CHAPTER

11 Series

Korean artist Do-Ho Suh (b 1962) is perhaps best known for his sculptures that use numerous miniature items that contribute to a greater, larger mass. Shown here is a detail from his installation *Floor* (1997–2000), made of thousands of plastic figurines that support a 40 m^2 floor of glass.

OBJECTIVES

In this chapter you will

- **learn about mathematical patterns called series, and distinguish between arithmetic and geometric series**
- **write recursive and explicit formulas for series**
- **find the sum of a finite number of terms of an arithmetic or geometric series**
- **determine when an infinite geometric series has a sum and find the sum if it exists**

LESSON 11.1

Arithmetic Series

According to the U.S. Environmental Protection Agency, each American produced an average of 978 lb of trash in 1960. This increased to 1336 lb in 1980. By 2000, trash production had risen to 1646 lb/yr per person. You have learned in previous chapters how to write a sequence to describe the amount of trash produced per person each year. If you added the terms in this sequence, you could find the amount of trash a person produced in his or her lifetime.

Environmental CONNECTION

Mount Everest, part of the Himalaya range of southern Asia, reaches an altitude of 29,035 ft and is the world's highest mountain above sea level. It has been nicknamed "the world's highest junkyard" because decades of litter—climbing gear, plastic, glass, and metal—have piled up along Mount Everest's trails and in its camps. An estimated 50 tons of junk remain. Environmental agencies like the World Wildlife Fund have cleared garbage from the mountain's base camp, but removing waste from higher altitudes is more challenging.

A 1998 cleanup of Mt. Everest

Series

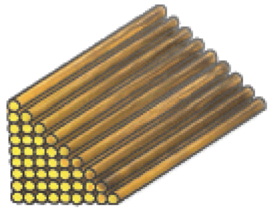
A sum of terms of a sequence is a **series.**

The sum of the first n terms in a sequence is represented by S_n. You can calculate S_n by adding the terms $u_1 + u_2 + u_3 + \cdots + u_n$.

History CONNECTION

Chu Shih-chieh (ca. 280–1303) was a celebrated mathematician from Beijing, China, known for his theories on arithmetic series, geometric series, and finite differences. His two mathematical works, *Introduction to Mathematical Studies* and *Precious Mirror of the Four Elements*, were discovered in the 19th century.

Finding the value of a series is a problem that has intrigued mathematicians for centuries. Chinese mathematician Chu Shih-chieh called the sum $1 + 2 + 3 + \cdots + n$ a "pile of reeds" because it can be pictured like the diagram at right. The diagram shows S_9, the sum of the first nine terms of this sequence, $1 + 2 + 3 + \cdots + 9$. The sum of any **finite,** or limited, number of terms is called a **partial sum** of the series.



The expressions S_9 and $\sum_{n=1}^{9} u_n$ are shorthand ways of writing $u_1 + u_2 + u_3 + \cdots + u_9$.

You can express the partial sum S_9 with sigma notation as $\sum_{n=1}^{9} n$.

The expression $\sum_{n=1}^{9} n$ tells you to substitute the integers 1 through 9 for n in the explicit formula $u_n = n$, and then sum the resulting nine values. You get $1 + 2 + 3 + \cdots + 9 = 45$.

How could you find the sum of the integers 1 through 100? The most obvious method is to add the terms, one by one. You can use a recursive formula and a calculator to do this quickly.

First, write the sequence recursively as

$$\begin{aligned} u_1 &= 1 \\ u_n &= u_{n-1} + 1 \quad \text{where } n \geq 2 \end{aligned}$$

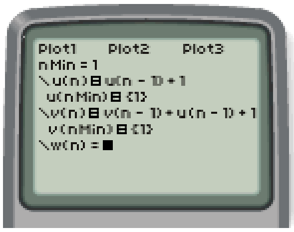
You can write a recursive formula for the series S_n like this:

$$\begin{aligned} S_1 &= 1 \\ S_n &= S_{n-1} + u_n \quad \text{where } n \geq 2 \end{aligned}$$

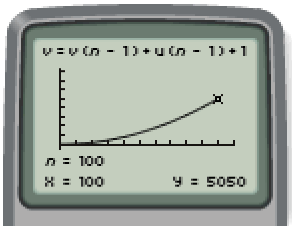
This states that the sum of the first n terms is equal to the sum of the first $(n - 1)$ terms, plus the nth term. From the recursive formula for the sequence you know that u_n is equivalent to $u_{n-1} + 1$, so the recursive formula for the series is

$$\begin{aligned} S_1 &= 1 \\ S_n &= S_{n-1} + u_{n-1} + 1 \quad \text{where } n \geq 2 \end{aligned}$$

Enter the recursive formulas into your calculator as shown. A table shows each term in the sequence and the sequence of partial sums.



The graph of S_n appears to form a solid curve, but it is actually a discrete set of 100 points representing each partial sum from S_1 through S_{100}. Each point is in the form (n, S_n) for integer values of n, for $1 \leq n \leq 100$. You can trace to find that the sum of the first 100 terms, S_{100}, is 5050. [See Calculator Note 11A for more information on graphing and calculating partial sums.]



[0, 110, 10, –3000, 10000, 1000]

When you compute this sum recursively, you or the calculator must compute each of the individual terms. The investigation will give you an opportunity to discover at least one explicit formula for calculating the partial sum of an **arithmetic series** without finding all terms and adding.

Investigation
Arithmetic Series Formula

Select three integers between 2 and 9 for your group to use. Each person should write his or her own arithmetic sequence using one of the three values for the first term and another value for the common difference. Make sure that each person uses a different sequence.

Step 1 Find the first ten terms of your sequence. Then find the first ten partial sums of the corresponding series. For example, using $u_1 = 7$ and $d = 8$, you would write

Sequence: $u_n = \{7, 15, 23, 31, \ldots\}$
Partial Sums: $S_n = \{7, 22, 45, 76, \ldots\}$

Step 2 Use finite differences to find the degree of a polynomial function that would fit data points in the form (n, S_n). Then find a polynomial function to fit the data.

Step 3 Create a new series by replacing either the first term or the common difference with another one of the three integers that your group selected. Repeat Steps 1 and 2.

Step 4 Combine the results from all members of your group into a table like the one below. For the partial sum column, enter the polynomial functions found in Step 2.

First term u_1	Common difference d	Partial sum S_n

Step 5 Look for a relationship between the coefficients of each polynomial and the values of u_1 and d. Then write an explicit formula for S_n in terms of u_1, d, and n.

In the investigation you found a formula for a partial sum of an arithmetic series. Use your formula to check that when you add $1 + 2 + 3 + \cdots + 100$ you get 5050.

History CONNECTION

According to legend, when German mathematician and astronomer Carl Friedrich Gauss (1777–1855) was 9 years old, his teacher asked the class to find the sum of the integers 1 through 100. The teacher was hoping to keep his students busy, but Gauss quickly wrote the correct answer, 5050. The example shows Gauss's solution method.

This stamp of Gauss was issued by Nicaragua in 1994 as part of a series about astronomers.

EXAMPLE Find the sum of the integers 1 through 100, without using a calculator.

Solution

Carl Friedrich Gauss solved this problem by adding the terms in pairs. Consider the series written in ascending and descending order, as shown.

$$\begin{array}{rcrcrcrcrcrcrcl} 1 & + & 2 & + & 3 & + \cdots + & 98 & + & 99 & + & 100 & = & S_{100} \\ 100 & + & 99 & + & 98 & + \cdots + & 3 & + & 2 & + & 1 & = & S_{100} \\ \hline 101 & + & 101 & + & 101 & + \cdots + & 101 & + & 101 & + & 101 & = & 2S_{100} \end{array}$$

The sum of every column is 101, and there are 100 columns. Thus, the sum of the integers 1 through 100 is

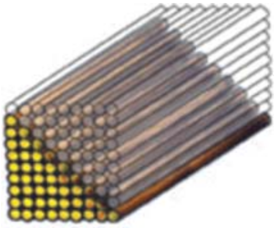
$$\frac{100(101)}{2} = 5050$$

You must divide the product 100(101) by 2 because the series was added twice.

You can extend the method in the example to any arithmetic series. Before continuing, take a moment to consider why the sum of the reeds in the original pile can be calculated using the expression

$$\frac{9(1+9)}{2}$$

What do the 9, 1, 9, and 2 represent in this context?



Partial Sum of an Arithmetic Series

The partial sum of an arithmetic series is given by the explicit formula

$$S_n = \left(\frac{d}{2}\right)n^2 + \left(u_1 - \frac{d}{2}\right)n$$

where n is the number of terms, u_1 is the first term, and d is the common difference.

An alternative formula is

$$S_n = \frac{n(u_1 + u_n)}{2}$$

where n is the number of terms, u_1 is the first term, and u_n is the last term.

In the exercises you will use the formulas for partial sums to find the sum of consecutive terms of an arithmetic sequence.

EXERCISES

Practice Your Skills

1. List the first five terms of this sequence. Identify the first term and the common difference.

$u_1 = -3$

$u_n = u_{n-1} + 1.5 \quad \text{where } n \geq 2$

2. Find S_1, S_2, S_3, S_4, and S_5 for this sequence: 2, 6, 10, 14, 18.

3. Write each expression as a sum of terms, then calculate the sum.

a. $\sum_{n=1}^{4}(n+2)$ **b.** $\sum_{n=1}^{3}(n^2-3)$

4. Find the sum of the first 50 multiples of 6: $\{6, 12, 18, \ldots, u_{50}\}$.

5. Find the sum of the first 75 even numbers, starting with 2.

Reason and Apply

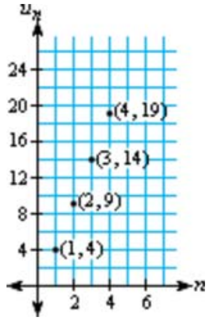
6. Find these values.

a. Find u_{75} if $u_n = 2n - 1$.

b. Find $\sum_{n=1}^{75}(2n-1)$.

c. Find $\sum_{n=20}^{75}(2n-1)$.

7. Consider the graph of the arithmetic sequence shown at right.



a. What is the 46th term?

b. Write a formula for u_n.

c. Find the sum of the heights from the horizontal axis of the first 46 points of the sequence's graph.

8. Suppose you practice the piano 45 min on the first day of the semester and increase your practice time by 5 min each day. How much total time will you devote to practicing during

a. The first 15 days of the semester?

b. The first 35 days of the semester?

American pianist Van Cliburn (b 1934) caused a sensation by winning the first International Tchaikovsky Competition in Moscow in 1958, during the height of the Cold War. When asked how many hours he practiced per day, he replied, "Mother always told me that if you knew how long you had practiced, you hadn't done anything. She believed you had to become engrossed . . . or it just didn't count for anything. Nothing happens if you watch the clock."

9. Jessica arranges a display of soup cans as shown.

a. List the number of cans in the top row, the second row, the third row, and so on, down to the tenth row.

b. Write a recursive formula for the terms of the sequence in 9a.

c. If the cans are to be stacked 47 rows high, how many cans will it take to build the display?

d. If Jessica uses six cases (288 cans), how tall can she make the display?

10. Find each value.

a. Find the sum of the first 1000 positive integers (the numbers 1 through 1000).

b. Find the sum of the second 1000 positive integers (1001 through 2000).

c. Guess the sum of the third 1000 positive integers (2001 through 3000).

d. Now calculate the sum for 10c.

e. Describe a way to find the sum of the third 1000 positive integers, if you know the sum of the first 1000 positive integers.

11. Suppose $y = 65 + 2(x - 1)$ is an explicit representation of an arithmetic sequence, for integer values $x \geq 1$. Express the partial sum of the arithmetic series as a quadratic expression, with x representing the term number.

12. It takes 5 toothpicks to build the top trapezoid shown at right. You need 9 toothpicks to build 2 adjoined trapezoids and 13 toothpicks for 3 trapezoids.

a. If 1000 toothpicks are available, how many trapezoids will be in the last complete row?

b. How many complete rows will there be?

c. How many toothpicks will you use to construct these rows?

d. Use the numbers in this problem to carefully describe the difference between a sequence and a series.

13. **APPLICATION** If an object falls from rest, then the distance it falls during the first second is about 4.9 m. In each subsequent second, the object falls 9.8 m farther than in the preceding second.

a. Write a recursive formula to describe the distance the object falls during each second of free fall.

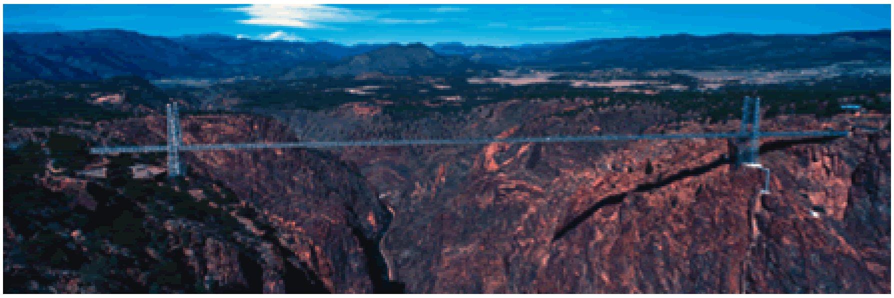
b. Find an explicit formula for 13a.

c. How far will the object fall during the 10th second?

d. How far does the object fall during the first 10 seconds?

e. Find an explicit formula for the distance an object falls in n seconds.

f. Suppose you drop a quarter from the Royal Gorge Bridge. How long will it take to reach the Arkansas River 331 m below?



The Royal Gorge Bridge (built 1929) near Cañon City, Colorado, is the world's highest suspension bridge, with length 384 m (1260 ft) and width 5 m (18 ft).

14. Consider these two geometric sequences:

i. 2, 4, 8, 16, 32, . . .

ii. $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

a. What is the long-run value of each sequence?

b. What is the common ratio of each sequence?

c. What will happen if you try to sum all of the terms of each sequence?

15. **APPLICATION** There are 650,000 people in a city. Every 15 minutes, the local radio and television stations broadcast a tornado warning. During each 15-minute time period, 42% of the people who had not yet heard the warning become aware of the approaching tornado. How many people have heard the news

a. After 1 hour?

b. After 2 hours?

In the central United States, an average of 800 to 1000 tornadoes occur each year. Tornado watches, forecasts, and warnings are announced to the public by the National Weather Service.

Review

16. Suppose you invest $500 in a bank that pays 5.5% annual interest compounded quarterly.

a. How much money will you have after five years?

b. Suppose you also deposit an additional $150 at the end of every three months. How much will you have after five years?

17. Consider the explicit formula $u_n = 81\left(\frac{1}{3}\right)^{n-1}$

a. List the first six terms, u_1 to u_6.

b. Write a recursive formula for the sequence.

18. Consider the recursive formula

$u_1 = 0.39$

$u_n = 0.01 \cdot u_{n-1}$ where $n \geq 2$

a. List the first six terms.

b. Write an explicit formula for the sequence.

19. Find the exact value for

a. cos 15°

b. cos 75°

20. Consider the rational equation $y = \frac{4x+3}{2x-1}$.

a. Rewrite the equation as a transformation of the parent function $y = \frac{1}{x}$.

b. What are the asymptotes of $y = \frac{4x+3}{2x-1}$?

c. The point (1, 1) is on the graph of the parent function. What is its image on the transformed function?

LESSON 11.2

Infinite Geometric Series

Beauty itself is but the sensible image of the infinite.

GEORGE BANCROFT

In Lesson 11.1, you developed an explicit formula for a partial sum of an arithmetic series. This formula works when you have a finite, or limited, number of terms. You also saw that as the number of terms, n, increases, the magnitude of the partial sum, S_n, increases in the long run. If the number of terms of an arithmetic series were **infinite,** or unending, then the magnitude of the sum would be infinite.

These nested photographs represent an infinite sequence.

But some geometric sequences have terms that get smaller. What happens to the partial sums of these sequences? For example, the geometric sequence

$0.4, 0.04, 0.004, \ldots$

has common ratio $\frac{1}{10}$, so the terms get smaller. Adding the terms creates a **geometric series.** Notice the pattern of repeating decimals that is formed.

$S_3 = 0.4 + 0.04 + 0.004 = 0.444$

$S_4 = 0.4 + 0.04 + 0.004 + 0.0004 = 0.4444$

$S_5 = 0.4 + 0.04 + 0.004 + 0.0004 + 0.00004 = 0.44444$

If you sum an infinite number of terms of this sequence, would the result be infinitely large?

An **infinite geometric series** is a geometric series with infinitely many terms. In this lesson you will specifically look at **convergent series,** for which the sequence of partial sums approaches a long-run value as the number of terms increases.

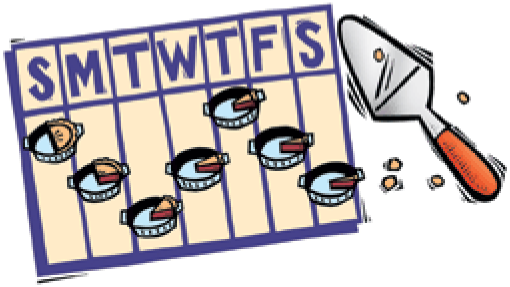
EXAMPLE A

Jack baked a pie and promptly ate one-half of it. Determined to make the pie last, he then decided to eat only one-half of the pie that remained each day.

a. Record the amount of pie eaten each day for the first seven days.

b. For each of the seven days, record the total amount of pie eaten since it was baked.

c. If Jack lives forever, then how much of this pie will he eat?



Solution

The amount of pie eaten each day is a geometric sequence with first term $\frac{1}{2}$ and common ratio $\frac{1}{2}$.

a. The first seven terms of this sequence are

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$$

b. Find the partial sums, S_1 through S_7, of the terms in part a.

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \frac{127}{128}$$

c. It may seem that eating pie "forever" would result in eating a lot of pie. However, if you look at the pattern of the partial sums, it seems as though for any finite number of days Jack's total is slightly less than 1. This leads to the conclusion that Jack would eat exactly one pie in the long run. This is a convergent infinite geometric series with long-run value 1.

Recall that a geometric sequence can be represented with an explicit formula in the form $u_n = u_1 \cdot r^{n-1}$ or $u_n = u_0 \cdot r^n$, where r represents the common ratio between the terms. The investigation will help you create an explicit formula for the sum of a convergent infinite geometric series.

Investigation

Infinite Geometric Series Formula

Select three integers between 2 and 9 for your group to use. Each person should write his or her own geometric sequence using one of the three values for the first term and one-tenth of another value for the common ratio. Make sure that each person uses a different sequence.

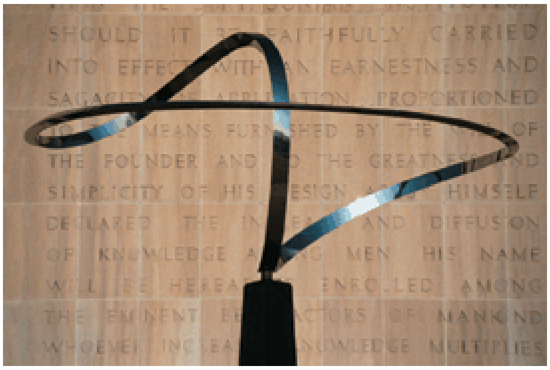
Step 1 Use your calculator to find the partial sum of the first 400 and the first 500 terms of your sequence. If your calculator rounds these partial sums to the same value, then use this number as the long-run value. If not, then continue summing terms until you find the long-run value. [▸ Revisit Calculator Note 11A to review how to calculate partial sums. ◂]

Step 2 Create a new series by replacing either the first term or the common ratio with another one of the three integers that your group selected. Remember that the common ratio is one-tenth of the integer. Repeat Step 1.

Step 3 Combine the results from all members of your group into a table like the one below. Your long-run value is equivalent to the sum of infinitely many terms, represented by S with no subscript.

First term u_1	Common ratio r	Sum S

Step 4 Find a formula for S in terms of u_1 and r. (*Hint:* Include another column for the ratio $\frac{u_1}{S}$ and look for relationships.) Will your formula work if the ratio is equal to 1? If the ratio is greater than 1? Justify your answers with examples.



Infinity by American sculptor José de Rivera (1904–1985)

In the investigation you used partial sums with large values of n to determine the long-run value of the sum of infinitely many terms. A graph of the sequence of partial sums is another tool you can use.

EXAMPLE B

Consider an ideal (frictionless) ball bouncing after it is dropped. The distances in inches that the ball falls on each bounce are represented by 200, 200(0.8), $200(0.8)^2$, $200(0.8)^3$, and so on. Summing these distances creates a series. Find the total distance the ball falls during an infinite number of bounces.

▶ Solution

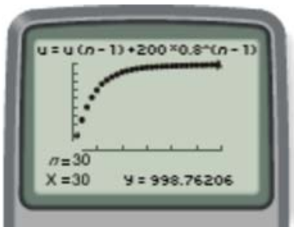
The sequence of partial sums is represented by the recursive formula

$$S_1 = 200$$
$$S_n = S_{n-1} + u_n \quad \text{where } n \geq 2$$

The explicit formula for the sequence of terms is $u_n = 200(0.8)^{n-1}$. So, the recursive formula for the series is equivalent to

$$S_1 = 200$$
$$S_n = S_{n-1} + 200(0.8)^{n-1} \quad \text{where } n \geq 2$$

The graph of S_n levels off as the number of bounces increases. This means the total sum of all the distances continues to grow, but seems to approach a long-run value for larger values of n.



[-5, 30, 5, -200, 1200, 100]

By looking at larger and larger values of n, you'll find the sum of this series is 1000 inches. So, the sum of the distances the ball falls is 1000 inches. Use the formula that you found in the investigation to verify this answer.

You now have several ways to determine the long-run value of an infinite geometric series. If the series is convergent, the formula that you found in the investigation gives you the sum of infinitely many terms.

Convergent Infinite Geometric Series

An infinite geometric series is a convergent series if the absolute value of the common ratio is less than 1, $|r| < 1$. The sum of infinitely many terms, S, of a convergent infinite geometric series is given by the explicit formula

$$S = \frac{u_1}{1-r}$$

where u_1 is the first term and r is the common ratio ($|r| < 1$).

In mathematics the symbol ∞ is used to represent infinity, or a quantity without bound. You can use ∞ and sigma notation to represent infinite series.

EXAMPLE C

Consider the infinite series

$$\sum_{n=1}^{\infty} 0.3(0.1)^{n-1}$$

a. Express this sum of infinitely many terms as a decimal.
b. Identify the first term, u_1, and the common ratio, r.
c. Express the sum as a ratio of integers.

Solution

When you substitute $n = \{1, 2, 3, \ldots\}$ into the expression $0.3(0.1)^{n-1}$, you get

$$0.3 + 0.03 + 0.003 + \cdots$$

a. The sum is the repeating decimal 0.333..., or $0.\overline{3}$.
b. $u_1 = 0.3$ and $r = 0.1$
c. Use the formula for the infinite sum and reduce to a ratio of integers:

$$S = \frac{0.3}{1-0.1} = \frac{0.3}{0.9} = \frac{1}{3}$$

You'll work with infinite geometric series and their sums further in the exercises.

EXERCISES

Practice Your Skills

1. Consider the repeating decimal 0.444..., or $0.\overline{4}$.

a. Express this decimal as the sum of terms of an infinite geometric series.
b. Identify the first term and the ratio.
c. Use the formula you learned in this lesson to express the sum as a ratio of integers.

2. Repeat the three parts of Exercise 1 with the repeating decimal 0.474747..., or $0.\overline{47}$.

3. Repeat the three parts of Exercise 1 with the repeating decimal 0.123123123..., or $0.\overline{123}$.

4. An infinite geometric sequence has a first term of 20 and a sum of 400. What are the first five terms?

Reason and Apply

5. An infinite geometric sequence contains the consecutive terms 128, 32, 8, and 2. The sum of the series is $43{,}690.\overline{6}$. What is the first term?

6. Consider the sequence $u_1 = 47$ and $u_n = 0.8u_{n-1}$ where $n \geq 2$. Find

   a. $\sum_{n=1}^{10} u_n$ **b.** $\sum_{n=1}^{20} u_n$ **c.** $\sum_{n=1}^{30} u_n$ **d.** $\sum_{n=1}^{\infty} u_n$

7. Consider the sequence $u_n = 96(0.25)^{n-1}$.

   a. List the first ten terms, u_1 to u_{10}.

   b. Find the sum $\sum_{n=1}^{10} 96(0.25)^{n-1}$.

   c. Make a graph of partial sums for $1 \leq n \leq 10$.

   d. Find the sum $\sum_{n=1}^{\infty} 96(0.25)^{n-1}$.

This digitally manipulated photo, depicting a roller coaster in the form of a Möbius strip, is titled *Infinite Fun*. For more information on Möbius strips, see the links at www.keymath.com/DAA.

8. A ball is dropped from an initial height of 100 cm. The rebound heights to the nearest centimeter are 80, 64, 51, 41, and so on. What is the total distance the ball will travel, both up and down?

9. **APPLICATION** A sporting event is to be held at the Superdome in New Orleans, Louisiana, which holds about 95,000 people. Suppose 50,000 visitors arrive in New Orleans and spend \$500 each. In the month after the event, the people in New Orleans spend 60% of the income from the visitors. The next month, 60% is spent again, and so on.

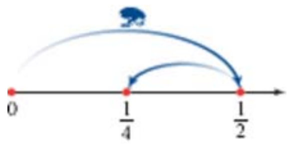
   a. What is the initial amount the visitors spent?

   b. In the long run, how much money does this sporting event seem to add to the New Orleans economy?

   c. The ratio of the long-run amount to the initial amount is called the economic multiplier. What is the economic multiplier in this example?

   d. If the initial amount spent by visitors is \$10,000,000 and the economic multiplier is 1.8, what percentage of the initial amount is spent again and again in the local economy?

10. A flea jumps $\frac{1}{2}$ ft right, then $\frac{1}{4}$ ft left, then $\frac{1}{8}$ ft right, and so on. To what point is the flea zooming in?



Magnified view of a flea in a dog's fur

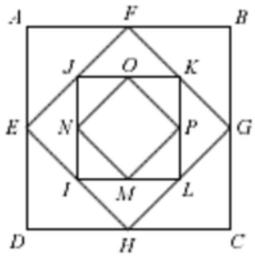
11. Suppose square *ABCD* with side length 8 in. is cut out of paper. Another square, *EFGH*, is placed with its corners at the midpoints of *ABCD*. A third square is placed with its corners at midpoints of *EFGH*, and so on.

a. What is the perimeter of the tenth square?

b. What is the area of the tenth square?

c. If the pattern could be repeated forever, what would be the sum of the perimeters of the squares?

d. What would be the sum of the areas?



12. The fractal known as the Sierpiński triangle begins as an equilateral triangle with side length 1 unit and area $\frac{\sqrt{3}}{4}$ square units. The fractal is created recursively by replacing the triangle with three smaller congruent equilateral triangles such that each smaller triangle shares a vertex with the larger triangle. This removes the area from the middle of the original triangle.

In the long run, what happens to

a. The perimeter of each of the smaller triangles?

b. The area of each of the smaller triangles?

c. The sum of the perimeters of the smaller triangles? (*Hint:* You can't use the sum formula from this lesson.)

d. The sum of the areas of the smaller triangles?

These Swedish stamps show the Koch snowflake, another fractal design that begins with an equilateral triangle. Can you determine the recursive procedure that creates the snowflake?

Review

13. A large barrel contains 12.4 gal of oil 18 min after a drain is opened. How many gallons of oil were in the barrel initially, if it drains at 4.2 gal/min?

14. Match each equation to a graph.

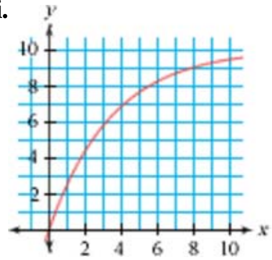
A. $y = 10(0.8)^x$

B. $y = 10 - 10(0.75)^x$

C. $y = 3 + 7(0.7)^x$

D. $y = 10 - 7(0.65)^x$

i.



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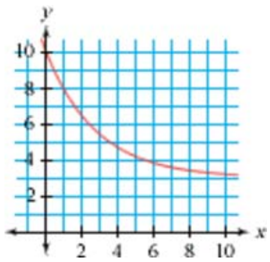

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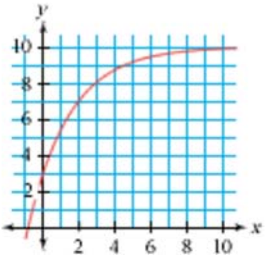

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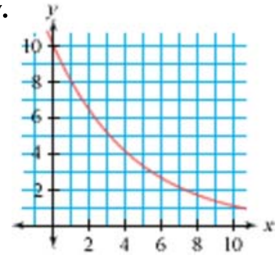


15. **APPLICATION** A computer software company decides to set aside $100,000 to develop a new video game. It estimates that development will cost $955 the first week and that expenses will increase by $65 each week.

a. After 25 weeks, how much of the development budget will be left?

b. How long can the company keep the development phase going before the budget will not support another week of expenses?

16. Hans sees a dog. The dog has four puppies. Four cats follow each puppy. Each cat has four kittens. Four mice follow each kitten. How many legs does Hans see? Express your answer using sigma notation.

IMPROVING YOUR VISUAL THINKING SKILLS

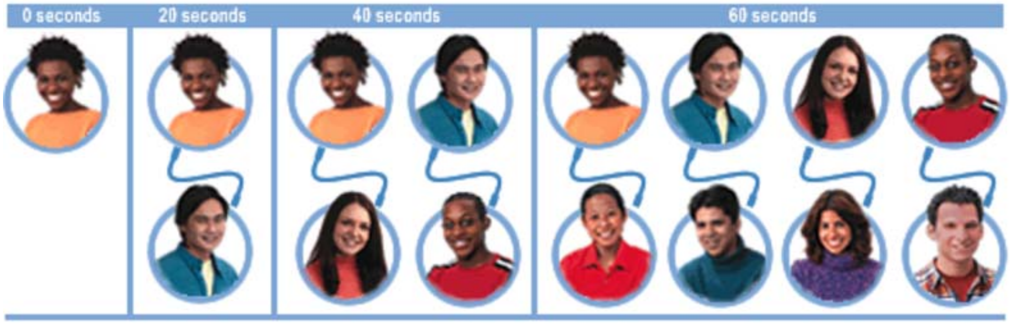
Toothpicks

You have eight toothpicks—four are short, and four are long. Each of the short toothpicks is one-half the length of any of the long toothpicks. Arrange the toothpicks to make exactly three congruent squares.

LESSON

11.3 Partial Sums of Geometric Series

If a pair of calculators can be linked and a program transferred from one calculator to the other in 20 s, how long will it be before everyone in a lecture hall of 250 students has the program? During the first time period, the program is transferred to one calculator; during the second time period, to two calculators; during the third time period, to four more calculators; and so on. The number of students who have the program doubles every 20 s. To solve this problem, you must determine the maximum value of n before S_n exceeds 250. This problem is an example of a partial sum of a geometric series. It requires the sum of a finite number of terms of a geometric sequence.



EXAMPLE A

Consider the sequence 2, 6, 18, 54, ...

a. Find u_{15}.

b. Graph the partial sums S_1 through S_{15}, and find the partial sum S_{15}.

▸ Solution

The sequence is geometric with $u_1 = 2$ and $r = 3$.

a. A recursive formula for the sequence is $u_1 = 2$ and $u_n = 3u_{n-1}$ where $n \geq 2$. The sequence can also be defined explicitly as $u_n = 2(3)^{n-1}$. Substituting 15 for n into either equation gives $u_{15} = 9{,}565{,}938$.

b. Use your calculator to graph the partial sums. (You'll see the data points better if you turn the axes off.) Trace to find that S_{15} is 14,348,906.

[0, 18, 1, –5000000, 20000000, 5000000]

In the example you used a recursive method to find the partial sum of a geometric series. For some partial sums, especially those involving a large number of terms, it can be faster and easier to use an explicit formula. The investigation will help you develop an explicit formula.

Investigation
Geometric Series Formula

Select three integers between 2 and 9 for your group to use. Each person should write his or her own geometric sequence using one of the three values for the first term and one-tenth of another value for the common ratio. Make sure that each person uses a different sequence.

Step 1 Find the first ten terms of your sequence. Then find the first ten partial sums of the corresponding series.

Step 2 Graph data points in the form (n, S_n), and find a translated exponential equation to fit the data. The equation will be in the form $S_n = L - a \cdot b^n$. (*Hint:* L is the long-run value of the partial sums. In Lesson 11.2, how did you find this value?)

Step 3 Rewrite your equation from Step 2 in terms of n, u_1, and r. Use algebraic techniques to write your explicit formula as a single rational expression.

Step 4 Create a new series, this time using an integer for the common ratio. Repeat Step 1.

Step 5 Use your formula to find S_{10}, and compare the result to the partial sum from Step 4. Does your formula work when the ratio is greater than 1? If not, then what changes do you have to make?

In the investigation you found an explicit formula for a partial sum of a geometric series that uses only three pieces of information—the first term, the common ratio, and the number of terms. Now you do not need to write out the terms to find a sum.

EXAMPLE B

Find S_{10} for the series $16 + 24 + 36 + \cdots$.

Solution

The first term, u_1, is 16. The common ratio, r, is 1.5. The number of terms, n, is 10. Use the formula you developed in the investigation to calculate S_{10}.

$$S_{10} = \frac{16\left(1 - 1.5^{10}\right)}{1 - 1.5} = 1813.28125$$

EXAMPLE C

Each day, the imaginary caterpillarsaurus eats 25% more leaves than it did the day before. If a 30-day-old caterpillarsaurus has eaten 151,677 leaves in its brief lifetime, how many will it eat the next day?

Solution To solve this problem, you must find u_{31}. The information in the problem tells you that r is (1 + 0.25), or 1.25, and when n equals 30, S_n equals 151,677. Substitute these values into the formula for S_n and solve for the unknown value, u_1.

$$\frac{u_1\left(1-1.25^{30}\right)}{1-1.25} = 151677$$

$$u_1 = 151677 \cdot \frac{1-1.25}{1-1.25^{30}}$$

$$u_1 \approx 47$$

Now you can write an explicit formula for the terms of the geometric sequence, $u_n = 47(1.25)^{n-1}$. Substitute 31 for n to find that on the 31st day, the caterpillarsaurus will consume 37,966 leaves.

The explicit formula for the sum of a geometric series can be written in several ways, but they are all equivalent. You probably found these two ways during the investigation:

Partial Sum of a Geometric Series

A partial sum of a geometric series is given by the explicit formula

$$S_n = \left(\frac{u_1}{1-r}\right) - \left(\frac{u_1}{1-r}\right)r^n \qquad \text{or} \qquad S_n = \frac{u_1(1-r^n)}{1-r}$$

where n is the number of terms, u_1 is the first term, and r is the common ratio ($r \neq 1$).

EXERCISES

Practice Your Skills

1. For each partial sum equation, identify the first term, the ratio, and the number of terms.

a. $\frac{12}{1-0.4} - \frac{12}{1-0.4}0.4^8 \approx 19.9869$

b. $\frac{75\left(1-1.2^{15}\right)}{1-1.2} \approx 5402.633$

c. $\frac{40-0.4 6117}{1-0.8} \approx 197.69$

d. $-40 + 40(2.5)^6 = 9725.625$

2. Consider the geometric sequence

256, 192, 144, 108, . . .

a. What is the eighth term?

b. Which term is the first one smaller than 20?

c. Find u_7.

d. Find S_7.

3. Find each partial sum of this sequence.

$u_1 = 40$

$u_n = 0.6u_{n-1}$ where $n \geq 2$

a. S_5

b. S_{15}

c. S_{25}

4. Identify the first term and the common ratio or common difference of each series. Then find the partial sum.

a. $3.2 + 4.25 + 5.3 + 6.35 + 7.4$

b. $3.2 + 4.8 + 7.2 + \cdots + 36.45$

c. $\sum_{n=1}^{27} (3.2 + 2.5n)$

d. $\sum_{n=1}^{10} 3.2(4)^{n-1}$

Reason and Apply

5. Find the missing value in each set of numbers.

a. $u_1 = 3$, $r = 2$, $S_{10} = \underline{?}$

b. $u_1 = 4$, $r = 0.6$, $S_{\underline{?}} \approx 9.999868378$

c. $u_1 = \underline{?}$, $r = 1.4$, $S_{15} \approx 1081.976669$

d. $u_1 = 5.5$, $r = \underline{?}$, $S_{18} \approx 66.30642497$

6. Find the nearest integer value for n if $\frac{3.2(1-0.8^n)}{1-0.8}$ is approximately 15.

7. Consider the sequence $u_1 = 8$ and $u_n = 0.5u_{n-1}$ where $n \geq 2$. Find

a. $\sum_{n=1}^{10} u_n$

b. $\sum_{n=1}^{20} u_n$

c. $\sum_{n=1}^{30} u_n$

d. Explain what is happening to these partial sums as you add more terms.

8. Suppose you begin a job with an annual salary of \$17,500. Each year, you can expect a 4.2% raise.

a. What is your salary in the tenth year after you start the job?

b. What is the total amount you earn in ten years?

c. How long must you work at this job before your total earnings exceed \$1 million?

9. An Indian folktale, recounted by Arab historian and geographer Ahmad al-Yaqubi in the 9th century, begins, "It is related by the wise men of India that when Husiya, the daughter of Balhait, was queen . . . ," and goes on to tell how the game of chess was invented. The queen was so delighted with the game that she told the inventor, "Ask what you will." The inventor asked for one grain of wheat on the first square of the chessboard, two grains on the second, four grains on the third, and so on, so that each square contained twice the number of grains as on the square before. (There are 64 squares on a chessboard.)

a. How many grains are needed

i. For the 8th square?

ii. For the 64th square?

iii. For the first row?

iv. To fill the board?

b. In sigma notation, write the series you used to fill the board.

Sonfonisba Anguissola's (ca. 1531–1625) painting, titled *The Chess Game* (1555), includes a self-portrait of the Italian artist (far left).

10. As a contest winner, you are given the choice of two prizes. The first choice awards \$1000 the first hour, \$2000 the second hour, \$3000 the third hour, and so on. For one entire year, you will be given \$1000 more each hour than you were given during the previous hour. The second choice awards 1¢ the first week, 2¢ the second week, 4¢ the third week, and so on. For one entire year, you will be given double the amount you received during the previous week. Which of the two prizes will be more profitable, and by how much?

11. Consider the geometric series

$$5 + 10 + 20 + 40 + \cdots$$

a. Find the first seven partial sums, $S_1, S_2, S_3, \ldots, S_7$.

b. Do the partial sums create a geometric sequence?

c. If u_1 is 5, find value(s) of r such that the partial sums form a geometric sequence.

12. Consider the series

$$\sum_{n=1}^{8} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{8}$$

a. Is this series arithmetic, geometric, or neither?

b. Find the sum of this series.

13. List terms to find

a. $\sum_{n=1}^{7} n^2$

b. $\sum_{n=3}^{7} n^2$

14. The 32 members of the Greeley High chess team are going to have a tournament. They need to decide whether to have a round-robin tournament or an elimination tournament. (Read the Recreation Connection.)

a. If the tournament is round-robin, how many games need to be scheduled?

b. If it is an elimination tournament, how many games need to be scheduled?

Recreation CONNECTION

In setting up tournaments, organizers have to decide the type of play. Most intramural sports programs are set up in "round-robin" format, in which every player or team plays every other player or team. Scheduling is different for odd and even numbers of teams, and it can be tricky if there is to be a minimum number of rounds. Another method is the elimination tournament, in which teams or players are paired and only the winners progress to the next round. For this format, scheduling difficulties arise when the initial number of teams is not a power of 2.

Review

15. What monthly payment is required to pay off an $80,000 mortgage at 8.9% interest in 30 years?

16. Develop the parametric equations of a hyperbola by following these steps.

a. Write the equation of a unit hyperbola in standard form using x and y.

b. Replace x with $\frac{1}{\cos t}$ and solve for y in terms of t. You should get a square root equation.

c. Add the terms under the radical by finding a common denominator.

d. Use the identity $(\sin t)^2 + (\cos t)^2 = 1$ to rewrite the numerator under the radical.

e. Use the identity $\tan t = \frac{\sin t}{\cos t}$ to rewrite the equation.

f. State the parametric equations for a unit hyperbola. (Look back at 16a to determine the parametric equation for x.)

g. Modify the parametric equations in 16f to incorporate horizontal and vertical translations (h and k) and horizontal and vertical scale factors (a and b).

17. The Magic Garden Seed Catalog advertises a bean with unlimited growth. It guarantees that with proper watering, the bean will grow 6 in. the first week and the height increase each subsequent week will be three-fourths of the previous week's height increase. "Pretty soon," the catalog claims, "Your beanstalk will touch the clouds!" Is this misleading advertising?

18. Write the polynomial equation of least degree that has integer coefficients and zeros $-3 + 2i$ and $\frac{2}{3}$.

IMPROVING YOUR REOMETRY SKILLS

Planning for the Future

Pamela and Candice are identical 20-year-old twins with identical jobs and identical salaries, and they receive identical bonuses of $2000 yearly.

Pamela is immediately concerned with saving money for her retirement. She invests her $2000 bonus each year at an interest rate of 9% compounded annually. At age 30, when she receives her tenth bonus, she decides it is time to see the world, and from that point on she spends her annual bonus on a trip.

Candice is immediately concerned with enjoying her income while she is young. She spends her $2000 bonus every year until she reaches 30. On her 30th birthday, when she receives her tenth bonus, she starts to worry about what will happen when she retires, so she starts saving her bonus money at an interest rate of 9% compounded annually.

How much will each twin have in her retirement account when she is 30 years old? Compare the value of each investment account when Pamela and Candice are 65 years old.

EXPLORATION

Seeing the Sum of a Series

The sum of an infinite geometric sequence is sometimes hard to visualize. Some sums clearly converge, whereas other sums do not. In this exploration you will use The Geometer's Sketchpad to simulate the sum of an infinite geometric series.

Activity

A Geometric Series

Procedure Note

1. Mark center B.
2. Dilate point A by $\frac{-6}{10}$ about center B. Label the new point A'. Construct $\overline{BA'}$.
3. Construct lines perpendicular to $\overline{AA'}$ through point B and A'. These will help you see how the series develops.

Step 1 Start Sketchpad. In a new sketch, construct horizontal segment $\overline{AB}$. Follow the procedure note to extend this segment by 60%.

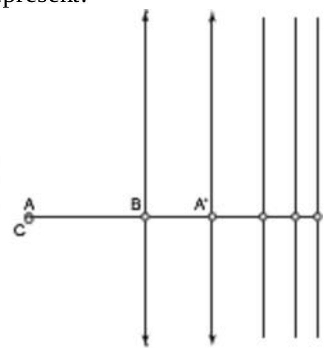
Step 2 Measure the distance between A and B.

Step 3 Construct point C, not on $\overline{AA'}$. Measure the distance between C and B.

Step 4 Select point A, then point B. Choose **Iterate** from the Transform menu. Map point A to point B, and point B to point A'. When you iterate, you'll get a table of values for AB and CB. What do the values for AB represent? If you move point C to coincide with point A, what do the values for CB represent?

$AB = 3.00$ cm
$CB = 3.00$ cm

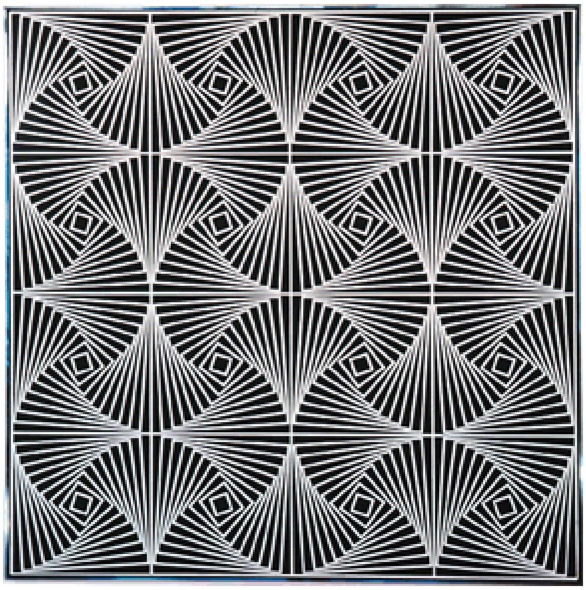
n	AB	CB
0	3.00 cm	3.00 cm
1	1.80 cm	4.80 cm
2	1.08 cm	5.88 cm
3	0.65 cm	6.53 cm



Step 5 Choose **Select All** from the Edit menu. Use the "+" key on your keyboard to add iterations to your sketch. Do approximately 25 iterations. Describe what happens as you add iterations.

Questions

1. What geometric series does your model represent? Write your answer in sigma notation.
2. According to the table values, what is the sum of your infinite series? Use the explicit formula $S = \frac{u_1}{1-r}$ to confirm this sum.
3. Vary the length of $\overline{AB}$ by dragging point B. What conclusions can you draw? Does the sum always converge for this common ratio?
4. Repeat the activity, but this time extend the segment by 120% (dilate point A by a factor of $\frac{-12}{10}$ about B). What happens with this series model?
5. Consider any geometric series in the form $\sum_{n=1}^{\infty} ar^{n-1}$. Explain how to model the series with Sketchpad, using the values a and r.



This untitled painting (1965) by French artist Jean-Pierre Yvaral (b 1934) uses geometric forms and symmetry to give the illusion of movement.

CHAPTER 11 REVIEW

Keymath.com Links to Resources

A **series** is a sum of terms of a sequence, defined recursively with the rule $S_1 = u_1$ and $S_n = S_{n-1} + u_n$ where $n \geq 2$. Series can also be defined explicitly. With an explicit formula, you can find any **partial sum** of a sequence without knowing the preceding term(s). Explicit formulas for **arithmetic series** are

$$S_n = \left(\frac{d}{2}\right)n^2 + \left(u_1 - \frac{d}{2}\right)n \quad \text{and} \quad S_n = \frac{n(u_1 + u_n)}{2}$$

Explicit formulas for **geometric series** are

$$S_n = \left(\frac{u_1}{1-r}\right) - \left(\frac{u_1}{1-r}\right)r^n \quad \text{and} \quad S_n = \frac{u_1(1-r^n)}{1-r}$$

where $r \neq 1$. If a geometric series is **convergent,** then you can calculate the sum of infinitely many terms with the formula $S = \frac{u_1}{1-r}$.

EXERCISES

1. Consider the arithmetic sequence: 3, 7, 11, 15, . . .

a. What is the 128th term?

b. Which term has the value 159?

c. Find u_{20}.

d. Find S_{20}.

2. Consider the geometric sequence: 100, 84, 70.56, . . .

a. Which term is the first one smaller than 20?

b. Find the sum of all the terms that are greater than 20.

c. Find the value of $\sum_{n=1}^{20} u_n$.

d. What happens to S_n as n gets very large?

3. Given plenty of food and space, a particular bug species will reproduce geometrically, with each pair hatching 24 young at age 5 days. (Assume that half the newborn bugs are male and half female, and are ready to reproduce in five days. Also assume each female bug can only reproduce once.) Initially, there are 12 bugs, half male and half female.

a. How many bugs are born on the 5th day? On the 10th day? On the 15th day? On the 35th day?

b. Write a recursive formula for the sequence in 3a.

c. Write an explicit formula for the sequence in 3a.

d. Find the total number of bugs after 60 days.

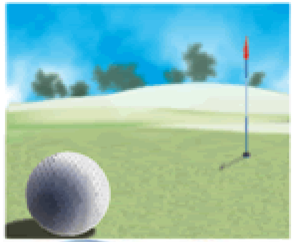
A swarm of ladybugs

4. Consider the series

125.3 + 118.5 + 111.7 + 104.9 + · · ·

a. Find S_{67}.

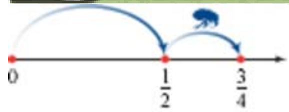
b. Write an expression for S_{67} using sigma notation.



5. Emma's golf ball lies 12 ft from the last hole on the golf course. She putts and, unfortunately, the ball rolls to the other side of the hole, $\frac{2}{3}$ as far away as it was before. On her next putt, the same thing happens.

a. If this pattern continues, how far will her ball travel in seven putts?

b. How far will the ball travel in the long run?



6. A flea jumps $\frac{1}{2}$ ft, then $\frac{1}{4}$ ft, then $\frac{1}{8}$ ft, and so on. It always jumps to the right.

a. Do the jump lengths form an arithmetic or geometric sequence? What is the common difference or common ratio?

b. How long is the flea's eighth jump, and how far is the flea from its starting point?

c. How long is the flea's 20th jump? Where is it after 20 jumps?

d. Write explicit formulas for jump length and the flea's location for any jump.

e. To what point is the flea zooming in?

7. For 7a–c, use $u_1 = 4$. Round your answers to the nearest thousandth.

a. For a geometric series with $r = 0.7$, find S_{10} and S_{40}.

b. For $r = 1.3$, find S_{10} and S_{40}.

c. For $r = 1$, find S_{10} and S_{40}.

d. Graph the partial sums of the series in 7a-c.

e. For which value of r (0.7, 1.3, or 1) do you have a convergent series?

8. Consider the series

$$0.8 + 0.08 + 0.008 + \cdots$$

a. Find S_{10}.

b. Find S_{15}.

c. Express the sum of infinitely many terms as a ratio of integers.

TAKE ANOTHER LOOK

1. You know how to write the equation of a continuous function that passes through the discrete points of a sequence, (n, u_n). For example, the function $y = 200(0.8)^{x-1}$ passes through the sequence of points representing the distance in inches that a ball falls on each bounce. Write a continuous function that passes through the points representing the *total* distance the ball has fallen on each bounce, (n, S_n). How can you use the function to find any partial sum or the sum of infinitely many terms? In general, what continuous function passes through the points of a geometric series? Through the points of an arithmetic series?

2. The explicit formula for a partial sum of a geometric series is $S_n = \frac{u_1(1 - r^n)}{1 - r}$. To find the sum of an infinite geometric series, you can imagine substituting ∞ for n. Explain what happens to the expression $\frac{u_1(1 - r^n)}{1 - r}$ when you do this substitution.

3. Since Chapter 1, you have solved problems about monthly payments, such as auto loans and home mortgages. You've learned how to find the monthly payment, P, required to pay off an initial amount, A_0, over n months with a monthly percentage rate, r. With series, you can find an explicit formula to calculate P. The recursive rule $A_n = A_{n-1}(1 + r) - P$ creates a sequence of the unpaid balances after the nth payment. The expanded equations for the unpaid balances are

$$A_1 = A_0(1 + r) - P$$

$$A_2 = A_0(1 + r)^2 - P(1 + r) - P$$

$$A_3 = A_0(1 + r)^3 - P(1 + r)^2 - P(1 + r) - P$$

and so on. Find the expanded equation for the last unpaid balance, A_n. Look at this equation for a partial sum of a geometric series, and use the explicit formula, $S_n = \frac{u_1(1 - r^n)}{1 - r}$, to simplify the equation for A_n. Then, substitute 0 for A_n (because after the last payment, the loan balance should be zero) and solve for P. This gives you an explicit formula for P in terms of A_0, n, and r. Test your explicit formula by solving these problems.

a. What monthly payment is required for a 60-month auto loan of $11,000 at an annual interest rate of 4.9% compounded monthly? (Answer: $207.08)

b. What is the maximum home mortgage for which Tina Fetzer can qualify if she can only afford a monthly payment of $620? Assume the annual interest rate is fixed at 7.5%, compounded monthly, and that the loan term is 30 years. (Answer: $88,670.93)

Assessing What You've Learned

WRITE TEST ITEMS Write a few test questions that explore series. You may use sequences that are arithmetic, geometric, or perhaps neither. You may want to include problems that use sigma notation. Be sure to include detailed solution methods and answers.

GIVE A PRESENTATION By yourself or with a partner, do a presentation showing how to find the partial sum of an arithmetic or geometric series using both a recursive formula and an explicit formula. Discuss the advantages and disadvantages of each method. Are there series that can be summed using one method, but not the other?

UPDATE YOUR PORTFOLIO Pick one of the three investigations from this chapter to include in your portfolio. Explain in detail the methods that you explored, and how you derived a formula for the partial sum of an arithmetic series, the partial sum of a geometric series, or the sum of an infinite convergent geometric series.