\overline{AB}$ and $\overline{AC}$ intersect the smaller circle. Locate point E at the intersection of $\overline{BF}$ and $\overline{CD}$. Choose a point G and draw quadrilateral $DEFG$. Use A as the center of the dilation and a scale factor of $\frac{1}{2}$. Dilate $DEFG$, $\triangle DBE$, and $\triangle CEF$ two times. Sketch each image on the circles. *Describe* the result.

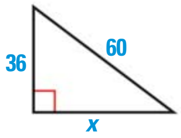


MIXED REVIEW

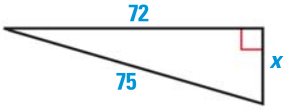
PREVIEW
Prepare for Lesson 10.1 in Exs. 43–45.

Find the unknown leg length x. *(p. 433)*

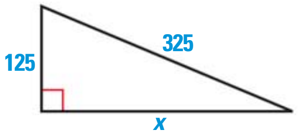
43.



44.



45.



Find the sum of the measures of the interior angles of the indicated convex polygon. *(p. 507)*

46. Hexagon

47. 13-gon

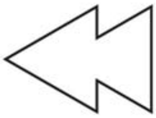
48. 15-gon

49. 18-gon

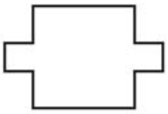
QUIZ *for Lessons 9.6–9.7*

Determine whether the figure has *line symmetry* and/or *rotational symmetry*. Identify the number of lines of symmetry and/or the rotations that map the figure onto itself. *(p. 619)*

1.



2.

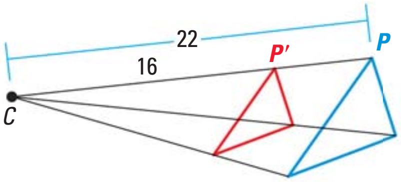


3.

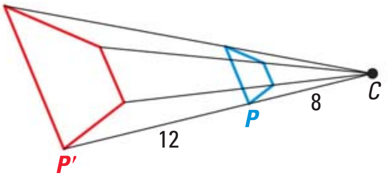
4.

Tell whether the dilation is a *reduction* or an *enlargement* and find its scale factor. *(p. 626)*

5.



6.



7. The vertices of $\triangle RST$ are $R(3, 1)$, $S(0, 4)$, and $T(-2, 2)$. Use scalar multiplication to find the image of the triangle after a dilation centered at the origin with scale factor $4\frac{1}{2}$. *(p. 626)*

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Keystrokes

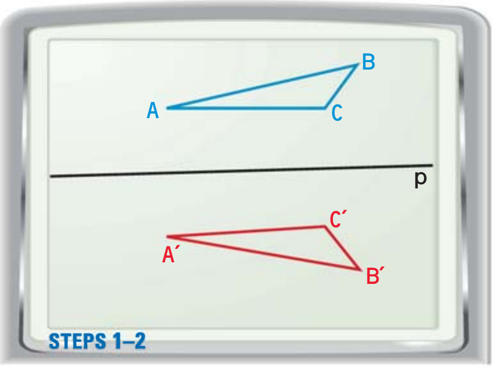
9.7 Compositions With Dilations

MATERIALS • graphing calculator or computer

QUESTION How can you graph compositions with dilations?

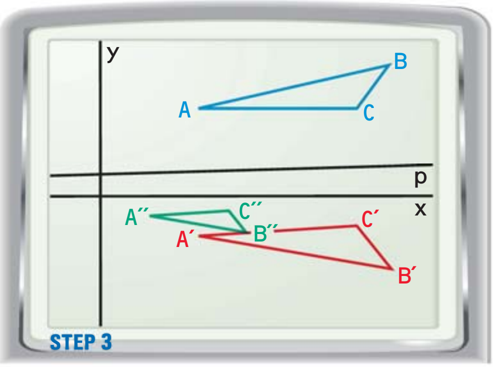
You can use geometry drawing software to perform compositions with dilations.

EXAMPLE Perform a reflection and dilation



STEP 1 ***Draw triangle*** Construct a scalene triangle like $\triangle ABC$ at the right. Label the vertices A, B, and C. Construct a line that does not intersect the triangle. Label the line p.

STEP 2 ***Reflect triangle*** Select Reflection from the F4 menu. To reflect $\triangle ABC$ in line p, choose the triangle, then the line.



STEP 3 ***Dilate triangle*** Select Hide/Show from the F5 menu and show the axes. To set the scale factor, select Alpha-Num from the F5 menu, press ENTER when the cursor is where you want the number, and then enter 0.5 for the scale factor.

Next, select Dilation from the F4 menu. Choose the image of $\triangle ABC$, then choose the origin as the center of dilation, and finally choose 0.5 as the scale factor to dilate the triangle. Save this as “DILATE”.

PRACTICE

1. Move the line of reflection. How does the final image change?
2. To change the scale factor, select the Alpha-Num tool. Place the cursor over the scale factor. Press ENTER, then DELETE. Enter a new scale. How does the final image change?
3. Dilate with a center not at the origin. How does the final image change?
4. Use $\triangle ABC$ and line p, and the dilation and reflection from the Example. Dilate the triangle first, then reflect it. How does the final image change?

MIXED REVIEW *of Problem Solving*

STATE TEST PRACTICE
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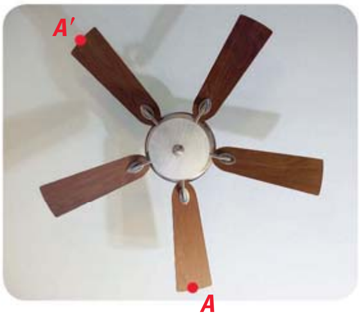
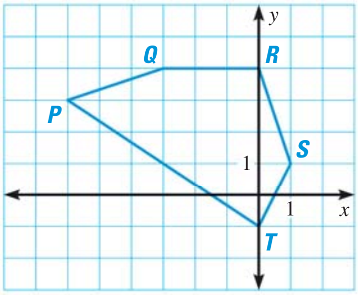
Lessons 9.4–9.7

1. GRIDDED ANSWER What is the angle of rotation, in degrees, that maps A to A' in the photo of the ceiling fan below?



2. SHORT RESPONSE The vertices of $\triangle DEF$ are $D(-3, 2)$, $E(2, 3)$, and $F(3, -1)$. Graph $\triangle DEF$. Rotate $\triangle DEF$ 90° about the origin. Compare the slopes of corresponding sides of the preimage and image. What do you notice?

3. MULTI-STEP PROBLEM Use pentagon $PQRST$ shown below.

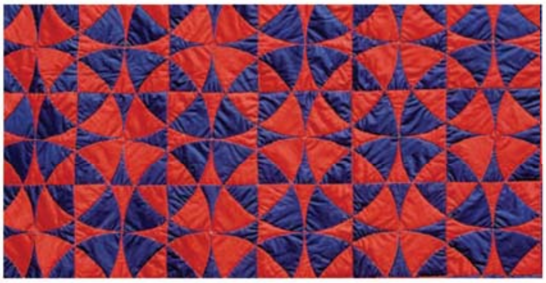


a. Write the polygon matrix for $PQRST$.

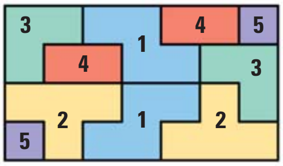
b. Find the image matrix for a 270° rotation about the origin.

c. Graph the image.

4. SHORT RESPONSE *Describe* the transformations that can be found in the quilt pattern below.



5. MULTI-STEP PROBLEM The diagram shows the pieces of a puzzle.



a. Which pieces are translated?

b. Which pieces are reflected?

c. Which pieces are glide reflected?

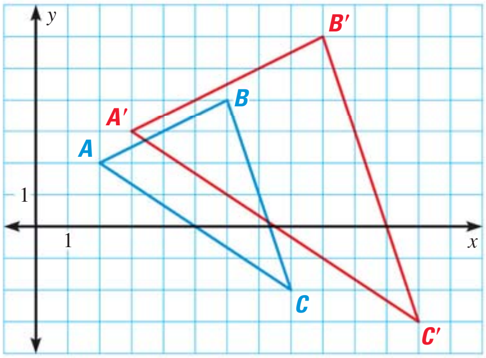
6. OPEN-ENDED Draw a figure that has the given type(s) of symmetry.

a. Line symmetry only

b. Rotational symmetry only

c. Both line symmetry and rotational symmetry

7. EXTENDED RESPONSE In the graph below, $\triangle A'B'C'$ is a dilation of $\triangle ABC$.



a. Is the dilation a *reduction* or an *enlargement*?

b. What is the scale factor? *Explain* your steps.

c. What is the polygon matrix? What is the image matrix?

d. When you perform a composition of a dilation and a translation on a figure, does order matter? *Justify* your answer using the translation $(x, y) \rightarrow (x + 3, y - 1)$ and the dilation of $\triangle ABC$.

9 CHAPTER SUMMARY

BIG IDEAS

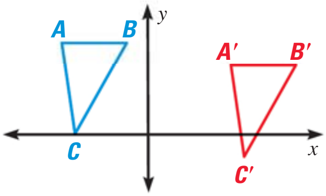
For Your Notebook

Big Idea ①

Performing Congruence and Similarity Transformations

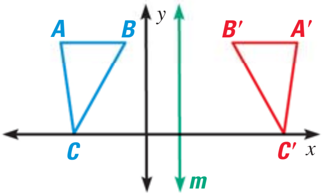
Translation

Translate a figure right or left, up or down.



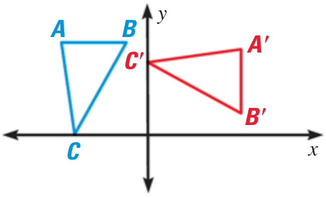
Reflection

Reflect a figure in a line.



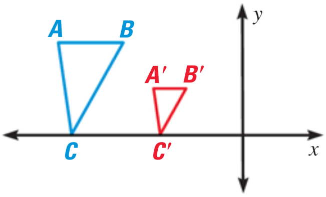
Rotation

Rotate a figure about a point.



Dilation

Dilate a figure to change the size but not the shape.

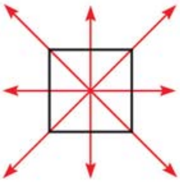


You can combine congruence and similarity transformations to make a composition of transformations, such as a glide reflection.

Big Idea ②

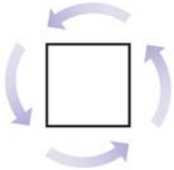
Making Real-World Connections to Symmetry and Tessellations

Line symmetry



4 lines of symmetry

Rotational symmetry



90° rotational symmetry

Big Idea ③

Applying Matrices and Vectors in Geometry

You can use matrices to represent points and polygons in the coordinate plane. Then you can use matrix addition to represent translations, matrix multiplication to represent reflections and rotations, and scalar multiplication to represent dilations. You can also use vectors to represent translations.

9 CHAPTER REVIEW

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- Multi-Language Glossary
- Vocabulary practice

REVIEW KEY VOCABULARY

For a list of postulates and theorems, see pp. 926–931.

- image, *p. 572*
- preimage, *p. 572*
- isometry, *p. 573*
- vector, *p. 574*
  initial point, terminal point, horizontal component, vertical component
- component form, *p. 574*
- matrix, *p. 580*
- element, *p. 580*
- dimensions, *p. 580*
- line of reflection, *p. 589*
- center of rotation, *p. 598*
- angle of rotation, *p. 598*
- glide reflection, *p. 608*
- composition of transformations, *p. 609*
- line symmetry, *p. 619*
- line of symmetry, *p. 619*
- rotational symmetry, *p. 620*
- center of symmetry, *p. 620*
- scalar multiplication, *p. 627*

VOCABULARY EXERCISES

1. Copy and complete: A(n) _?_ is a transformation that preserves lengths.

2. Draw a figure with exactly one line of symmetry.

3. **WRITING** *Explain* how to identify the dimensions of a matrix. Include an example with your explanation.

Match the point with the appropriate name on the vector.

4. *T* **A.** Initial point

5. *H* **B.** Terminal point

REVIEW EXAMPLES AND EXERCISES

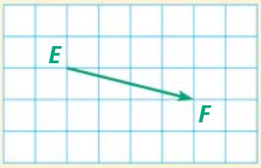
Use the review examples and exercises below to check your understanding of the concepts you have learned in each lesson of Chapter 9.

9.1 Translate Figures and Use Vectors *pp. 572–579*

EXAMPLE

Name the vector and write its component form.

The vector is $\overrightarrow{EF}$. From initial point *E* to terminal point *F*, you move 4 units right and 1 unit down. So, the component form is ⟨4, 1⟩.



EXERCISES

EXAMPLES 1 and 4 on pp. 572, 574 for Exs. 6–7

6. The vertices of $\triangle ABC$ are $A(2, 3)$, $B(1, 0)$, and $C(-2, 4)$. Graph the image of $\triangle ABC$ after the translation $(x, y) \rightarrow (x + 3, y - 2)$.

7. The vertices of $\triangle DEF$ are $D(-6, 7)$, $E(-5, 5)$, and $F(-8, 4)$. Graph the image of $\triangle DEF$ after the translation using the vector $\langle -1, 6\rangle$.

9.2 Use Properties of Matrices

pp. 580–587

EXAMPLE

Add $\begin{bmatrix} -9 & 12 \\ 5 & -4 \end{bmatrix} + \begin{bmatrix} 20 & 18 \\ 11 & 25 \end{bmatrix}$.

These two matrices have the same dimensions, so you can perform the addition. To add matrices, you add corresponding elements.

$$\begin{bmatrix} -9 & 12 \\ 5 & -4 \end{bmatrix} + \begin{bmatrix} 20 & 18 \\ 11 & 25 \end{bmatrix} = \begin{bmatrix} -9+20 & 12+18 \\ 5+11 & -4+25 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 16 & 21 \end{bmatrix}$$

EXERCISES

EXAMPLE 3
on p. 581
for Exs. 8–9

Find the image matrix that represents the translation of the polygon. Then graph the polygon and its image.

8. $\begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 2 & 8 & 1 \\ 4 & 3 & 2 \end{bmatrix} \end{array}$; 5 units up and 3 units left

9. $\begin{array}{c} \begin{matrix} D & E & F & G \end{matrix} \\ \begin{bmatrix} -2 & 3 & 4 & -1 \\ 3 & 6 & 4 & -1 \end{bmatrix} \end{array}$; 2 units down

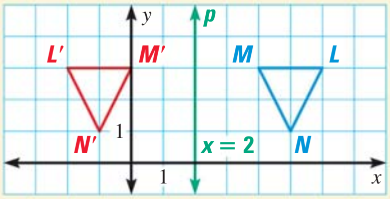
9.3 Perform Reflections

pp. 589–596

EXAMPLE

The vertices of $\triangle MLN$ are $M(4, 3)$, $L(6, 3)$, and $N(5, 1)$. Graph the reflection of $\triangle MLN$ in the line p with equation $x = 2$.

Point M is 2 units to the right of p, so its reflection M' is 2 units to the left of p at (0, 3). Similarly, L' is 4 units to the left of p at (−2, 3) and N' is 3 units to the left of p at (−1, 1).

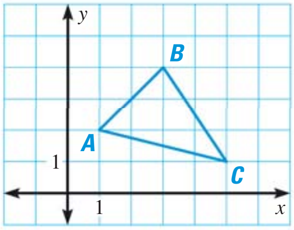


EXERCISES

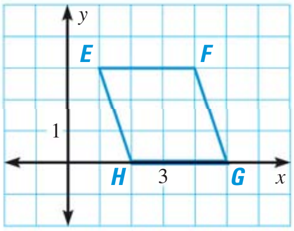
EXAMPLES 1 and 2
on pp. 589–590
for Exs. 10–12

Graph the reflection of the polygon in the given line.

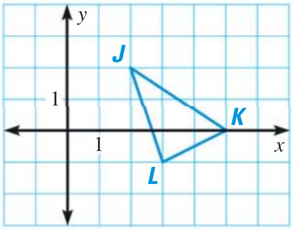
10. $x = 4$



11. $y = 3$



12. $y = x$



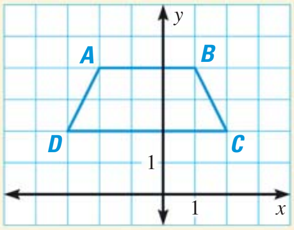
9 CHAPTER REVIEW

9.4 Perform Rotations

pp. 598–605

EXAMPLE

Find the image matrix that represents the 90° rotation of *ABCD* about the origin.



The polygon matrix for *ABCD* is $\begin{bmatrix} -2 & 1 & 2 & -3 \\ 4 & 4 & 2 & 2 \end{bmatrix}$.

Multiply by the matrix for a 90° rotation.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} -2 & 1 & 2 & -3 \\ 4 & 4 & 2 & 2 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} A' & B' & C' & D' \end{array} \\ \begin{bmatrix} -4 & -4 & -2 & -2 \\ -2 & 1 & 2 & -3 \end{bmatrix} \end{array}$$

EXERCISES

EXAMPLE 3
on p. 600
for Exs. 13–14

Find the image matrix that represents the given rotation of the polygon about the origin. Then graph the polygon and its image.

13. $\begin{array}{c} \begin{array}{ccc} Q & R & S \end{array} \\ \begin{bmatrix} 3 & 4 & 1 \\ 0 & 5 & -2 \end{bmatrix} \end{array}$; 180°

14. $\begin{array}{c} \begin{array}{cccc} L & M & N & P \end{array} \\ \begin{bmatrix} -1 & 3 & 5 & -2 \\ 6 & 5 & 0 & -3 \end{bmatrix} \end{array}$; 270°

9.5 Apply Compositions of Transformations

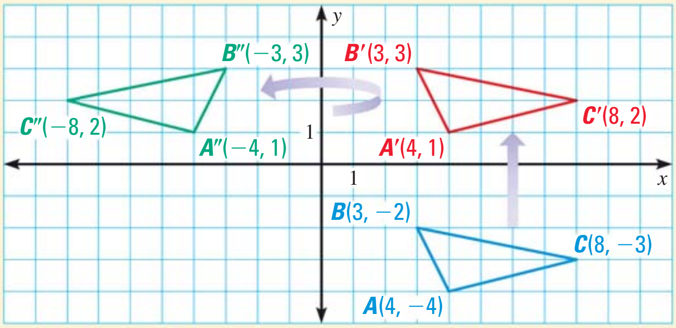
pp. 608–615

EXAMPLE

The vertices of △*ABC* are *A*(4, −4), *B*(3, −2), and *C*(8, −3). Graph the image of △*ABC* after the glide reflection.

Translation: $(x, y) \to (x, y + 5)$
Reflection: in the *y*-axis

Begin by graphing △*ABC*. Then graph the image △*A′B′C′* after a translation of 5 units up. Finally, graph the image △*A″B″C″* after a reflection in the *y*-axis.



EXERCISES

EXAMPLE 1
on p. 608
for Exs. 15–16

Graph the image of *H*(−4, 5) after the glide reflection.

15. **Translation:** $(x, y) \to (x + 6, y - 2)$
Reflection: in $x = 3$

16. **Translation:** $(x, y) \to (x - 4, y - 5)$
Reflection: in $y = x$

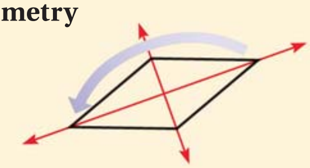
9.6 Identify Symmetry

pp. 619–624

EXAMPLE

Determine whether the rhombus has *line symmetry* and/or *rotational symmetry*. Identify the number of lines of symmetry and/or the rotations that map the figure onto itself.

The rhombus has two lines of symmetry. It also has rotational symmetry, because a 180° rotation maps the rhombus onto itself.



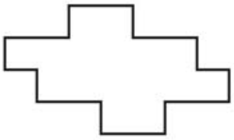
EXERCISES

EXAMPLES 1 and 2 on pp. 619–620 for Exs. 17–19

Determine whether the figure has *line symmetry* and/or *rotational symmetry*. Identify the number of lines of symmetry and/or the rotations that map the figure onto itself.

17.

18.



19.

9.7 Identify and Perform Dilations

pp. 626–632

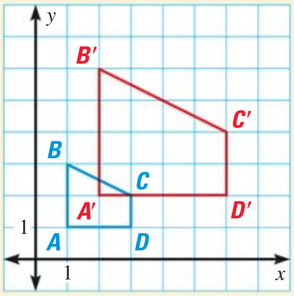
EXAMPLE

Quadrilateral *ABCD* has vertices *A*(0, 0), *B*(0, 3), *C*(2, 2), and *D*(2, 0). Use scalar multiplication to find the image of *ABCD* after a dilation with its center at the origin and a scale factor of 2. Graph *ABCD* and its image.

To find the image matrix, multiply each element of the polygon matrix by the scale factor.

$$2\begin{array}{c} A \quad B \quad C \quad D \\ \begin{bmatrix} 1 & 1 & 3 & 3 \\ 1 & 3 & 2 & 1 \end{bmatrix} \end{array} = \begin{array}{c} A' \quad B' \quad C' \quad D' \\ \begin{bmatrix} 2 & 2 & 6 & 6 \\ 2 & 6 & 4 & 2 \end{bmatrix} \end{array}$$

Scale factor **Polygon matrix** **Image matrix**



EXERCISES

EXAMPLE 4 on p. 628 for Exs. 20–21

Find the image matrix that represents a dilation of the polygon centered at the origin with the given scale factor. Then graph the polygon and its image.

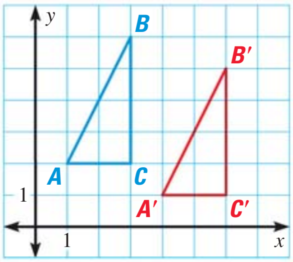
20. $\begin{array}{c} Q \quad R \quad S \\ \begin{bmatrix} 2 & 4 & 8 \\ 2 & 4 & 2 \end{bmatrix} \end{array}$; $k = \frac{1}{4}$

21. $\begin{array}{c} L \quad M \quad N \\ \begin{bmatrix} -1 & 1 & 2 \\ -2 & 3 & 4 \end{bmatrix} \end{array}$; $k = 3$

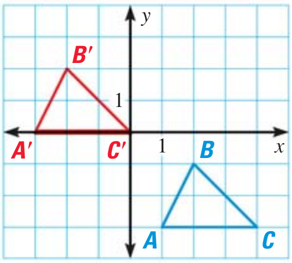
9 CHAPTER TEST

Write a rule for the translation of $\triangle ABC$ to $\triangle A'B'C'$. Then verify that the translation is an isometry.

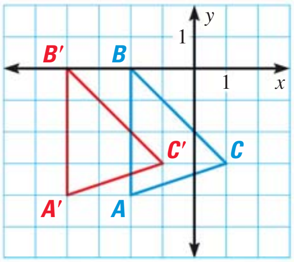
1.



2.



3.



Add, subtract, or multiply.

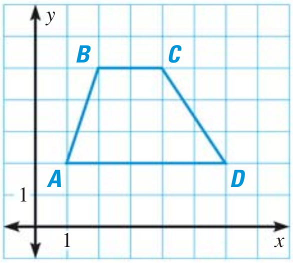
4. $\begin{bmatrix} 3 & -8 \\ 9 & 4.3 \end{bmatrix} + \begin{bmatrix} -10 & 2 \\ 5.1 & -5 \end{bmatrix}$

5. $\begin{bmatrix} -2 & 2.6 \\ 0.8 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ -1 & 3 \end{bmatrix}$

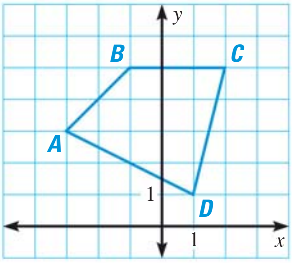
6. $\begin{bmatrix} 7 & -3 & 2 \\ 5 & 1 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

Graph the image of the polygon after the reflection in the given line.

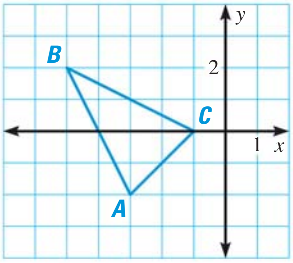
7. x-axis



8. $y = 3$



9. $y = -x$



Find the image matrix that represents the rotation of the polygon. Then graph the polygon and its image.

10. $\triangle ABC$: $\begin{bmatrix} 2 & 4 & 6 \\ 2 & 5 & 1 \end{bmatrix}$; 90° rotation

11. $KLMN$: $\begin{bmatrix} -5 & -2 & -3 & -5 \\ 0 & 3 & -1 & -3 \end{bmatrix}$; 180° rotation

The vertices of $\triangle PQR$ are $P(-5, 1)$, $Q(-4, 6)$, and $R(-2, 3)$. Graph $\triangle P''Q''R''$ after a composition of the transformations in the order they are listed.

12. **Translation:** $(x, y) \rightarrow (x - 8, y)$
Dilation: centered at the origin, $k = 2$

13. **Reflection:** in the y-axis
Rotation: 90° about the origin

Determine whether the flag has *line symmetry* and/or *rotational symmetry*. Identify all lines of symmetry and/or angles of rotation that map the figure onto itself.

14.

15.

16.

9 ALGEBRA REVIEW

Animated Algebra
classzone.com

MULTIPLY BINOMIALS AND USE QUADRATIC FORMULA

EXAMPLE 1 Multiply binomials

Find the product (2x + 3)(x − 7).

Solution

Use the **FOIL** pattern: Multiply the **F**irst, **O**uter, **I**nner, and **L**ast terms.

$$\begin{aligned}(2x+3)(x-7) &= \overset{\text{First}}{\overset{\downarrow}{2x(x)}} + \overset{\text{Outer}}{\overset{\downarrow}{2x(-7)}} + \overset{\text{Inner}}{\overset{\downarrow}{3(x)}} + \overset{\text{Last}}{\overset{\downarrow}{3(-7)}} && \text{Write the products of terms.}\\ &= 2x^2 - 14x + 3x - 21 && \text{Multiply.}\\ &= 2x^2 - 11x - 21 && \text{Combine like terms.}\end{aligned}$$

EXAMPLE 2 Solve a quadratic equation using the quadratic formula

Solve $2x^2 + 1 = 5x$.

Solution

Write the equation in standard form to be able to use the quadratic formula.

$$\begin{aligned} 2x^2 + 1 &= 5x && \text{Write the original equation.}\\ 2x^2 - 5x + 1 &= 0 && \text{Write in standard form.}\\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Write the quadratic formula.}\\ x &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(1)}}{2(2)} && \text{Substitute values in the quadratic formula: } a = 2, b = -5, \text{ and } c = 1.\\ x &= \frac{5 \pm \sqrt{25 - 8}}{4} = \frac{5 \pm \sqrt{17}}{4} && \text{Simplify.}\end{aligned}$$

▶ The solutions are $\frac{5 + \sqrt{17}}{4} \approx 2.28$ and $\frac{5 - \sqrt{17}}{4} \approx 0.22$.

EXERCISES

EXAMPLE 1
for Exs. 1–9

Find the product.

1. $(x + 3)(x - 2)$

2. $(x - 8)^2$

3. $(x + 4)(x - 4)$

4. $(x - 5)(x - 1)$

5. $(7x + 6)^2$

6. $(3x - 1)(x + 9)$

7. $(2x + 1)(2x - 1)$

8. $(-3x + 1)^2$

9. $(x + y)(2x + y)$

EXAMPLE 2
for Exs. 10–18

Use the quadratic formula to solve the equation.

10. $3x^2 - 2x - 5 = 0$

11. $x^2 - 7x + 12 = 0$

12. $x^2 + 5x - 2 = 0$

13. $4x^2 + 9x + 2 = 0$

14. $3x^2 + 4x - 10 = 0$

15. $x^2 + x = 7$

16. $3x^2 = 5x - 1$

17. $x^2 = -11x - 4$

18. $5x^2 + 6 = 17x$

9 ★ Standardized TEST PREPARATION

Scoring Rubric

Full Credit
- solution is complete and correct

Partial Credit
- solution is complete but has errors,

or
- solution is without error but incomplete

No Credit
- no solution is given,

or
- solution makes no sense

SHORT RESPONSE QUESTIONS

PROBLEM

The vertices of $\triangle PQR$ are $P(1, -1)$, $Q(4, -1)$, and $R(0, -3)$. What are the coordinates of the image of $\triangle PQR$ after the given composition? *Describe* your steps. Include a graph with your answer.

Translation: $(x, y) \rightarrow (x - 6, y)$
Reflection: in the x-axis

Below are sample solutions to the problem. Read each solution and the comments in blue to see why the sample represents full credit, partial credit, or no credit.

SAMPLE 1: Full credit solution

The reasoning is correct, and the graphs are correct.

First, graph $\triangle PQR$. Next, to translate $\triangle PQR$ 6 units left, subtract 6 from the x-coordinate of each vertex.

$P(1, -1) \rightarrow P'(-5, -1)$

$Q(4, -1) \rightarrow Q'(-2, -1)$

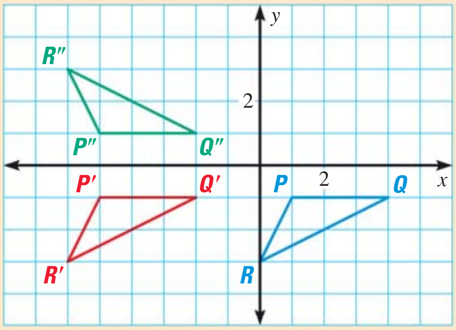
$R(0, -3) \rightarrow R'(-6, -3)$

Finally, reflect $\triangle P'Q'R'$ in the x-axis by multiplying the y-coordinates by -1.

$P'(-5, -1) \rightarrow P''(-5, 1)$

$Q'(-2, -1) \rightarrow Q''(-2, 1)$

$R'(-6, -3) \rightarrow R''(-6, 3)$

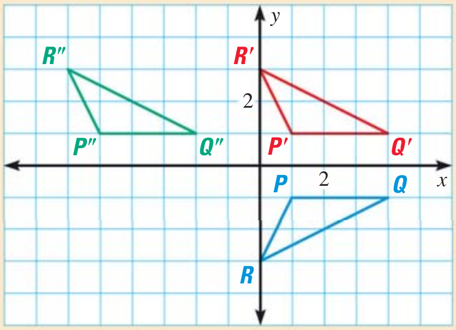


SAMPLE 2: Partial credit solution

Each transformation is performed correctly. However, the transformations are not performed in the order given in the problem.

First, graph $\triangle PQR$. Next, reflect $\triangle PQR$ over the x-axis by multiplying each y-coordinate by -1. Finally, to translate $\triangle P'Q'R'$ 6 units left, subtract 6 from each x-coordinate.

The coordinates of the image of $\triangle PQR$ after the composition are $P''(-2, 1)$, $Q''(-5, 1)$, and $R''(-6, 3)$.



SAMPLE 3: Partial credit solution

The reasoning is correct, but the student does not show a graph.

First subtract 6 from each x-coordinate. So, $P'(1 - 6, -1) = P'(-5, -1)$, $Q'(4 - 6, -1) = Q'(-2, -1)$, and $R'(0 - 6, -3) = R'(-6, -3)$. Then reflect the triangle in the x-axis by multiplying each y-coordinate by -1. So, $P''(-5, -1 \cdot (-1)) = P''(-5, 1)$, $Q''(-2, -1 \cdot (-1)) = Q''(-2, 1)$, and $R''(-6, -1 \cdot (-3)) = R''(-6, 3)$.

SAMPLE 4: No credit solution

The reasoning is incorrect, and the student does not show a graph.

Translate $\triangle PQR$ 6 units by adding 6 to each x-coordinate. Then multiply each x-coordinate by -1 to reflect the image over the x-axis. The resulting $\triangle P'Q'R'$ has vertices $P'(-7, -1)$, $Q'(-10, -1)$, and $R'(-6, -3)$.

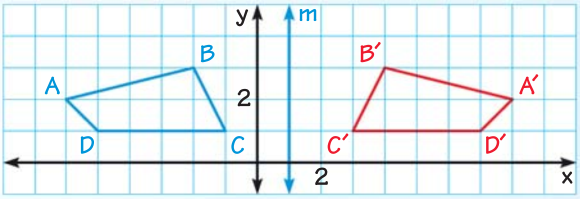
PRACTICE

Apply Scoring Rubric

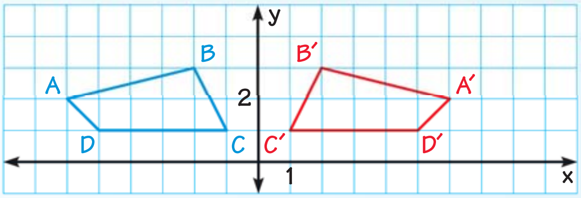
Use the rubric on page 642 to score the solution to the problem below as *full credit*, *partial credit*, or *no credit*. *Explain* your reasoning.

PROBLEM The vertices of $ABCD$ are $A(-6, 2)$, $B(-2, 3)$, $C(-1, 1)$, and $D(-5, 1)$. Graph the reflection of $ABCD$ in line m with equation $x = 1$.

1. First, graph ABCD. Because m is a vertical line, the reflection will not change the y-coordinates. A is 7 units left of m, so A′ is 7 units right of m, at A′(8, 2). Since B is 3 units left of m, B′ is 3 units right of m, at B′(4, 3). The images of C and D are C′(3, 1) and D′(7, 1).



2. First, graph ABCD. The reflection is in a vertical line, so only the x-coordinates change. Multiply the x-coordinates in ABCD by −1 to get A′(6, 2), B′(2, 3), C′(−1, 1), and D′(5, 1). Graph A′B′C′D′.



9 ★ Standardized TEST PRACTICE

SHORT RESPONSE

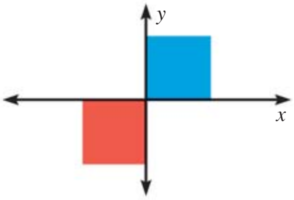
1. Use the square window shown below.

   a. Draw a sketch showing all the lines of symmetry in the window design.
   b. Does the design have rotational symmetry? If so, *describe* the rotations that map the design onto itself.

2. The vertices of a triangle are $A(0, 2)$, $B(2, 0)$, and $C(-2, 0)$. What are the coordinates of the image of $\triangle ABC$ after the given composition? Include a graph with your answer.

   Dilation: $(x, y) \rightarrow (3x, 3x)$
   Translation: $(x, y) \rightarrow (x - 2, y - 2)$

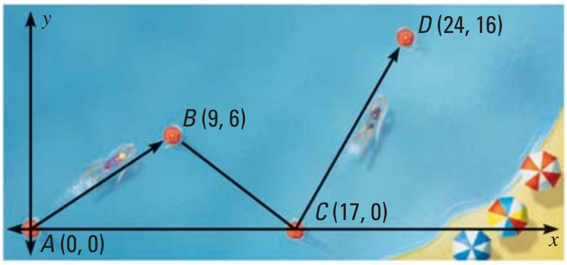
3. The red square is the image of the blue square after a single transformation. *Describe* three different transformations that could produce the image.



4. At a stadium concession stand, a hotdog costs \$3.25, a soft drink costs \$2.50, and a pretzel costs \$3. The Johnson family buys 5 hotdogs, 3 soft drinks, and 1 pretzel. The Scott family buys 4 hotdogs, 4 soft drinks, and 2 pretzels. Use matrix multiplication to find the total amount spent by each family. Which family spends more money? *Explain*.

5. The design below is made of congruent isosceles trapezoids. Find the measures of the four interior angles of one of the trapezoids. *Explain* your reasoning.

6. Two swimmers design a race course near a beach. The swimmers must move from point A to point B. Then they swim from point B to point C. Finally, they swim from point C to point D. Write the component form of the vectors shown in the diagram, $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CD}$. Then write the component form of $\overrightarrow{AD}$.



7. A polygon is reflected in the x-axis and then reflected in the y-axis. *Explain* how you can use a rotation to obtain the same result as this composition of transformations. Draw an example.

8. In rectangle $PQRS$, one side is twice as long as the other side. Rectangle $P'Q'R'S'$ is the image of $PQRS$ after a dilation centered at P with a scale factor of 0.5. The area of $P'Q'R'S'$ is 32 square inches.

   a. Find the lengths of the sides of $PQRS$. *Explain*.
   b. Find the ratio of the area of $PQRS$ to the area of $P'Q'R'S'$.

MULTIPLE CHOICE

9. Which matrix product is equivalent to the product $\begin{bmatrix} 3 & -1 \end{bmatrix}\begin{bmatrix} 7 \\ 4 \end{bmatrix}$?

Ⓐ $\begin{bmatrix} -3 & 1 \end{bmatrix}\begin{bmatrix} -7 \\ 4 \end{bmatrix}$

Ⓑ $\begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} -4 \\ 7 \end{bmatrix}$

Ⓒ $\begin{bmatrix} -1 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 4 \end{bmatrix}$

Ⓓ $\begin{bmatrix} 1 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ -7 \end{bmatrix}$

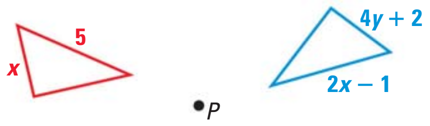
10. Which transformation is *not* an isometry?

Ⓐ Translation
Ⓑ Reflection
Ⓒ Rotation
Ⓓ Dilation

GRIDDED ANSWER

11. Line p passes through points $J(2, 5)$ and $K(-4, 13)$. Line q is the image of line p after line p is reflected in the x-axis. Find the slope of line q.

12. The red triangle is the image of the blue triangle after it is rotated about point P. What is the value of y?



13. The vertices of $\triangle PQR$ are $P(1, 4)$, $Q(2, 0)$, and $R(4, 5)$. What is the x-coordinate of Q' after the given composition?

Translation: $(x, y) \rightarrow (x - 2, y + 1)$
Dilation: centered at $(0, 0)$ with $k = 2$

EXTENDED RESPONSE

14. An equation of line ℓ is $y = 3x$.

a. Graph line ℓ. Then graph the image of line ℓ after it is reflected in the line $y = x$.

b. Find the equation of the image.

c. Suppose a line has an equation of the form $y = ax$. Make a conjecture about the equation of the image of that line when it is reflected in the line $y = x$. Use several examples to support your conjecture.

15. The vertices of $\triangle EFG$ are $E(4, 2)$, $F(-2, 1)$, and $G(0, -3)$.

a. Find the coordinates of the vertices of $\triangle E'F'G'$, the image of $\triangle EFG$ after a dilation centered at the origin with a scale factor of 2. Graph $\triangle EFG$ and $\triangle E'F'G'$ in the same coordinate plane.

b. Find the coordinates of the vertices of $\triangle E''F''G''$, the image of $\triangle E'F'G'$ after a dilation centered at the origin with a scale factor of 2.5. Graph $\triangle E''F''G''$ in the same coordinate plane you used in part (a).

c. What is the dilation that maps $\triangle EFG$ to $\triangle E''F''G''$?

d. What is the scale factor of a dilation that is equivalent to the composition of two dilations described below? *Explain.*

Dilation: centered at $(0, 0)$ with a scale factor of a
Dilation: centered at $(0, 0)$ with a scale factor of b

CUMULATIVE REVIEW Chapters 1–9

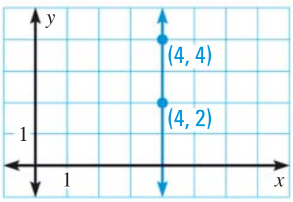
Tell whether the lines through the given points are *parallel*, *perpendicular*, or *neither*. *(p. 171)*

1. **Line 1:** (3, 5), (−2, 6)
Line 2: (−3, 5), (−4, 10)

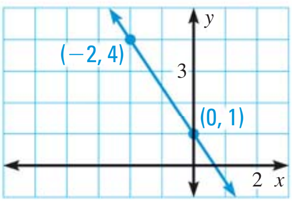
2. **Line 1:** (2, −10), (9, −8)
Line 2: (8, 6), (1, 4)

Write an equation of the line shown. *(p. 180)*

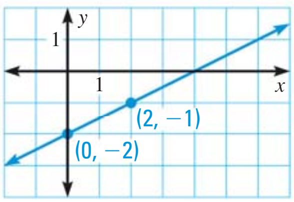
3.



4.

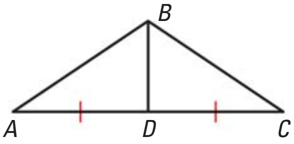


5.

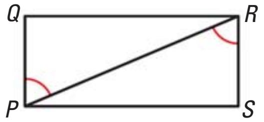


State the third congruence that must be given to prove that the triangles are congruent using the given postulate or theorem. *(pp. 234, 240, and 249)*

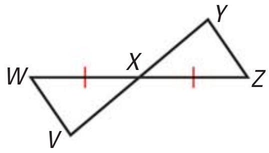
6. SSS Congruence Post.



7. SAS Congruence Post.

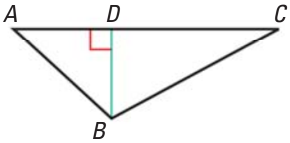


8. AAS Congruence Thm

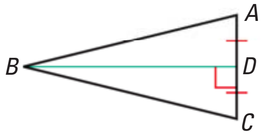


Determine whether $\overline{BD}$ is a *perpendicular bisector*, *median*, or *altitude* of △*ABC*. *(p. 319)*

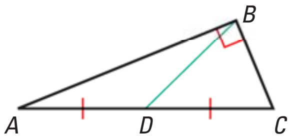
9.



10.



11.



Determine whether the segment lengths form a triangle. If so, would the triangle be *acute*, *right*, or *obtuse*? *(pp. 328 and 441)*

12. 11, 11, 15

13. 33, 44, 55

14. 9, 9, 13

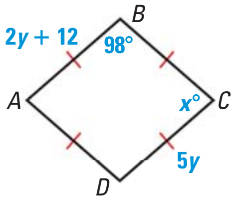
15. 7, 8, 16

16. 9, 40, 41

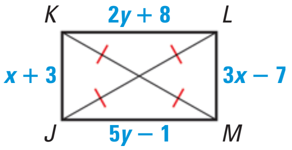
17. 0.5, 1.2, 1.3

Classify the special quadrilateral. *Explain* your reasoning. Then find the values of *x* and *y*. *(p. 533)*

18.



19.



20.