CHAPTER

10 Trigonometric Functions

In 1880, English-American photographer Eadweard J. Muybridge (1830-1904) created the zoopraxiscope, an early motion picture machine that projected a series of images on a spinning disk. The series of photographs shown here depicts a mule bucking and kicking. When the disk is spun quickly, it creates the illusion of a cyclical, repetitive motion. Muybridge spent many years studying animal and human movement, and perfecting his method of photographing motion.

Courtesy George Eastman House

OBJECTIVES

In this chapter you will

- **identify the relationship between circular motion and the sine and cosine functions**
- **use a unit circle to find values of sine and cosine for various angles**
- **learn a new unit of measurement for angles, called radians**
- **apply your knowledge of transformations to the graphs of trigonometric functions**
- **model real-world phenomena with trigonometric functions**
- **study trigonometric identities**

LESSON

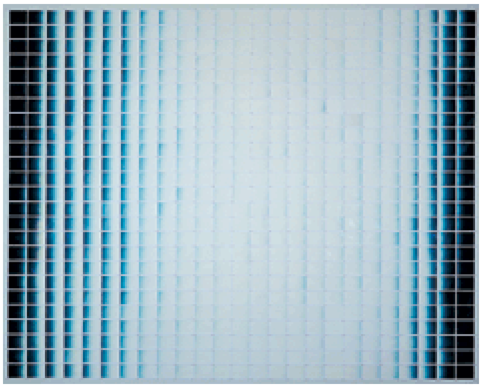
10.1

Defining the Circular Functions

It is by will alone I set my mind in motion.

MENTAT CHANT

Many phenomena are predictable because they are repetitive, or cyclical. The water depths caused by the tides, the motion of a person on a swing, your height above ground as you ride a Ferris wheel, and the number of hours of daylight each day throughout the year are all examples of cyclical patterns. In this lesson you will discover how to model phenomena like these with the sine and cosine functions.

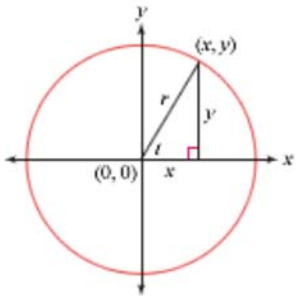


In *Light Graphs: Winter Solstice 2000*, American photographer Erika Blumenfeld (b 1971) used Polaroid photos, placed together in a grid pattern, to show the amount of light present every minute, from sunrise to sunset, on the shortest day of the year.

Science CONNECTION

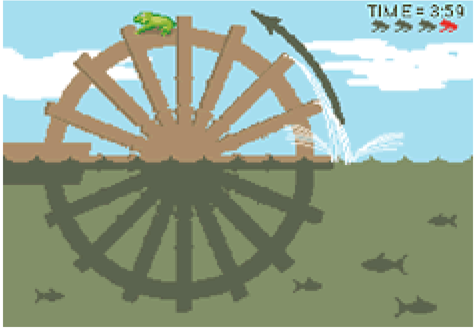
High and low ocean tides repeat in a continuous cycle, with two high tides and two low tides every 24.84 hours. The highest tide is an effect of the gravitational pull of the Moon, which causes a dome of water to travel under the Moon as it orbits Earth. The entire Earth experiences the Moon's gravitational pull, but because water is less rigid than land, it flows more easily in response to this force. The lowest tide occurs when this dome of water moves away from the shoreline. There are several theories about what causes the second high tide each day. For more information about tides, see the links at www.keymath.com/DAA.

The circle at right has radius r and center at the origin. A central angle of t degrees is shown. You can use your knowledge of right triangle trigonometry to write the equations $\sin t = \frac{y}{r}$ and $\cos t = \frac{x}{r}$.



In this investigation you'll see how to use the sine and cosine functions to model circular motion.

Investigation
Paddle Wheel



Procedure Note

1. Set your calculator in *degree* and *parametric* modes.
2. Graph the equations in a friendly window with a factor of $\frac{1}{2}$. Use t-values of $0° \leq t \leq 900°$ and t-step 15°. [► **See Calculator Note 10A.** ◄]

While swimming along, a frog reaches out and grabs onto the rim of a paddle wheel with radius 1 m. The center of the wheel is at water level. The frog, clinging tightly to the wheel, is immediately lifted from the surface of the river.

Step 1 The wheel spins slowly counterclockwise at a rate of one rotation every 6 minutes. Through how many degrees does the frog rotate each minute? Each second?

Step 2 Follow the procedure note to graph the parametric equations $x = \cos t$ and $y = \sin t$. Sketch this graph, and explain how to find the x- and y-values of any point on the graph.

Step 3 Create a table recording the frog's x- and y-positions every 15°, relative to the center of the wheel. Use domain $0° \leq t \leq 510°$. Note that the wheel is turning through 1° per second, so the number of degrees equals the number of seconds. Explain any patterns you find in your table values.

Step 4 Answer these questions by looking for patterns in your table.

a. What is the frog's location after 1215°, or 1215 s? When, during the first three rotations of the wheel, is the frog at that same location?

b. When is the frog at a height of -0.5 m during the first three rotations?

c. What are the maximum and minimum x- and y-values?

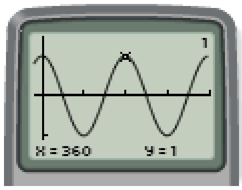
Step 5 Plot data in the form (t, x) to create a graph of the function $x = \cos t$. Use domain $0° \leq t \leq 360°$. Plot data in the form (t, y) on a different graph to create a graph of the function $y = \sin t$. How do these graphs compare? How can you use these graphs to find the frog's position at any time? Why do you think the sine and cosine functions are sometimes called circular functions?

A circle with radius 1 unit centered at the origin is called a unit circle. While using the unit circle created during the investigation, you discovered that values for sine and cosine repeat in a regular pattern. When output values of a function repeat at regular intervals, the function is **periodic.** The **period** of a function is the smallest distance between values of the independent variable before the cycle begins to repeat.

EXAMPLE A Find the period of the cosine function.

▶ Solution

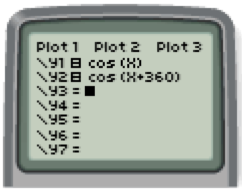
In the investigation the frog returned to the same position each time the paddle wheel made one complete rotation, or every 360°. So, a period of 360° seems reasonable. You can verify this guess by looking at a graph of $y = \cos x$.

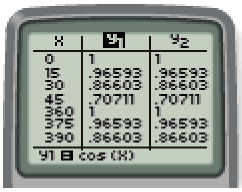


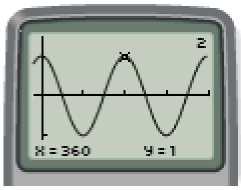
[–30, 720, 90, –1.5, 1.5, 1]

The graph shows that from $x = 0°$ to $x = 360°$, the function completes one full cycle.

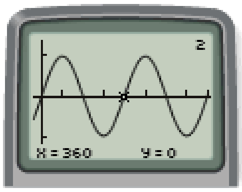
Notice that the graphs of $y = \cos x$ and $y = \cos(x + 360°)$, shown below, are the same. And the table shows that $\cos 0° = \cos 360°$, $\cos 15° = \cos 375°$, and so on. So the period of cosine is 360°.



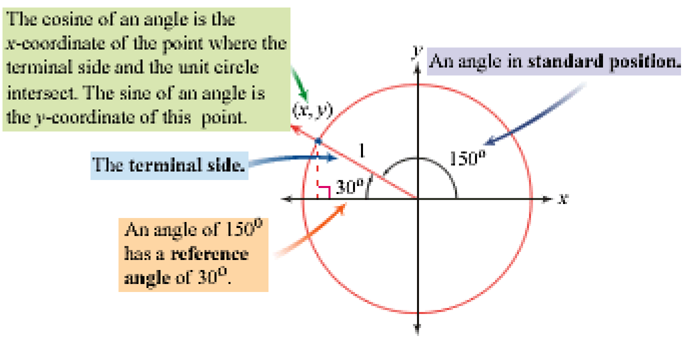




You can also verify that the sine function has a period of 360°. Use the table and graph below to convince yourself that this is true.



[– 30, 720, 90, –1.5, 1.5, 1]



Notice that the domains of the sine and cosine functions include all positive and negative numbers. In a unit circle, angles are measured from the positive x-axis. Positive angles are measured in a counterclockwise direction, and negative angles are measured in a clockwise direction. An angle in **standard position** has one side on the positive x-axis, and the other side is called the **terminal side.**

The x- and y-coordinates of the point where the terminal side touches the unit circle determine the values of the cosine and sine of the angle. Identifying a **reference angle,** the acute angle between the terminal side and the x-axis, and drawing a **reference triangle** can help you find these values.

EXAMPLE B

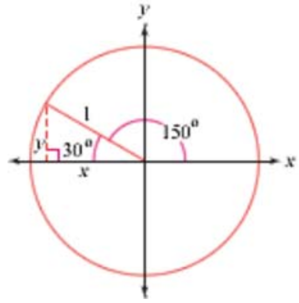
Find the value of the sine or cosine for each angle. Explain your process.

a. sin 150° **b.** cos 150° **c.** sin 210° **d.** cos 320°

▸ Solution

For each angle in a–d, rotate the terminal side counterclockwise from the positive *x*-axis, then draw a right triangle by dropping a line perpendicular to the *x*-axis. Then identify the reference angle.

a. For 150°, the reference angle measures 30°.



The value of *y* determines the sine of the angle. Use your knowledge of the ratios of side lengths in a 30°-60°-90° triangle to find the value of *y*. The sides have ratio $\frac{1}{2}$:$\frac{\sqrt{3}}{2}$:1. So, the length, *y*, of the leg opposite the 30° angle is $\frac{1}{2}$, and the length, *x*, of the adjacent leg is $\frac{\sqrt{3}}{2}$. Therefore

$$y = \sin 150° = \frac{1}{2}$$

Notice that *y* remains positive in Quadrant II. Use your calculator to verify that sin 150° equals 0.5.

b. You can use the same unit circle diagram as above. To find the value of cos 150°, find the value of *x*. As stated in part a, the length of the adjacent leg is $\frac{\sqrt{3}}{2}$. However, in the second quadrant *x* is negative, so

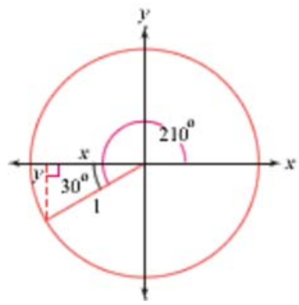
$$x = \sin 150° = -\frac{\sqrt{3}}{2}$$

The calculator gives – 0.866, which is approximately equal to $-\frac{\sqrt{3}}{2}$, as a decimal approximation of cos 150°.

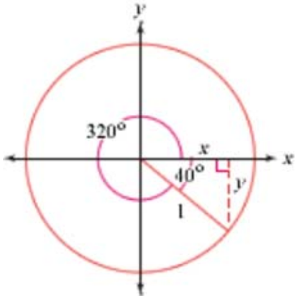
c. Rotate the terminal side counterclockwise 210°, then draw a right triangle by drawing a line perpendicular to the *x*-axis. The reference angle in this triangle again measures 30°. Because this angle is in Quadrant III, the *y*-value is negative, so

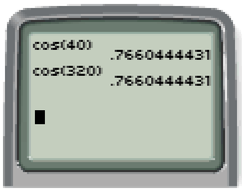
$$\sin 210° = -\frac{1}{2}$$

Use your calculator to verify this result.

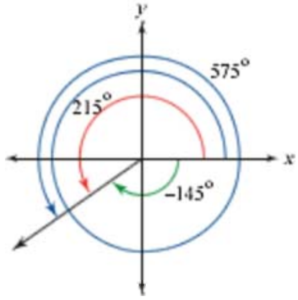


d. If you draw a reference triangle for 320°, you will find a reference angle of 40°, in Quadrant IV. In this quadrant, x-values are positive. So, cos 320° = cos 40°. An angle measuring 40° is not a special angle, so you don't know its exact trigonometric values. According to the calculator, either cos 320° or cos 40° is approximately 0.766.





Angles in standard position are **coterminal** if they share the same terminal side. For example, the angles measuring –145°, 215°, and 575° are coterminal, as shown. Coterminal angles have the same trigonometric values. When a variable is chosen to represent the measure of an unknown angle, it is common to use a Greek letter. When you see a Greek letter like θ (theta) or α (alpha), it is likely that the variable quantity is an angle measure.



EXERCISES

Practice Your Skills

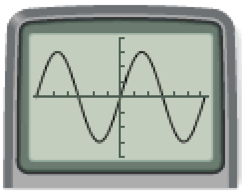
1. After 300 s, the paddle wheel in the investigation has rotated 300°. Draw a reference triangle and find the frog's height at this time.

2. Use your calculator to find each value, approximated to four decimal places. Then draw diagrams in a unit circle to show the meaning of the value. Name the reference angle.

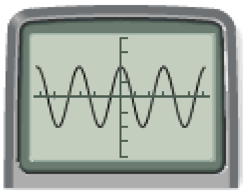
a. sin (–175°) **b.** cos 147° **c.** sin 280° **d.** cos 310° **e.** sin (–47°)

3. The functions $y = \sin x$ and $y = \cos x$ are periodic. How many cycles of each function are pictured?

a.



b.

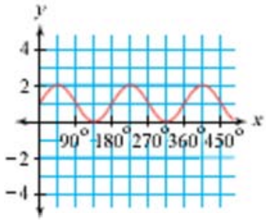


4. Create a sine or cosine graph, and trace to find the value of each expression.

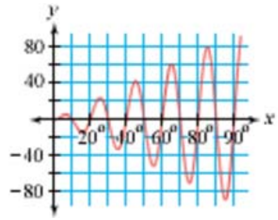
a. $\sin 120°$ **b.** $\sin(-120°)$ **c.** $\cos(-150°)$ **d.** $\cos 150°$

5. Which of the following functions are periodic? For each periodic function, identify the period.

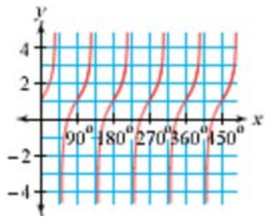
a.



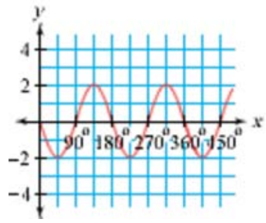
b.



c.



d.

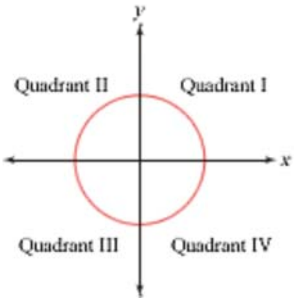


Keymath.com More Practice Your Skills

Reason and Apply

6. Identify an angle θ that is coterminal with the given angle. Use domain $0° \leq \theta \leq 360°$.

a. $-25°$ **b.** $-430°$ **c.** $435°$ **d.** $1195°$



7. For each Quadrant, I – IV, shown at right, identify whether the values of $\cos\theta$ and $\sin\theta$ are positive or negative.

8. Carefully sketch a graph of the function $y = \sin x$ over the domain $-360° \leq x \leq 360°$. Identify all values of x in this interval for which $\sin x = 0$.

9. Carefully sketch a graph of the function $y = \cos x$ over the domain $-360° \leq x \leq 360°$. Identify all values of x in this interval for which $\cos x = 0$.

10. Suppose $\sin\theta \approx -0.7314$ and $180° \leq \theta \leq 270°$.

a. Locate the point where the terminal side of θ intersects the unit circle.

b. Find θ and $\cos\theta$.

c. What other angle α has the same cosine value? Use domain $0° \leq \alpha \leq 360°$.

11. Find an angle in standard position, θ, for a plane flying on these bearings. Use domain $-180° \leq \theta \leq 180°$.

a. 105° **b.** 325° **c.** 180° **d.** 42°

12. Find $\sin\theta$ and $\cos\theta$ for each angle in standard position described. (*Hint:* You may want to use the Pythagorean Theorem.)

a. The terminal side of angle θ passes through the point (2, 3).

b. The terminal side of angle θ passes through the point (-2, 3).

13. Find each angle θ with the given trigonometric value. Use domain $0 \leq \theta \leq 360°$.

a. $\cos\theta = -\frac{\sqrt{3}}{2}$ **b.** $\cos\theta = -\frac{\sqrt{2}}{2}$ **c.** $\sin\theta = -\frac{3}{5}$ **d.** $\sin\theta = 1$

14. ***Mini-Investigation*** Make a table of the values of $\sin\theta$, $\cos\theta$, $\tan\theta$, and $\frac{\sin\theta}{\cos\theta}$, using values of θ at intervals of 30° over the domain $0° \leq \theta \leq 360°$. What do you notice? Use the definitions of trigonometric ratios to explain your conjecture.

15. **APPLICATION** For the past several hundred years, astronomers have kept track of the number of sunspots. This table shows the average number of sunspots each year from 1972 to 1999.

Year	Number of sunspots	Year	Number of sunspots	Year	Number of sunspots
1972	68.9	1982	115.9	1992	94.3
1973	38.0	1983	66.6	1993	54.6
1974	34.5	1984	45.9	1994	29.9
1975	15.5	1985	17.9	1995	17.5
1976	12.6	1986	3.4	1996	8.6
1977	27.5	1987	29.4	1997	21.5
1978	92.5	1988	100.2	1998	64.3
1979	155.4	1989	157.6	1999	93.3
1980	154.6	1990	142.6		
1981	140.4	1991	145.7		

(*www.solarmovie.com*)

a. Make a scatter plot of the data and describe any patterns that you notice.

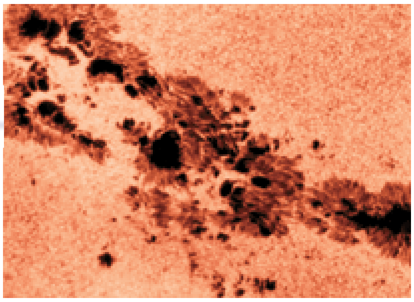
b. Estimate the length of a cycle.

c. Predict the next period of maximum solar activity after 1999.

Science CONNECTION

Sunspots are dark regions on the Sun's surface that are cooler than the surrounding areas. They are caused by magnetic fields on the Sun and seem to follow short-term and long-term cycles.

Sunspot activity affects conditions on Earth. The particles emitted by the Sun during periods of high sunspot activity disrupt radio communications, and have an impact on Earth's magnetic field and climate. Some scientists theorize that ice ages are caused by relatively low solar activity over a period of time.



This photo of several sunspots shows powerful eruptions occurring on the Sun's surface.

Review

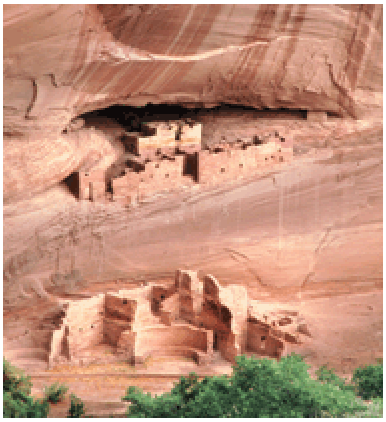
16. Annie is standing on a canyon floor 20 m from the base of a cliff. Looking through her binoculars, she sees the remains of ancient cliff dwellings in the cliff face. Annie holds her binoculars at eye level, 1.5 m above the ground.

a. Write an equation that relates the angle at which she holds the binoculars to the height above ground of the object she sees.

b. The top of the cliff is at an angle of 58° above horizontal when viewed from where Annie is standing. How high is the cliff, to the nearest tenth of a meter?

c. Lower on the cliff, Annie sees ruins at angles of 36° and 40° from the horizontal. How high are the ruins?

d. There is a nest of cliff swallows in the cliff face, 10 m above the canyon floor. At what angle should Annie point her binoculars to observe the nest? Round your answer to the nearest degree.



The Cliff Palace in Mesa Verde National Park, Colorado, was constructed by Ancestral Puebloan people around 1200 C.E.

17. Convert to the specified units using ratios. For example, to convert 0.17 meter to inches:

$$0.17\,\cancel{m}\cdot\frac{100\,\cancel{cm}}{1\,\cancel{m}}\cdot\frac{1\text{ in.}}{2.54\,\cancel{cm}}\approx 6.7\text{ in.}$$

a. 0.500 day to seconds

b. 3.0 mi/h to ft/s (There are 5280 feet per mile.)

18. Find the circumference and area of the circle with equation $2x^2 + 2y^2 - 2x + 7y - 38 = 0$.

19. Rewrite each expression as a single rational expression in factored form.

a. $\frac{x+1}{x-4}-\frac{x+2}{x+4}+\frac{4x}{16-x^2}$

b. $\frac{2x^2-2}{x^2+3x+2}\cdot\frac{x^2-x-6}{x^2-4x+3}$

c. $\frac{1+\frac{a}{3}}{1-\frac{a}{6}}$

20. Write an equation for a rational function, $f(x)$, that has vertical asymptotes $x = -4$ and $x = 1$, horizontal asymptote $y = 2$, and zeros $x = -2$ and $x = 5$. Check your answer by graphing the equation on your calculator.

LESSON

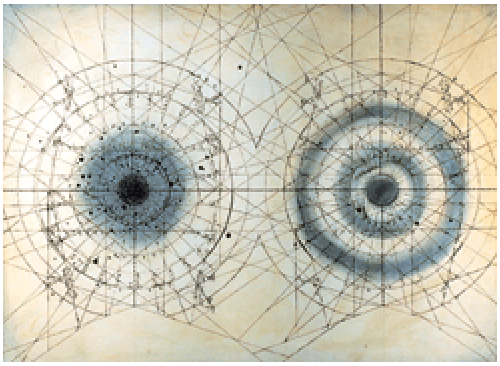
10.2

Radian Measure and Arc Length

The measure of our intellectual capacity is the capacity to feel less and less satisfied with our answers to better and better problems.

C. WEST CHURCHMAN

The choice to divide a circle into 360 degrees is rooted in the history of mathematics. The number 360 actually has no connection to any fundamental properties of a circle. In this lesson you will learn about a different angle measure.



American artist Penny Cerling (b 1946) created this piece, titled *Black Hole and Big Bang* (1999), using pen, ink, and oil on wood. Cerling explores themes of science and nature in many of her works.

History CONNECTION

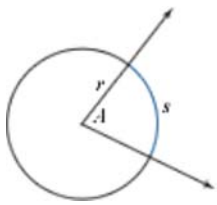
Over 5000 years ago, the Sumerians in Mesopotamia used a base-60 number system. They may have chosen this system because numbers like 30, 60, and 360 can be evenly divided by many numbers. The Babylonians and Egyptians then borrowed this system and divided the circle into 360 degrees. The Egyptians also devised the symbol for degrees and went on to divide both the Earth's equator and north-south great circles into 360 degrees, inventing latitude and longitude lines. The Greek astronomer and mathematican Hipparchus of Rhodes (ca. 190–120 B.C.E.) is credited with introducing the Babylonian division of the circle to Greece and producing a table of chords, the earliest known trigonometric table. Hipparchus is often called the "founder of trigonometry."

Investigation

A Circle of Radians

Recall from geometry that the measure of an arc is not the same as the length of an arc. For example, all 90° arcs have the same measure, but they can have different lengths.

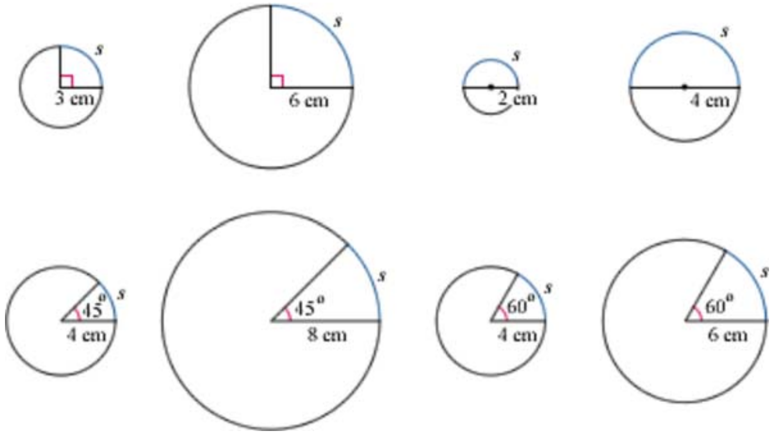
To find the length of an arc, you need to know the radius of the circle, as well as the measure of the central angle intercepted by the arc.



Step 1 Make a table to record angle degree measure A, radius r, arc length s, and radian measure θ. You will not need to fill in the last column until Step 2.

A (deg)	r (cm)	s (cm)	θ (radians)

For each figure, use the radius to calculate the circumference of the circle. Then use the angle measure and the circumference to determine the arc length. Give exact answers in terms of π. Record your results in your table.



Using degrees, a full circle, or one rotation, is 360°. A half circle, or a half rotation, is 180°. A quarter rotation is 90°, and so on.

Suppose you divide a circle a different way so that one full rotation is 2π. Then a half rotation is π. A quarter rotation is $\frac{\pi}{2}$. These measures are called **radians.** As you will see in the next part of the investigation, radians have certain advantages over degrees.

The conversion formula

$$\frac{\text{angle in degrees}}{360} = \frac{\text{angle in radians}}{2\pi}$$

is based on the full circle, or one rotation. You can also use an equivalent formula based on a half rotation.

$$\frac{\text{angle in degrees}}{180} = \frac{\text{angle in radians}}{\pi}$$

Radian measures of some common angles are irrational numbers, but you can express them as exact values in terms of π.

Step 2 Find a formula to convert degrees to radians. Convert the degree measures in your table to radians, and fill in the last column of your table. Give exact values in terms of π.

Step 3 Look for a relationship between r, s, and θ in your table. Express s in terms of r and θ. Check a few values from your table to make sure your relationship works.

Step 4 What is the advantage of radians over degrees in calculating arc length?

Radian measure is based on the properties of a circle. For this reason, it is often preferred in advanced mathematics and in physics. You can learn to recognize, compare, and use radians and degrees, just as in the past you have worked with inches and centimeters and Fahrenheit and Celsius.

Degree measures are always labeled with the symbol °. Radian measures do not need to be labeled, but they can be labeled for clarity.

EXAMPLE A

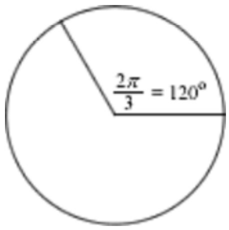
Convert degrees to radians or radians to degrees.

a. $\frac{2\pi}{3}$ **b.** n radians

c. $225°$ **d.** $n°$

Solution

You can use either conversion formula. Here we will use the equivalence $180° = \pi$ radians.



a. You can think of $\frac{2\pi}{3}$ as $\frac{2}{3}(\pi)$ or $\frac{(2\pi)}{3}$. Although no units are given, it is clear that these are radians.

$$\frac{x}{180}=\frac{\frac{2\pi}{3}}{\pi} \text{ or } \frac{x}{180}=\frac{2}{3}$$

So, $x = 120°$. Therefore $\frac{2\pi}{3} = 120°$.

b. $\frac{x}{180}=\frac{n}{\pi}$ or $x=\frac{n}{\pi}\cdot 180$, so n radians $=\left(\frac{180n}{\pi}\right)^\circ$.

c. $\frac{225}{180}=\frac{x}{\pi}$ or $x=\frac{225}{180}\cdot\pi=\frac{5\pi}{4}$, so $225^\circ=\frac{5\pi}{4}$ radians.

d. $\frac{n}{180}=\frac{x}{\pi}$ or $x=\frac{n}{180}\cdot\pi$, so $n^\circ=\frac{\pi n}{180}$ radians.

You can use the relations you found in parts b and d in Example A to convert easily between radians and degrees.

Another way to convert between degrees and radians is dimensional analysis. Dimensional analysis is a procedure based on multiplying by fractions formed of conversion factors to change units. For example, to convert 140° to radian measure, you write it as a fraction over 1 and multiply it by $\frac{\pi \text{ radians}}{180^0}$, which is a fraction equal to 1.

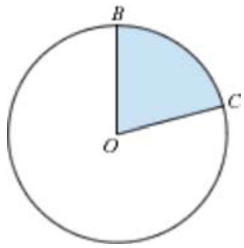
$$\frac{140\,\cancel{\text{degrees}}}{1} \cdot \frac{\pi \text{ radians}}{180\,\cancel{\text{degrees}}} = \frac{140\pi \text{ radians}}{180} = \frac{7\pi}{9} \text{ radians}$$

You can use your calculator to check or approximate conversions between radians and degrees. [► See Calculator Note 10B. ◄]

You can also write the relationship $s = r\theta$ as $\theta = \frac{s}{r}$. So, the radian measure of a central angle, θ, is the quotient , $\frac{s}{r}$ where s is the arc length and r is the radius. You can use this equation to find the length of an arc of a circle. You can also use radian measure to write a simple formula for the area of a sector.

EXAMPLE B

Circle O has diameter 10 m. The measure of central angle BOC is 1.4 radians. What is the length of its intercepted arc, $\overset{\frown}{BC}$? What is the area of the shaded sector?



► Solution

In the formula $s = r\theta$, substitute 5 for r and 1.4 for θ, and solve for s. The length of $\overset{\frown}{BC}$ is $5 \cdot 1.4$, or 7 m.

The ratio of the area of the sector to the total area of the circle is the same as the ratio of the measure of ∠BOC to 2π radians.

$$\frac{A_{\text{sector}}}{A_{\text{circle}}} = \frac{1.4}{2\pi}$$

The area of the circle is πr^2, or 25π m^2.

$$\frac{A_{\text{sector}}}{25\pi} = \frac{1.4}{2\pi}$$

Solving the equation, you find that the area of the sector is 17.5 m^2.

As you saw in the example, the area of a sector with radius r and central angle θ can be found using the proportion

$$\frac{A_{\text{sector}}}{\pi r^2} = \frac{\theta}{2\pi}$$

You can rewrite this relationship as

$$A_{\text{sector}} = \frac{\pi r^2 \theta}{2\pi} = \frac{r^2\theta}{2} = \frac{1}{2}r^2\theta$$

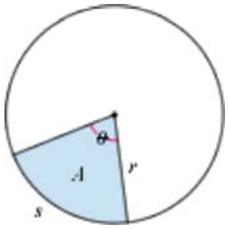
Length of an Arc and Area of a Sector

When a central angle, θ, of a circle with radius r is measured in radians, the length of the intercepted arc, s, is given by the equation

$$s = r\theta$$

and the area of the intercepted sector, A, is given by the equation

$$A = \frac{1}{2}r^2\theta$$



When an object follows a circular path, the distance it travels is the arc length. You can calculate its speed as distance traveled per unit of time. The amount of rotation, or angle traveled per unit of time, is called the **angular speed**.

EXAMPLE C

The Cosmo Clock 21 Ferris wheel at the Cosmo World amusement park in Yokohama, Japan, has a 100 m diameter. This giant Ferris wheel, with 60 gondolas and 8 people per gondola, makes one complete rotation every 15 minutes. The wheel reaches a maximum height of 112.5 m from the ground.

a. Find the speed of a person on this Ferris wheel as it is turning.

b. Find the angular speed of this person.

The Cosmo Clock 21 Ferris wheel is the world's largest Ferris wheel and also a giant clock.

▸Solution

The speed is the distance traveled per unit of time, measured in units such as meters per second. The angular speed is the rate of rotation, measured in units like radians per second or degrees per second.

a. The person travels one complete circumference every 15 min or

$$\frac{2\pi r\text{ m}}{15\text{ min}} = \frac{2\pi \cdot 50\text{ m}}{15\text{ min}} \approx 20.94\frac{\text{m}}{\text{min}}$$

So, the person travels at about 21 m/min. You can also use dimensional analysis to express this speed as approximately 1.26 km/h.

$$\frac{20.94\cancel{\text{m}}}{1\cancel{\text{min}}} \cdot \frac{60\cancel{\text{min}}}{1\text{ h}} \cdot \frac{1\text{ km}}{1000\cancel{\text{m}}} \approx \frac{1.26\text{ km}}{1\text{ h}}$$

b. The person completes one rotation, or 2π radians, every 15 min.

$$\frac{2\pi\text{ radians}}{15\text{ min}} \approx 0.42\text{ radian/min}$$

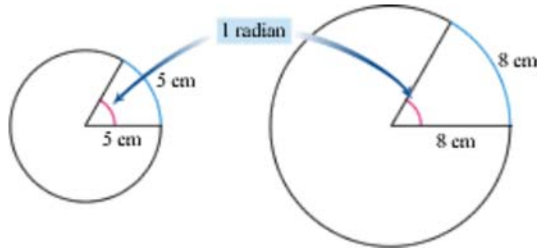
So, the person's angular speed is 0.42 radian/min.

Using dimensional analysis, you can convert between units to express answers in any form.

$$\frac{2\pi\ \cancel{\text{radians}}}{15\ \text{min}} \cdot \frac{360^\circ}{2\pi\ \cancel{\text{radians}}} = \frac{24^\circ}{1\ \text{min}}$$

So, you can also express the angular speed as 24°/min.

A central angle has a measure of 1 radian when its intercepted arc is the same length as the radius. Similarly, the number of radians in an angle measure is the number of radii in the arc length of its intercepted arc. One radian is $\frac{180^\circ}{\pi}$, or approximately 57.3°.



EXERCISES

Practice Your Skills

1. Convert between radians and degrees. Give exact answers. (Remember that degree measures are always labeled °, and radians generally are not labeled.)

a. 80° **b.** 570° **c.** $-\frac{4\pi}{3}$ **d.** $\frac{11\pi}{9}$

e. $-\frac{3\pi}{4}$ **f.** 3π **g.** $-900°$ **h.** $\frac{5\pi}{6}$

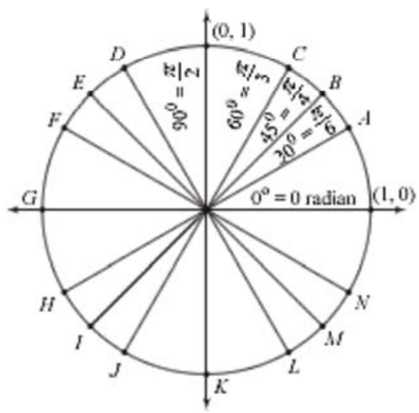
2. Find the length of the intercepted arc for each central angle.

a. $r = 3$ and $\theta = \frac{2\pi}{3}$

b. $r = 1$ and $\theta = 1$

c. $d = 5$ and $\theta = \frac{\pi}{6}$

3. Draw a large copy of this diagram on your paper. Each angle shown has a reference angle of 0°, 30°, 45°, 60°, or 90°.



a. Find the counterclockwise degree rotation of each segment from the positive x-axis. Write your answers in both degrees and radians.

b. Find the exact coordinates of points A–N.

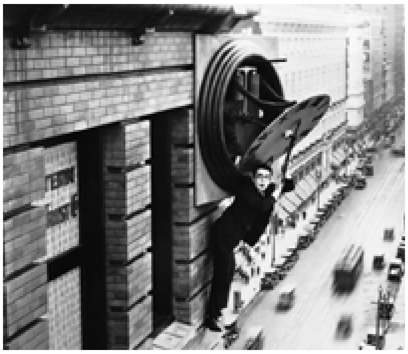
Keymath.com More Practice Your Skills

4. One radian is equivalent to how many degrees? One degree is equivalent to how many radians?

Reason and Apply

5. Are 6 radians more than, less than, or the same as one rotation about a circle? Explain.

6. The minute hand of a clock is 15.2 cm long.
   - **a.** What is the distance the tip of the minute hand travels during 40 minutes?
   - **b.** At what speed is the tip moving, in cm/min?
   - **c.** What is the angular speed of the tip in radians/minute?



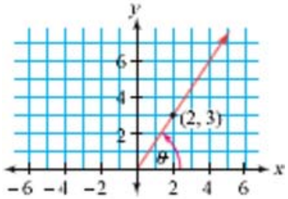
American actor Harold Lloyd (1893–1971), shown here hanging from a clock in the film *Safety Last* (1923), often performed daring physical feats in his more than 500 comedic films.

7. ***Mini-Investigation*** On your paper, graph $y = \sin x$ over the domain $0 \le x \le 2\pi$.
   - **a.** On the x-axis, label all the x-values that are multiples of $\frac{\pi}{6}$.
   - **b.** On the x-axis, label all the x-values that are multiples of $\frac{\pi}{4}$.
   - **c.** What x-values in this domain correspond to a maximum value of $\sin x$? A minimum value of $\sin x$? $\sin x = 0$?

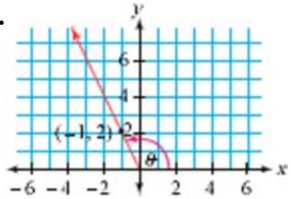
8. ***Mini-Investigation*** On your paper, graph $y = \cos x$ over the domain $0 \le x \le 2\pi$.
   - **a.** On the x-axis, label all the x-values that are multiples of $\frac{\pi}{6}$.
   - **b.** On the x-axis, label all the x-values that are multiples of $\frac{\pi}{4}$.
   - **c.** What x-values in this domain correspond to a maximum value of $\cos x$? A minimum value of $\cos x$? $\cos x = 0$?

9. ***Mini-Investigation*** Follow these steps to explore the relationship between the tangent ratio and the slope of a line. For 9a and b, find $\tan\theta$ and the slope of the line.

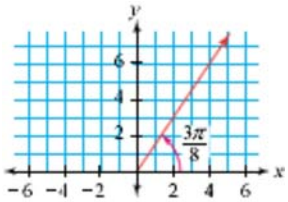
a.



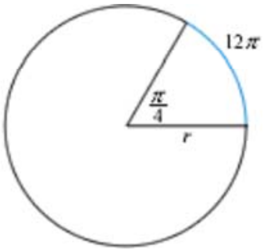
b.



   - **c.** What is the relationship between the tangent of an angle in standard position, and the slope of its terminal side?
   - **d.** Find the slope of this line.



10. Find the value of r in the circle at right.



11. Solve for θ. Express your answers in radians.

a. $\cos\theta = -\frac{1}{2}$ and $\pi \le \theta \le \frac{3\pi}{2}$

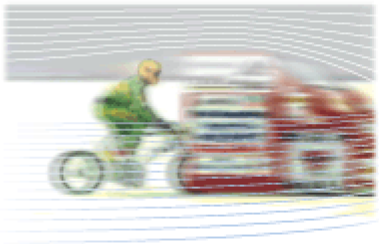
b. $\cos\theta = \frac{\sqrt{2}}{2}$ and $\frac{3\pi}{2} \le \theta \le 2\pi$

c. $\frac{\sin\theta}{\cos\theta} = \sqrt{3}$ and $\le \theta \le \frac{\pi}{2}$

12. Suppose you are biking down a hill at 24 mi/h. What is the angular speed, in radians per second, of your 27-inch-diameter bicycle wheel?

Recreation CONNECTION

Fred Rompelberg of the Netherlands broke the world record in 1995 for the fastest recorded bicycle speed, 167.043 mi/h. He rode a lead bicycle behind a race car, which propelled him forward by slipstream, an airstream that reduces air pressure. The record-breaking event took place at Bonneville Salt Flats in Utah.



Dutch cyclist Fred Rompelberg (b 1945)

13. A sector of a circle with radius 8 cm has central angle $\frac{4\pi}{7}$.

a. Find the area of the sector.

b. Set up a proportion of the area of the sector to the total area of the circle, and a proportion of the central angle of the sector to the total central angle measure.

c. Solve your proportion from 13b to show that the formula you used in 13a is correct.

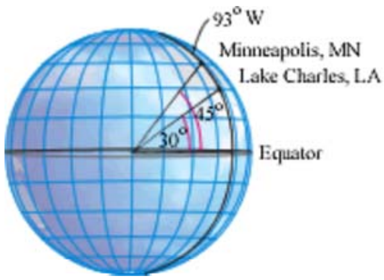
14. Sitting at your desk, you are approximately 6350 km from the center of Earth. Consider your motion relative to the center of Earth, as Earth rotates on its axis.

a. What is your angular speed?

b. What is your speed in km/h?

c. What is your speed in mi/h?

15. **APPLICATION** The two cities Minneapolis, Minnesota, and Lake Charles, Louisiana, lie on the 93° W longitudinal line. The latitude of Minneapolis is 45° N (45° north of the equator) whereas the latitude of Lake Charles is 30° N. The radius of Earth is approximately 3960 mi.



a. Calculate the distance between the two cities.

b. Like hours, degrees can be divided into minutes and seconds for more precision. (There are 60 seconds in a minute and 60 minutes in a degree.) For example, 61° 10′ means 61 degrees 10 minutes. Write this measurement as a decimal.

c. If you know the latitudes and longitudes of two cities, you can find the distance in miles between them, D, using this formula:

$$D = \frac{\pi \cdot r}{180}\cos^{-1}\left(\sin\phi_A\ \sin\phi_B + \cos\phi_A\ \cos\phi_B \cos(\theta_A - \theta_B)\right)$$

In the formula, r is the radius of Earth in miles, ϕ_A and θ_A are the latitude and longitude of city A in degrees, and ϕ_B and θ_B are the latitude and longitude of city B in degrees. North and east are considered positive angles, and south and west are considered negative. Using this formula, find the distance between Anchorage, Alaska (61° 10′ N, 150° 1′ W), and Tucson, Arizona (32° 7′ N , 110° 56′ W).

History CONNECTION

To find the shortest path between two points on a sphere, you connect the two points with a great arc, an arc of a circle that has the same center as the center of Earth. Pilots fly along great circle routes to save time and fuel. Charles Lindbergh's carefully planned 1927 flight across the Atlantic was along a great circle route that saved about 473 miles compared to flying due east.

Review

16. List the transformations of each graph from its parent function.

a. $y = 2 + (x + 4)^2$

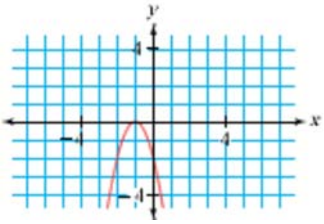
b. $\frac{y}{3} = \left(\frac{x-5}{4}\right)^2$

c. $y + 1 = |x - 3|$

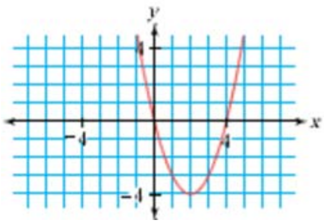
d. $y = 3 - 2|x + 1|$

17. Write an equation for each graph.

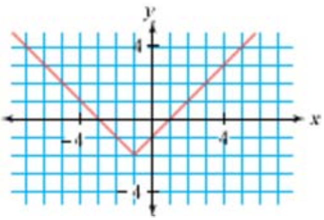
a.



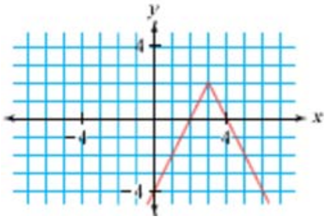
b.



c.



d.



18. Find a second value of θ that gives the same trigonometric value as the angle given. Use domain $0° \leq \theta \leq 360°$.

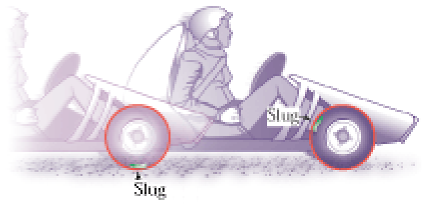
a. $\sin 23° = \sin \theta$ **b.** $\sin 216° = \sin \theta$

c. $\cos 342° = \cos \theta$ **d.** $\cos 246° = \cos \theta$

19. While Yolanda was parked at her back steps, a slug climbed onto the wheel of her go-cart just above where the wheel makes contact with the ground. When Yolanda hopped in and started to pull away, she did not notice the slug until her wheels had rotated $2\frac{1}{3}$ times. Yolanda's wheels have a 12 cm radius.

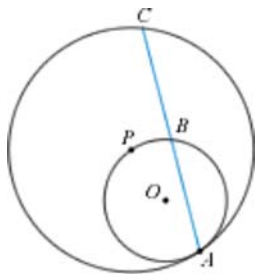
a. How far off the ground is the slug when Yolanda notices it?

b. How far horizontally has the slug moved?



20. Sketch a rectangle with length a and width $2a$. Inscribe an ellipse in the rectangle. What is the eccentricity of the ellipse? Why?

21. Circles O and P, shown at right, are tangent at A. Explain why $AB = BC$. (*Hint*: One method uses congruence.)

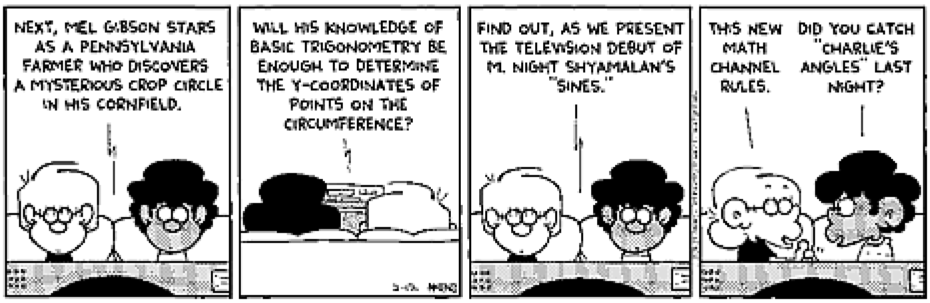


22. A river flowing at 1.50 m/s runs over an 80 m high cliff and into a lake below.

a. Write and graph parametric equations to model the water's path from the cliff to the lake.

b. Use the trace feature to approximate the time it takes for the water to reach the surface of the lake. Then find the distance from the bottom of the cliff to the bottom of the waterfall.

c. Write an equation and solve it to find the time and distance in 22b.



LESSON

10.3

Graphing Trigonometric Functions

The best way
to predict
the future is
to invent it.
BENJAMIN IMENTEL

Graphs of the functions $y = \sin x$ and $y = \cos x$, and transformations of these graphs, are collectively called **sine waves** or **sinusoids.** You will find that reflecting, translating, and stretching or shrinking sinusoidal functions and other **trigonometric functions** is very much like transforming any other function. In this lesson you will explore many real-world situations in which two variables can be modeled by a sinusoidal function in the form $y = k + b\sin\left(\frac{x-h}{a}\right)$ or $\frac{y-k}{b} = \sin\left(\frac{x-h}{a}\right)$.

The wavy terraces of the Hsinbyume Pagoda in Mingum, Myanmar, may represent the seven surrounding hills or the seven seas of the universe.

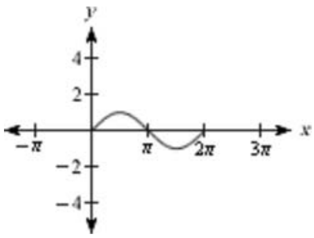
EXAMPLE A

The graph of one cycle ($0 \leq x \leq 2\pi$) of $y = \sin x$ is shown at right. Sketch the graph of one cycle of

a. $y = 2 + \sin x$

b. $y = \sin(x - \pi)$

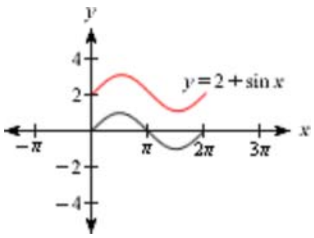
c. $y = 3 + 2\sin(x + \pi)$



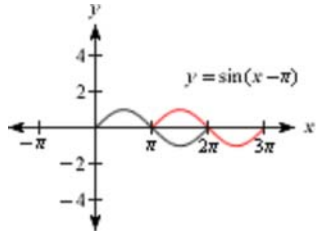
▸ Solution

Use your knowledge of transformations of functions.

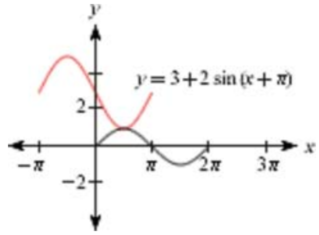
a. In relation to the graph of any parent function, $y = f(x)$, the graph of $y = 2 + f(x)$ is a translation up 2 units. The graph of one cycle of $y = 2 + \sin x$ is therefore a translation up 2 units from the graph of $y = \sin x$.



b. When x is replaced by $(x - \pi)$ in any function, the graph translates right π units.

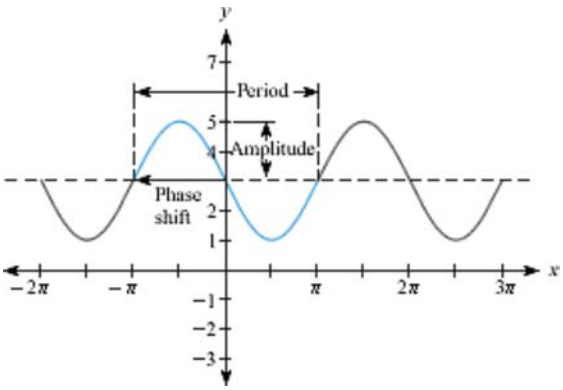


c. The coefficient 2 means the graph $y = \sin x$ must be stretched vertically by a factor of 2. The graph must also be translated left π units and up 3 units.



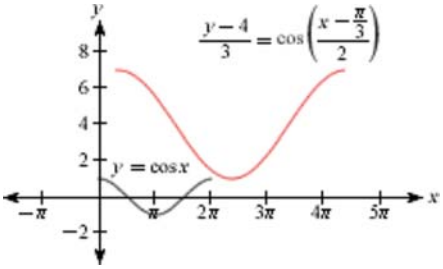
Recall that the period of a function is the smallest distance between values of the independent variable before the cycle begins to repeat. In Lesson 10.1, you discovered that the period of both $y = \sin x$ and $y = \cos x$ is 2π, or 360°. The period of each of the three functions in Example A is also 2π, because they were not stretched horizontally.

The **amplitude** of a sinusoid is half the difference of the maximum and minimum function values, or $\frac{maximum - minimum}{2}$. This is the same as the absolute value of the vertical scale factor, or $|b|$. The amplitude of $y = \sin x$ and $y = \cos x$ is 1 unit. In Example A, the amplitude of $y = 2 + \sin x$ and $y = \sin(x - \pi)$ is also 1. The amplitude of $y = 3 + 2\sin(x + \pi)$, however, is 2.



The horizontal translation of a sine or cosine graph is called the **phase shift.** In Example A, the phase shift of $y = 3 + 2\sin(x + \pi)$ is $-\pi$.

Cosine function sinusoids are transformed in the same way as sine function sinusoids. In fact, a cosine function is simply a horizontal translation of a sine function. Below is the graph of one cycle of two cosine functions:



The parent cosine function is shown in black. In relation to the parent cosine function, the graph of the second function has been stretched vertically by a factor of 3, stretched horizontally by a factor of 2, translated right $\frac{\pi}{3}$ units, and translated up 4 units.

Values of the second function vary from a minimum of 1 to a maximum of 7, so it has amplitude 3. The horizontal scale factor 2 means its period is stretched to 4π. The horizontal translation makes the phase shift $\frac{\pi}{3}$, so that the cycle of this graph starts at $\frac{\pi}{3}$ and ends at $4\pi + \frac{\pi}{3}$, or $\frac{13\pi}{3}$.

In the investigation you will transform sinusoidal functions to fit real-world, periodic data.

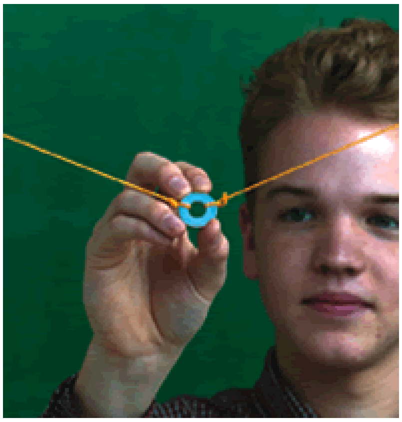
Investigation

The Pendulum II

You will need

- a washer
- string
- a motion sensor

Suspend a washer from two strings so that it hangs 10 to 15 cm from the floor between two tables or desks. Place the motion sensor on the floor about 1 m in front of the washer hanging at rest. Pull the washer back about 20 to 30 cm and let it swing. Collect data points for 2 s of time. Model your data with both a sine function and a cosine function. Give real-world meanings for all numerical values in each equation.



History CONNECTION

Italian scientist Galileo Galilei (1564-1642) studiedpendulums of the same length, but with different amplitudes of motion. He found that the back-and-forth motion occurred at a set period of time, regardless of the amplitude. Called isochronism, this principle made way for devices, such as clocks, that rely on the pendulum as a regulator.

Russian-French painter Marc Chagall's (1887–1985) style, with its multiple points of view and geometric shapes, was partially influenced by the Cubist movement, but originated independently out of his own spiritual interests and works of poetry.

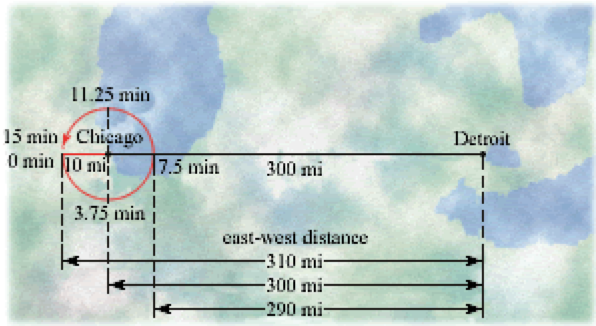
You can use sinusoidal functions to model many kinds of situations.

EXAMPLE B

After flying 300 mi west from Detroit to Chicago, a plane is put in a circular holding pattern above Chicago's O'Hare International Airport. The plane flies an additional 10 mi west past the airport and then starts flying in a circle with diameter 20 mi. The plane completes one circle every 15 min. Model the east-west component of the plane's distance from Detroit as a function of time.

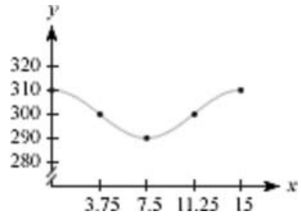
▶ Solution

To help understand the situation, first sketch a diagram. The plane flies 300 mi to Chicago and then 10 mi past Chicago. At this time, call it 0 min, the plane begins to make a circle every 15 minutes. The diagram helps you see at least five data points, recorded in the table.



Time from beginning of circle (min) x	East-west distance from Detroit (mi) y
0	310
3.75	300
7.5	290
11.25	300
15	310

A quick plot of these points suggests that a cosine function might be a good model.



The period of $y = \cos x$ is 2π (from 0 to 2π). The period of this model should be 15 s (from 0 to 15). The horizontal scale factor that stretches 2π to 15 is $a = \frac{15}{2\pi}$.

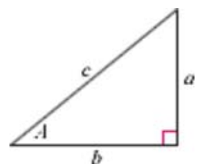
The amplitude of the model should be 10 mi, so use the vertical scale factor $b = 10$.

The plane's initial flight of 300 mi means that the function must be translated up 300 units, so use $k = 300$.
One possible model is

$$y = 300 + 10\cos\left(\frac{2\pi x}{15}\right)$$

Although the sine and cosine functions describe many periodic phenomena, such as the motion of a pendulum or the number of hours of daylight each day, there are other periodic functions that you can create from the unit circle.

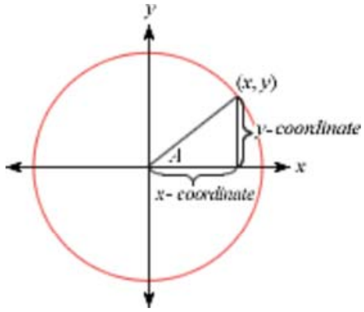
You may recall studying the tangent ratio for right triangles. In right triangle trigonometry, the tangent of angle A is the ratio of the length of the opposite leg to the length of the adjacent leg.



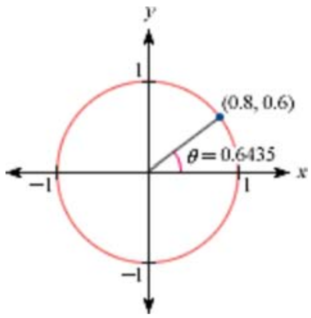
$$\tan A = \frac{a}{b}$$

The definition of tangent can be extended to apply to any angle. Here, the tangent of angle A is the ratio of the y-coordinate to the x-coordinate of a point rotated $A°$ (or radians) counterclockwise about the origin from the positive ray of the x-axis.

$$\tan A = \frac{y\text{–coordinate}}{x\text{–coordinate}}$$



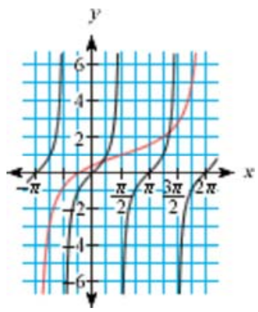
The tangent ratio describes the slope of the segment that connects the origin to any point on a circle centered at the origin. For example, the unit circle at right shows the point (0.8, 0.6) on the circle, which is rotated 0.6435 radian. The slope of the segment is $\frac{0.6}{0.8}$, or 0.75. This slope is equivalent to tan 0.6435, which is also 0.75.



You can transform the graph of the tangent function in the same way as you transform sinusoidal functions.

EXAMPLE C

The graph of $y = \tan x$ is shown in black. Find an equation for the red curve, which is a transformation of the graph of $y = \tan x$.



Solution

Notice that the parent tangent curve, $y = \tan x$, has a period of π $\left(\text{from } -\frac{\pi}{2} \text{ to } \frac{\pi}{2}\right)$.
The red curve appears to run from $-\pi$ to 2π, a distance of 3π, so the horizontal scale factor is 3.

Notice how the parent curve bends as it passes through the origin, (0, 0). The red curve appears to bend in the same way at $\left(\frac{\pi}{2}, 1\right)$. That is a translation right $\frac{\pi}{2}$ units and up 1 unit. So an equation for the red curve is

$$y = 1 + \tan\left(\frac{x - \frac{\pi}{2}}{3}\right)$$

In Example C, the point $\left(\frac{\pi}{2}, 1\right)$ on the red curve could have been considered the image of $(\pi, 0)$, rather than (0, 0). This would indicate a translation *left* $\frac{\pi}{2}$ units and up 1 unit. So the equation

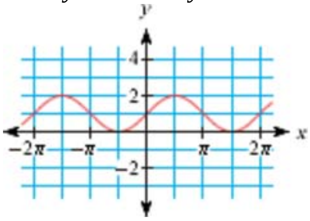
$$y = 1 + \tan\left(\frac{x + \frac{\pi}{2}}{3}\right)$$

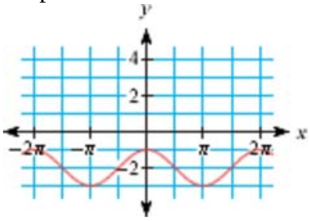
will also model the red curve. Periodic graphs can always be modeled with many different, but equivalent, equations.

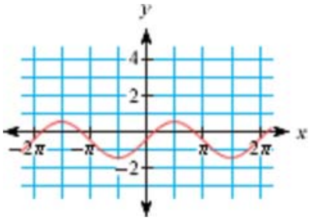
EXERCISES

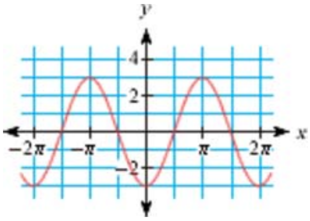
Practice Your Skills

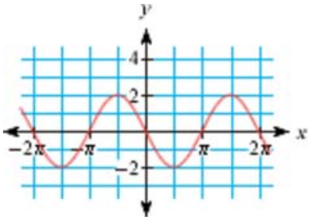
1. For 1a–f, write an equation for each sinusoid as a transformation of the graph of either $y = \sin x$ or $y = \cos x$. More than one answer is possible.

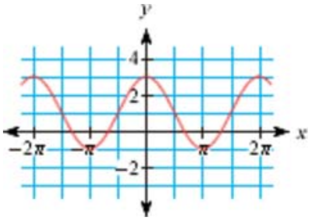
a.


b.


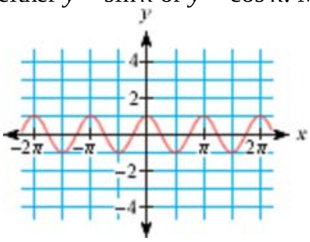
c.


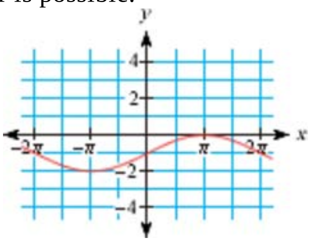
d.


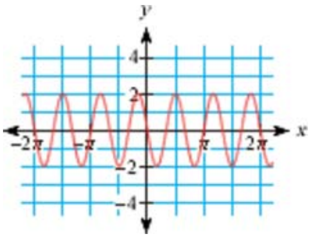
e.


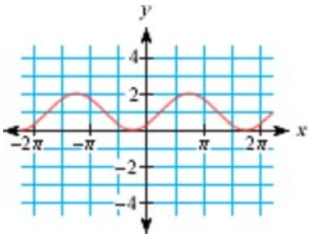
f.


2. For 2a–f, write an equation for each sinusoid as a transformation of the graph of either $y = \sin x$ or $y = \cos x$. More than one answer is possible.

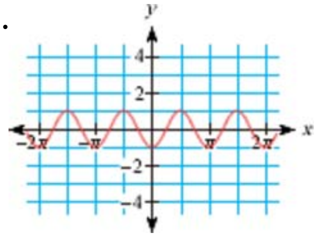
a.


b.


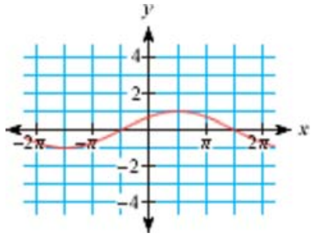
c.


d.


e.



f.



3. Consider the graph of $y = k + b \sin\left(\frac{x-h}{a}\right)$.

a. What effect does k have on the graph of $y = k + \sin x$?

b. What effect does b have on the graph of $y = b \sin x$? What is the effect if b is negative?

c. What effect does a have on the graph of $y = \sin\left(\frac{x}{a}\right)$?

d. What effect does h have on the graph of $y = \sin(x - h)$?

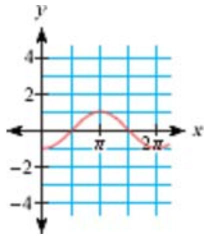
Keymath.com More Practice Your Skills

Reason and Apply

4. Sketch the graph of $y = 2\sin\left(\frac{x}{3}\right) - 4$. Use a calculator to check your sketch.

5. Describe a transformation of the graph of $y = \sin x$ to obtain an image that is equivalent to the graph of $y = \cos x$.

6. Write three different equations for the graph at right.



7. **APPLICATION** The percentage of the lighted surface of the Moon that is visible from Earth can be modeled with a sinusoid. Assume that tonight the Moon is full (100%) and in 14 days it will be a new moon (0%).

a. Define variables and find a sinusoidal function that models this situation.

b. What percentage will be visible 23 days after the full moon?

c. What is the first day after a full moon that shows less than 75% of the lit surface?

Science CONNECTION

Half of the Moon's surface is always lit by the Sun. The phases of the Moon, as visible from Earth, result from the Moon's orientation changing as it orbits Earth. A full moon occurs when the Moon is farther away from the Sun than Earth, and the lighted side faces Earth. A new moon is mostly dark because its far side is receiving the sunlight, and only a thin crescent of the lighted side is visible from Earth. For more information about the phases of the Moon, see the links at www.keymath.com/DAA.

This engraving from the *Harmonia Macrocosmica Atlas* (ca. 1661) by Dutch-German mathematician Andreas Cellarius (ca. 1596-1665) depicts how astronomers of the 17th century interpreted the phases of the Moon.

8. You may have noticed that a fluorescent light flickers, especially when it is about to blow out. Fluorescent lights do not produce constant illumination like incandescent lights. Ideally, fluorescent light cycles sinusoidally from dim to bright 60 times per second. At a certain distance from the light, the maximum brightness is measured at 50 watts per square centimeter (W/cm^2) and the minimum brightness at 20 W/cm^2.

a. To collect data on light brightness from three complete cycles, how much total time should you record data?

b. Sketch a graph of the sinusoidal model for the data collected in 8a if the light climbed to its mean value of 35 W/cm^2 at 0.003 s.

c. Write an equation for the sinusoidal model.

9. Imagine a unit circle in which a point is rotated *A* radians counterclockwise about the origin from the positive *x*-axis. Copy this table and record the *x*-coordinate and *y*-coordinate for each angle. Then use the definition of tangent to find the slope of the segment that connects the origin to each point.

Angle *A*	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$
***x*-coordinate**									
***y*-coordinate**									
Slope or tan *A*									

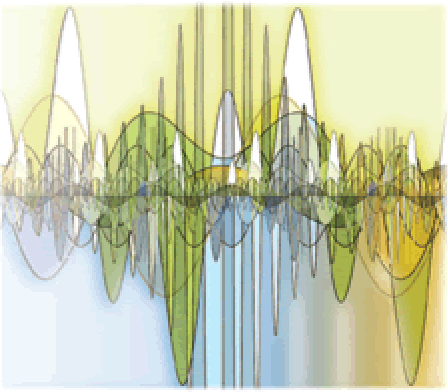
10. Graph $y = \tan x$ on your calculator. Use Radian mode with $-2\pi \leq x \leq 2\pi$ and $-5 \leq y \leq 5$.

a. What happens at $x = \frac{\pi}{2}$? Explain why this is so, and name other values when this occurs.

b. What is the period of $y = \tan x$?

c. Explain, in terms of the definition of tangent, why the values of $\tan\frac{\pi}{5}$ and $\tan\frac{6\pi}{5}$ are the same.

d. Carefully graph two cycles of $y = \tan x$ on paper. Include vertical asymptotes at *x*-values where the graph is undefined.



11. Write equations for sinusoids with these characteristics:

a. a cosine function with amplitude 1.5, period π, and phase shift $-\frac{\pi}{2}$

b. a sine function with minimum value -5, maximum value -1 and one cycle starting at $x = \frac{\pi}{4}$ and ending at $x = \frac{3\pi}{4}$

c. a cosine function with period 6π, phase shift π, vertical translation 3, and amplitude 2

12. **APPLICATION** The table gives the number of hours between sunrise and sunset for the period between December 21, 1995, and July 29, 1996, in New Orleans, Louisiana, which is located at approximately 30° N latitude.

Date	Hours	Date	Hours	Date	Hours	Date	Hours	Date	Hours
21 Dec	10.217	30 Jan	10.717	11 Mar	11.833	20 May	13.750	29 June	14.050
26 Dec	10.233	4 Feb	10.850	16 Mar	11.983	25 May	13.833	4 July	14.017
31 Dec	10.250	9 Feb	10.967	21 Mar	12.133	30 May	13.917	9 July	13.983
5 Jan	10.283	14 Feb	11.100	26 Mar	12.283	4 June	13.983	14 July	13.900
10 Jan	10.350	19 Feb	11.250	31 Mar	12.433	9 June	14.017	24 July	13.833
15 Jan	10.417	24 Feb	11.400	5 Apr	12.583	14 June	14.050	29 July	13.750
20 Jan	10.517	1 Mar	11.533	10 Apr	12.733	19 June	14.083		
25 Jan	10.617	6 Mar	11.683	15 Apr	12.833	24 June	14.083		

a. Assign December 31 as day zero, let x represent the number of days after December 31, and let y represent the number of hours between sunrise and sunset. Graph these data and find a sinusoidal model.

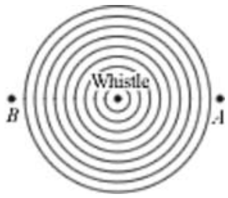
b. Which day had the least amount of daylight? How many hours of daylight were there on this day?

c. For locations that are far north, the tilt of the Earth as it orbits the Sun causes days to be very long in summer and very short in winter. (For more information, see the links at www.keymath.com/DAA.) Homer, Alaska, is located at approximately 60° N latitude. How does its daylight graph compare to your graph in 12a?

The Mississippi River Bridge in New Orleans, Louisiana, at sunset

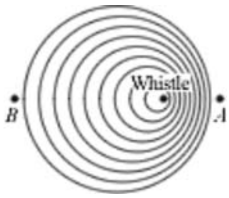
13. A train whistle produces a sound at 440 Hz.

a. When the train is not moving, the sound waves emanate evenly in all directions. This diagram shows how the sound waves travel toward persons A and B. If the sound wave completes 8 cycles over a distance of 2π meters, what is its wavelength (period)?



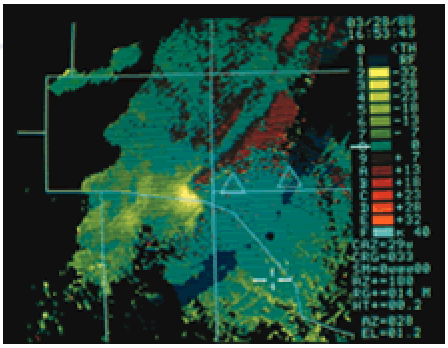
b. When the train heads east at 20 m/s, the sound waves are more compressed in the direction of motion. This diagram shows how the sound waves travel toward persons A and B in this situation. What is the approximate wavelength of the sound wave traveling toward person A if it completes 8.5 cycles in 2π meters?

c. In the situation in 13b, what is the approximate wavelength of the sound wave traveling toward person B if it completes 7.75 cycles in 2π meters?



Science CONNECTION

As a vehicle speeds toward you, the sound of its horn, whistle, or engine is high in pitch. This is because the sound waves in front of the vehicle are being compressed as it moves. This motion causes more vibrations to reach your ear per second, resulting in a higher pitched sound. As the vehicle moves away from you, there are fewer vibrations per second, and thus you hear a lower pitched sound. This change in pitch is known as the Doppler effect, discovered by the Austrian mathematician and physicist Christian Doppler (1803-1853).

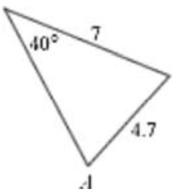


The Doppler effect applies to light and radio waves, as well as sound waves. This effect is the principle behind Doppler radar, shown here recording wind velocity. Doppler radar has been used to forecast weather patterns since the 1970s.

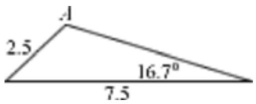
Review

14. Find the measure of the labeled angle in each triangle.

a.



b.



15. Make a table of angle measures from 0° to 360° by 15° increments. Then find the radian measure of each angle. Express the radian measure as a multiple of π.

16. The second hand of a wristwatch is 0.5 cm long.

a. What is the speed, in meters per hour, of the tip of the second hand?

b. How long would the minute hand of the same watch have to be for its tip to have the same speed as the second hand?

c. How long would the hour hand of the same watch have to be for its tip to have the same speed as the second hand?

d. What is the angular speed, in radians per hour, of the three hands in 16a–c?

e. Make an observation about the speeds you found.

17. Consider these three functions:

i. $f(x) = -\frac{3}{2}x + 6$ **ii.** $g(x) = (x + 2)^2 - 4$ **iii.** $h(x) = 1.3^{x+6} - 8$

a. Find the inverse of each function.

b. Graph each function and its inverse.

c. Which of the inverses, if any, are functions?

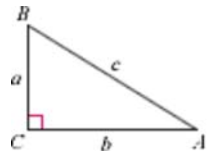
18. Use what you know about the unit circle to find possible values of θ in each equation. Use domain $0° \leq \theta \leq 360°$ or $0 \leq \theta \leq 2\pi$.

a. $\cos\theta = \sin 86°$ **b.** $\sin\theta = \cos\frac{19\pi}{12}$ **c.** $\sin\theta = \cos 123°$ **d.** $\cos\theta = \sin\frac{7\pi}{6}$

LESSON 10.4 Inverses of Trigonometric Functions

As you learned in Chapter 8, trigonometric functions of an angle in a right triangle give the ratios of sides when you know an angle. The inverses of these functions give the measure of the angle when you know the ratio of sides.

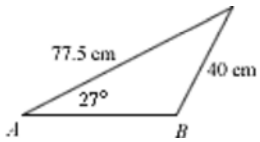
$$\sin^{-1}\left(\frac{a}{c}\right) = A$$
$$\cos^{-1}\left(\frac{b}{c}\right) = A$$
$$\tan^{-1}\left(\frac{a}{b}\right) = A$$



In Lesson 10.1, you learned that the trigonometric ratios apply to any angle measure, not just acute measures. The sine, cosine, and tangent functions have repeating values. So, if you want to find an angle whose sine is 0.5, there will be many answers. For this reason, the calculator answer for an inverse trigonometric function is not always the angle that you're looking for.

EXAMPLE A

Find the measure of $\angle B$.



▶ Solution

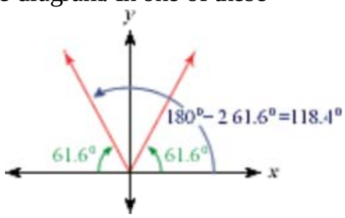
This is not a right triangle, but you can use the Law of Sines to find $m\angle B$.

$$\frac{\sin 27°}{40} = \frac{\sin B}{77.5}$$

$$\sin B = \frac{77.5 \sin 27°}{40}$$

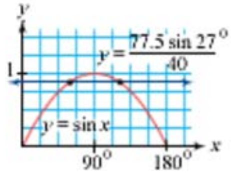
$$B = \sin^{-1}\left(\frac{77.5 \sin 27°}{40}\right) \approx 61.6°$$

Recall from geometry that two sides and a non-included angle do not determine a specific triangle. This situation is an ambiguous case. So, two different triangles can be constructed with the information given in the diagram. In one of these cases, $\angle B$ is acute; in the other, $\angle B$ is obtuse. The calculator has given you the reference angle, or the acute angle. The obtuse angle with the same sine as 61.6° measures 180° – 61.6°, or 118.4°.



$$\sin 118.4° = \sin 61.6° \approx 0.8796$$

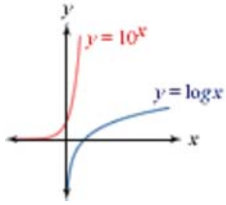
Shown at right are the graphs of $y = \sin x$ and $y = \frac{77.5 \sin 27°}{40}$. In the interval $0° \leq x \leq 180°$, the graphs intersect at 61.6° and again at 118.4°. The two solutions are based on the symmetry of the sine graph—118.4° is the same distance from the x-intercept at 180° as 61.6° is from the x-intercept at 0°.



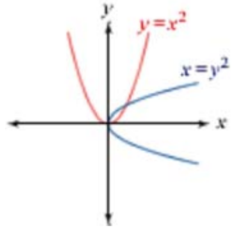
When you evaluate an inverse trigonometric function with your calculator, you only get one answer. A graph can help you make sense of a situation that has more than one solution.

You may recall that the inverse of a relation maps every point (x, y) to a point (y, x). You get the inverse of a relation by exchanging the x- and y-coordinates of all points. That's why a graph and its inverse are reflections of each other across the line $y = x$.

At right are the graphs of the exponential function $y = b^x$ and its inverse, $x = b^y$, which you've learned to call $y = \log_b x$. Notice that each is a function.



The graphs of the equation $y = x^2$ and its inverse, $x = y^2$, are also shown. Notice that in this case the inverse is not a function. When you solve the equation $4 = y^2$, how many solutions do you find?



Investigation

Exploring the Inverses

In this investigation you will explore the graphs of trigonometric functions and their inverses.

Step 1 On graph paper, carefully graph $y = \sin x$ on x- and y-axes ranging from -10 to 10. Mark the x-axis at intervals of $\frac{\pi}{2}$, and mark the y-axis at intervals of 1. Use the same scale for both axes. That is, the distance from 0 to 3.14 on your y-axis should be the same as the distance from 0 to π on your x-axis. Test a few points on your graph to make sure they fit the sine function.

Step 2 Add the line $y = x$ to your graph. Fold your paper along this line, and then trace the image of $y = \sin x$ onto your paper.

Verify that this transformation maps the point $\left(\frac{\pi}{2}, 1\right)$ onto $\left(1, \frac{\pi}{2}\right)$, $(\pi, 0)$ onto $(0, \pi)$, and $\left(\frac{3\pi}{2}, -1\right)$ onto $\left(-1, \frac{3\pi}{2}\right)$. In general, every point (x, y) should map onto (y, x).

Step 3 If your original graph is $y = \sin x$, then the equation of the inverse, when x and y are switched, is $x = \sin y$. Is the inverse of $y = \sin x$ a function? Why or why not?

Step 4 Darken the portion of the curve $x = \sin y$, between $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$. Is this portion of the graph a function? Why or why not?

Step 5 Carefully sketch graphs of $y = \cos x$ and its inverse, $x = \cos y$, on axes similar to those you used in Step 1. Then darken the portion of the curve $x = \cos y$ that is a function. What domain did you select?

When you solve an equation like $x^2 = 7$ with your calculator, the square root function gives only one value. But there are actually two solutions. This is because the inverse of the parabola $y = x^2$ is not a function.

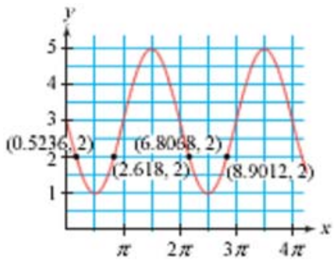
Similarly, the calculator command $\sin^{-1}$ is a function—it gives only one answer. However, the actual inverse of the sine function is not a function but a relation. So, the equation $\sin x = 0.7$ has many solutions, because x can be any angle whose sine is 0.7.

EXAMPLE B

Find the first four positive values of x for which $3 + 2\cos\left(x + \frac{\pi}{2}\right) = 2$.

▶ Solution

Graphically, this is equivalent to finding the first four intersections of the equations $y = 3 + 2\cos\left(x + \frac{\pi}{2}\right)$ and $y = 2$ for positive values of x. Examine this graph.



Solving the system of equations symbolically, you will find one answer.

$$3 + 2\cos\left(x + \frac{\pi}{2}\right) = 2$$

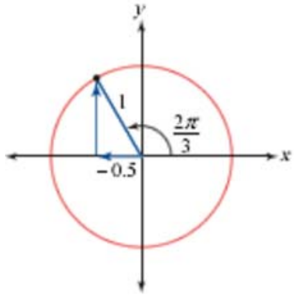
$$2\cos\left(x + \frac{\pi}{2}\right) = -1$$

$$\cos\left(x + \frac{\pi}{2}\right) = -\frac{1}{2}$$

$$x + \frac{\pi}{2} = \cos^{-1}\left(-\frac{1}{2}\right)$$

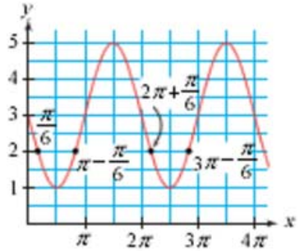
$$x = \cos^{-1}\left(-\frac{1}{2}\right) - \frac{\pi}{2}$$

The unit circle shows that the first positive solution for $\cos x = -\frac{1}{2}$ is $\frac{2\pi}{3}$. Substitute $\frac{2\pi}{3}$ for $\cos^{-1}\left(-\frac{1}{2}\right)$ and continue solving.



$$x = \cos^{-1}\left(-\frac{1}{2}\right) - \frac{\pi}{2}$$
$$x = \frac{2\pi}{3} - \frac{\pi}{2}$$
$$x = \frac{\pi}{6} \approx 0.5236$$

So, one solution is $x = \frac{\pi}{6}$, or approximately 0.5236. Graphs of trigonometric functions follow a pattern and have certain symmetries. You can look at the graph to find the other solutions.



So, the next solutions are

$$x = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \approx 2.6180$$
$$x = 2\pi + \frac{\pi}{6} = \frac{13\pi}{6} \approx 6.8068$$
$$x = 3\pi - \frac{\pi}{6} = \frac{17\pi}{6} \approx 8.9012$$

Consider the graphs of $y = \sin x$ and its inverse relation, $x = \sin y$. You can graph these on your calculator using the parametric equations $x_1 = t$ and $y_1 = \sin t$, and $x_2 = \sin t$ and $y_2 = t$. The graph of the relation $x = \sin y$ extends infinitely in the y-direction.

[–2π, 2π, π/2, –2π, 2π, π/2]

The function $y = \sin^{-1} x$ is the portion of the graph of $x = \sin y$ such that $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ (or $-90° \leq y \leq 90°$).

Similarly, the function $y = \cos^{-1} x$ is the portion of the graph of $x = \cos y$, such that $0 \leq y \leq \pi$ (or $0° \leq y \leq 180°$).

Because inverse cosine is defined as a function, the equation $x = \cos^{-1} 0.5$ has only one solution, whereas $\cos x = 0.5$ has infinitely many solutions. As you saw in Example B, some problems may have multiple solutions, but the inverse cosine and inverse sine functions on your calculator will give you only the value within the ranges given above. This value is called the **principal value.**

EXERCISES

Practice Your Skills

1. Find the principal value of each expression to the nearest tenth of a degree and then to the nearest hundredth of a radian.

a. $\sin^{-1} 0.4665$ **b.** $\sin^{-1}(-0.2471)$ **c.** $\cos^{-1}(-0.8113)$ **d.** $\cos^{-1} 0.9805$

2. Find all four values of x between -2π and 2π that satisfy each equation.

a. $\sin x = \sin\frac{\pi}{6}$ **b.** $\cos x = \cos\frac{3\pi}{8}$ **c.** $\cos x = \cos 0.47$ **d.** $\sin x = \sin 1.47$

3. Illustrate the answers to Exercise 2 by plotting the points on the graph of the sine or cosine curve.

4. Illustrate the answers to Exercise 2 by drawing segments on a graph of the unit circle.

Reason and Apply

5. Explain why your calculator can't find $\sin^{-1} 1.28$.

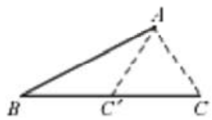
6. On the same coordinate axes, create graphs of $y = 2\cos\left(x + \frac{\pi}{4}\right)$ and its inverse.

7. Find values of x that satisfy the conditions given.
a. Find the first two positive solutions of $0.4665 = \sin x$.
b. Find the two negative solutions closest to zero of $-0.8113 = \cos x$.

8. How many solutions to the equation $-2.6 = 3\sin x$ occur in the first three positive cycles of the function $y = 3\sin x$? Explain your answer.

9. Find the measure of the largest angle of a triangle with sides 4.66 m, 5.93 m, and 8.54 m.

10. In $\triangle ABC$, $AB = 7$ cm, $CA = 3.9$ cm, and $m\angle B = 27°$. Find the two possible measurements for $\angle C$.



11. ***Mini-Investigation*** Consider the inverse of the tangent function.

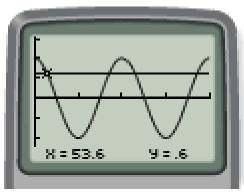
a. Find the values of $\tan^{-1} x$ for several positive and negative values of x.

b. Based on these answers, predict what the graph of $y = \tan^{-1} x$ will look like. Sketch your prediction.

c. How does the graph of $y = \tan^{-1} x$ compare to the graph of $x = \tan y$?

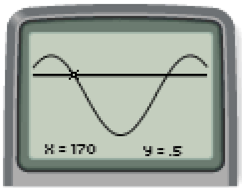
d. Use your calculator to verify your graph of the function $y = \tan^{-1} x$.

12. Shown at right are a constant function and two cycles of a cosine graph over the domain $0° \le x \le 720°$. The first intersection point is shown. What are the next three intersection values?



[0, 720, 90, −1.5, 1.5, 0.5]

13. The cosine graph at right shows two maximum values, at 50° and 770°. If the first point of intersection with the constant function occurs when $x = 170°$, what is the value of x at the second point of intersection?



14. Find the exact value of each expression.

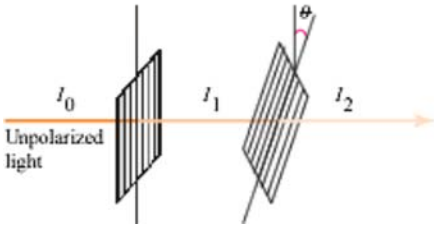
a. $\sin\left(\sin^{-1}\frac{4}{5}\right)$ **b.** $\sin\left(\sin^{-1}\left(-\frac{2}{3}\right)\right)$

c. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ **d.** $\cos^{-1}(\cos(-45°))$

15. **APPLICATION** When a beam of light passes through a polarized lens, its intensity is cut in half, or $I_1 = 0.5I_0$. To further reduce the intensity, you can place another polarized lens in front of it. The intensity of the beam after passing through the second lens depends on the angle of the second filter to the first. For instance, if the second lens is polarized in the same direction, it will have little or no additional effect on the beam of light. If the second lens is rotated so that its axis is $\theta°$ from the first lens's axis, then the intensity in watts per square meter (W/m^2) of the transmitted beam, I_2, is

$$I_2 = I_1 \cos^2\theta$$

where I_1 is the intensity in watts per square meter of the incoming beam.



a. A beam of light passes successively through two polarized sheets. The angle between the polarization axes of the filters is 30°. If the intensity of the incoming beam is 16.0×10^{-4} W/m^2, what is the intensity of the beam after passing through the first filter? The second filter?

b. A polarized beam of light has intensity 3 W/m^2. The beam then passes through a second polarized lens and intensity drops to 1.5 W/m^2. What is the angle between the polarization axes?

c. At what angle should the axes of two polarized lenses be placed to cut the intensity of a transmitted beam to 0 W/m^2?

Technology CONNECTION

Skiers, boaters, and photographers know that ordinary sunglasses reduce brightness but do not remove glare. Polarized lenses will eliminate the glare from reflective surfaces such as snow, water, sand, and roads. This is because glare is the effect of reflected light being polarized parallel to the reflective surface. Polarized lenses will filter, or block out, this polarized light. You can think of a polarized lens as a set of tiny slits. The slits block light at certain angles, while allowing light to pass through select angles.

The photo on the left was taken without a polarized lens and the one on the right was taken with a polarized lens. Notice how the polarized lens reduces glare.

Review

16. Convert

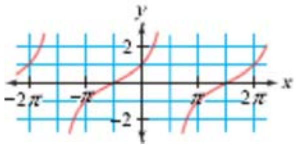
a. $\frac{7\pi}{10}$ radians to degrees

b. $-205°$ to radians

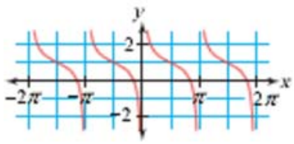
c. 5π radians per hour to degrees per minute

17. Write an equation for each graph.

a.



b.



18. Find all roots, real or nonreal. Give exact answers.

a. $2x^2 - 6x + 3 = 0$

b. $13x - 2x^2 = 6$

c. $3x^2 + 4x + 4 = 0$

19. Consider the equation $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$.

a. Describe the shape defined by the equation and write the equation in parametric form.

b. Write the nonparametric and parametric equations of the given equation after a translation left 1 unit and up 2 units.

c. Graph both sets of parametric equations on your calculator, and trace to approximate the coordinates of the intersection point(s).

d. Solve a system of equations to find the points of intersection.

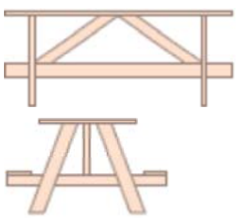
Project

DESIGN A PICNIC TABLE

A well-made piece of wood furniture is carefully designed so that the pieces fit together perfectly. If the design is not perfect, assembly will be more difficult, and the furniture may not be stable.

Your task is to design a picnic table and draw the plans for it. Decide on the shape and size of your table. Make scale drawings showing your design. Then make a separate drawing of each piece, labeling all the lengths and all the angle measures.

Your project should include



- Scale drawings of your table showing the front view, side view, and top view.
- A drawing of each piece.
- The calculations you made to determine the lengths and angles, clearly labeled and organized.
- A description of your design process, including any problems you encountered and how you solved them.

LESSON 10.5

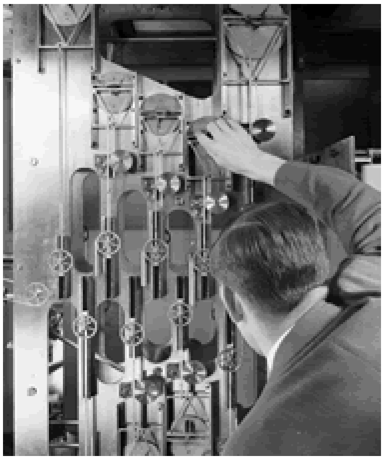
Modeling with Trigonometric Equations

Language exerts hidden power, like the moon on the tides.

RITA MAE BROWN

Tides are caused by gravitational forces, or attractions, between the Earth, Moon, and Sun as the Moon circles Earth. You can model the height of the ocean level in a seaport with a combination of sinusoidal functions of different phase shifts, periods, and amplitudes.

The wooden pillars below, shown at low tide, protect land in Winchelsea, East Sussex, England, from high waters.



The tide-predicting machine shown above records the times and heights of high and low tides. Designed in 1910, it combines sine functions to produce a graph that predicts tide levels over a period of time.

Social Science CONNECTION

The Moon's gravitational pull causes ocean tides, and tidal bores in rivers. Tidal bores occur during new and full moons when ocean waves rush upstream into a river passage, sometimes at speeds of more than 40 mi/h. The highest recorded river bore was 15 ft high, in China's Fu-ch'un River. To learn more about tides, see the links at www.keymath.com/DAA.

EXAMPLE A

The height of water at the mouth of a certain river varies during the tide cycle. The time in hours since midnight, t, and the height in feet, h, are related by the equation

$$h = 15 + 7.5\cos\left(\frac{2\pi(t-3)}{12}\right)$$

a. What is the length of a period modeled by this equation?

b. When is the first time the height of the water is 11.5 ft?

c. When will the water be at that height again?

Solution

One cycle of the parent cosine function has length 2π.

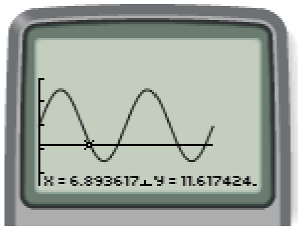
a. The cosine function in the height equation has been stretched horizontally by a scale factor of $\frac{12}{2\pi}$. So, the length of the cycle is $\frac{12}{2\pi} \cdot 2\pi = 12$ h.

b. To find when the height of the water is 11.5 ft, substitute 11.5 for *h*.

$11.5 = 15 + 7.5\cos\left(\frac{2\pi(t-3)}{12}\right)$	Substitute 11.5 for *h*.
$\frac{-3.5}{7.5} = \cos\left(\frac{\pi(t-3)}{6}\right)$	Subtract 15, and divide by 7.5.
$\cos^{-1}\left(\frac{-3.5}{7.5}\right) = \frac{\pi(t-3)}{6}$	Take the inverse cosine of both sides.
$\frac{6}{\pi} \cdot \cos^{-1}\left(\frac{-3.5}{7.5}\right) + 3 = t$	Multiply by $\frac{6}{\pi}$, and add 3.
$t \approx 6.9$ h	

So, the water height will be 11.5 ft after approximately 6.9 h, or at 6:54 A.M.

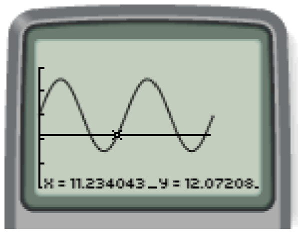
You can also find this answer by graphing the height function and the line $h = 11.5$ and by approximating the point of intersection. The graph verifies your solution. (*Note:* On your calculator, use *x* in place of *t* and *y* in place of *h*.)



[0, 24, 5, 0, 25, 5]

c. The graph in part b shows several different times when the water depth is 11.5 ft. The third occurrence is 12 hours after the first, because each cycle is 12 hours in length. So the third occurrence is at 6:54 P.M. But how can you find the second occurrence? First, consider that the graph is shifted left 3 units, so the cycle begins at $t = 3$. It has a period of 12, so it ends at $t = 15$.

The first height of 11.5 ft occurred when $t = 6.9$, or 3.9 h after the cycle's start. The next solution will be 3.9 h before the end of the cycle, that is, at $15 - 3.9 = 11.1$ h, or 11:01 A.M. This value will also repeat every 12 h. A graph confirms this approximation.



[0, 24, 5, 0, 25, 5]

You can use both sine and cosine functions to model sinusoidal patterns because they are just horizontal translations of each other. Often it is easier to use a cosine function, because you can identify the maximum value as the start value of a cycle.

In Example A, the height of the river has a cycle length, or period, of 12 hours. Sometimes it is useful to discuss the **frequency** of a function as well. Whereas a period tells you how long it takes to complete one cycle, the frequency tells you how many cycles are completed in one unit of time. The frequency is the reciprocal of the period. It is more useful to talk about frequency in motion with a short cycle. For example, if a wave has a period of 0.01 second, then it has a frequency of 100 cycles per second. In Example A, the period is 12 h, so the frequency is $\frac{1}{12}$ cycle per hour.

$$\textit{frequency} = \frac{1}{\textit{period}}$$

Investigation

A Bouncing Spring

You will need

- a motion sensor
- a spring
- a mass of 50 to 100 g
- a support stand

In this experiment you will suspend a mass from a spring. When you pull down on the mass slightly, and release, the mass will move up and down. In reality, the amount of motion gradually decreases, and eventually the mass returns to rest. However, if the initial motion is small, then the decrease in the motion occurs more slowly and can be ignored during the first few seconds.

Procedure Note

1. Attach a mass to the bottom of a spring. Position the motion sensor directly below the spring, so it is always at least 0.5 m from the mass.
2. Set the motion sensor to collect about 5 s of data. Pull the mass down slightly, and release at the same moment as you begin gathering data.

Step 1 Follow the procedure note to collect data on the height of the bouncing spring for a few seconds.

Step 2 Delete values from your lists to limit your data to about four cycles. Identify the phase shift, amplitude, period, frequency, and vertical shift of your function.

Step 3 Write a sine or cosine function that models the data.

Step 4 Answer these questions, based on your equation and your observations.

a. How does each of the numbers in your equation from Step 3 correspond to the motion of the spring?

b. How would your equation change if you moved the motion sensor 1 m farther away?

c. How would your equation change if you pulled the spring slightly lower when you started?

This car is performing in a car dance competition in Los Angeles, California. Springs are often used as shock absorbers.

In the investigation, you modeled cyclical motion with a sine or cosine function. As you read the next example, think about how the process of finding an equation based on given measurements compares to your process in the investigation.

EXAMPLE B

On page 577 you learned that the Cosmo Clock 21 Ferris wheel has a 100 m diameter, takes 15 min to rotate, and reaches a maximum height of 112.5 m.

Find an equation that models the height in meters, h, of a seat on the perimeter of the wheel as a function of time in minutes, t. Then determine when, in the first 30 min, a given seat is 47 m from the ground, if the seat is at its maximum height 10 min after the wheel begins rotating.

▶ Solution

The parent sinusoid curve has an amplitude of 1. The diameter of the Ferris wheel is 100 m, so the vertical stretch, or amplitude, is 50. The period of a parent sinusoid is 2π and the period of the Ferris wheel is 15 min, so the horizontal stretch is $\frac{15}{2\pi}$. The average value of the parent sinusoid is 0, but the average height of the wheel is 62.5 m, so the vertical translation is 62.5. The top of a sinusoid corresponds to the maximum height of a seat. The cosine curve starts at a maximum point, so it will be easiest to use a cosine function. Because the first maximum of the Ferris wheel occurs after 10 min, the phase shift of its equation is 10 to the right. Incorporate these values into a cosine function.

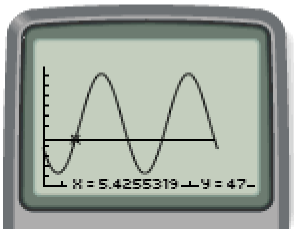
$$h = 50\cos\left(\frac{2\pi(t-10)}{15}\right) + 62.5$$

Now, to find when the height is 47 m, substitute 47 for h.

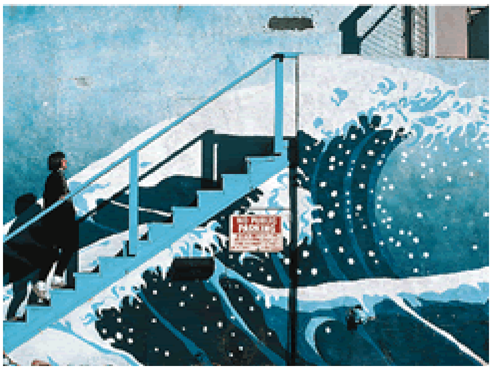
$50\cos\left(\frac{2\pi(t-10)}{15}\right) + 62.5 = 47$	Substitute 47 for h.
$\cos\left(\frac{2\pi(t-10)}{15}\right) = \frac{47-62.5}{50} = -0.31$	Subtract 62.5 and divide by 50. Evaluate.
$\frac{2\pi(t-10)}{15} = \cos^{-1}(-0.31)$	Take the inverse cosine of both sides.
$t - 10 = \frac{15}{2\pi}\cdot\cos^{-1}(-0.31)$	Multiply by $\frac{15}{2\pi}$ on both sides.
$t = \frac{15}{2\pi}\cdot\cos^{-1}(-0.31) + 10$	Add 10.
$t \approx 14.5$	Approximate the principal value of t.

So, the given seat is at a height of 47 m approximately 14.5 min after it starts rotating. But this is not the only time. The period is 15 min, so the seat will also reach 47 m after 29.5 min. Also, the height is 47 m once on the way up and once on the way down. The seat is at 47 m 4.5 min after its maximum point, which is at 10 min. It will also be at the same height 4.5 min before its maximum point, at 5.5 min, and 15 min after that, at 20.5 min.

So, in the first 30 min, the seat reaches a height of 47 m after 5.5, 14.5, 20.5, and 29.5 min. A graph confirms this.



[0, 30, 5, 0, 120, 10]



Using sinusoidal models, you can easily find y when given an x-value. As you saw in the examples, the more difficult task is finding x when given a y-value. There will be multiple answers, because the graphs are periodic. You should always check the values and number of solutions with a calculator graph.

This mural is painted in the style of *The Great Wave,* by Japanese artist Hokusai (1760–1849). Like tides, the motion of waves can be modeled with trigonometric functions.

EXERCISES

You will need

Geometry software for Exercise 13

Practice Your Skills

1. Find the first four positive solutions. Give exact values in radians.

a. $\cos x = 0.5$ **b.** $\sin x = -0.5$

2. Find all solutions for $0 \leq x \leq 2\pi$, rounded to the nearest thousandth.

a. $2\sin(x + 1.2) - 4.22 = -4$ **b.** $7.4\cos(x - 0.8) + 12.3 = 16.4$

3. Consider the graph of the function $h = 5 + 7\sin\left(\frac{2\pi(t-9)}{11}\right)$.

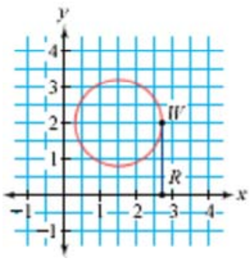
a. What are the vertical translation and average value?
b. What are the vertical stretch factor, minimum and maximum values, and amplitude?
c. What are the horizontal stretch factor and period?
d. What are the horizontal translation and phase shift?

4. Consider the graph of the function $h = 18 - 17\cos\left(\frac{2\pi(t+16)}{15}\right)$.

a. What are the vertical translation and average value?
b. What are the vertical stretch factor, minimum and maximum values, and amplitude?
c. What are the horizontal stretch factor and period?
d. What are the horizontal translation and phase shift?

Keymath.com More Practice Your Skills

Reason and Apply



5. A walker moves counterclockwise around a circle with center (1.5, 2) and radius 1.2 m and completes a cycle in 8 s. A recorder walks back and forth along the x-axis, staying even with the walker, with a motion sensor pointed toward the walker. What equation models the (*time*, *distance*) relationship? Assume the walker starts at point (2.7, 2).

6. A mass attached to a string is pulled down 3 cm from its resting position and then released. It makes ten complete bounces in 8 s. At what times during the first 2 s was the mass 1.5 cm above its resting position?

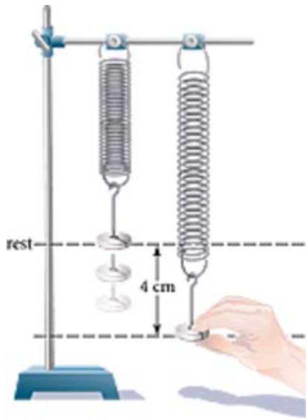
7. Household appliances are typically powered by electricity through wall outlets. The voltage provided varies sinusoidally between $-110\sqrt{2}$ volts and $110\sqrt{2}$ volts, with a frequency of 60 cycles per second.

   a. Use a sine or cosine function to write an equation for (*time*, *voltage*).
   b. Sketch and label a graph picturing three complete cycles.

8. The time between high and low tide in a river harbor is approximately 7 h. The high-tide depth of 16 ft occurs at noon and the average harbor depth is 11 ft.
   a. Write an equation modeling this relationship.
   b. If a boat requires a harbor depth of at least 9 ft, find the next two time periods when the boat will not be able to enter the harbor.

9. Two masses are suspended from springs, as shown. The first mass is pulled down 3 cm from its resting position and released. A second mass is pulled down 4 cm from its resting position. It is released just as the first mass passes its resting position on its way up. When released, each mass makes 12 bounces in 8 s.

   a. Write a function for the height of each mass. Use the moment the second mass is released as $t = 0$.
   b. At what times during the first 2 s will the two masses be at the same height? Solve graphically, and state your answers to the nearest 0.1 s.

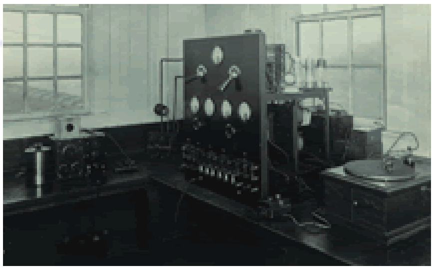


10. **APPLICATION** An AM radio transmitter generates a radio wave given by a function in the form $f(t) = A \sin 2000\pi nt$. The variable n represents the location on the broadcast dial, $550 \leq n \leq 1600$, and t is the time in seconds.

   a. For radio station WINS, located at 1010 on the AM radio dial, what is the period of the function that models its radio waves?
   b. What is the frequency of your function from 10a?
   c. Find a function that models the radio waves of an AM radio station near you. Find the period and frequency.

Technology CONNECTION

Amplitude modulation, or AM, is one way that a radio transmitter can send information over large distances. By varying the amplitude of a continuous wave, the transmitter adds audio information to the signal. The wave's amplitude is adjusted simply by changing the amount of energy put in. More energy causes a larger amplitude, whereas less energy causes a smaller amplitude. One problem with AM (as opposed to FM, or frequency modulation) is that the signal is affected by electrical fields, such as lighting. The resulting static and clicks decrease sound quality.



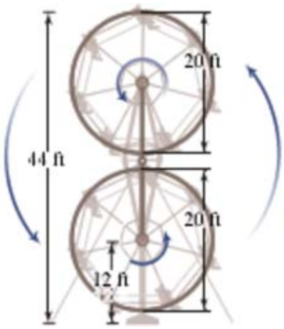
A radio transmitter from the early 1900s used electric pulses or vacuum tubes to create radio waves.

11. The number of hours of daylight, y, on any day of the year in Philadelphia can be modeled using the equation $y = 12 + 2.4 \sin\left(\frac{2\pi(x-80)}{365}\right)$, where x represents the day number (with January 1 as day 1).

a. Find the amount of sunlight in Philadelphia on day 354, the shortest day of the year (the winter solstice).

b. Find the dates on which Philadelphia has exactly 12 hours of daylight.

12. A popular amusement park ride is the double Ferris wheel. Each small wheel takes 20 s to make a single rotation. The two-wheel set takes 30 s to rotate once. The dimensions of the ride are given in the diagram.



The Sky Wheel, a double Ferris wheel, operated at the Cedar Point theme park in Sandusky, Ohio, from 1962 to 1981.

a. Sandra gets on at the foot of the bottom wheel. Write an equation that will model her height above the center of this wheel as the wheels rotate.

b. The entire ride (the two-wheel set) starts revolving at the same time that the two smaller wheels begin to rotate. Write an equation that models the height of the center of Sandra's wheel as the entire ride rotates.

c. Because the two motions occur simultaneously, you can add the two equations to write a final equation for Sandra's position. Write this equation.

d. During a 5 min ride, during how many distinct time periods is Sandra within 6 ft of the ground?

13. ***Technology*** Use geometry software to simulate Sandra's ride on the double ferris wheel in Exercise 12. Describe your steps.

Review

14. Solve $\tan\theta = -1.111$ graphically. Use domain $-180° \le \theta \le 360°$. Round answers to the nearest degree.

15. Which has the larger area, an equilateral triangle with side 5 cm or a sector of a circle with radius 5 cm and arc length 5 cm? Give the area of each shape to the nearest 0.1 cm.

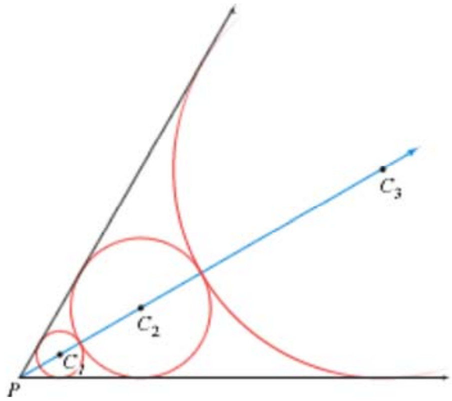
16. Find the equation of the circle with center $(-2, 4)$ that has tangent line $2x - 3y - 6 = 0$.

17. Consider the equation $P(x) = 2x^3 - x^2 - 10x + 5$.

a. List all possible rational roots of $P(x)$.
b. Find any actual rational roots.
c. Find exact values for all other roots.
d. Write $P(x)$ in factored form.

18. Circles $C_1, C_2, \ldots$ are tangent to the sides of $\angle P$ and to the adjacent circle(s). The radius of circle C_1 is 6. The measure of $\angle P$ is 60°.

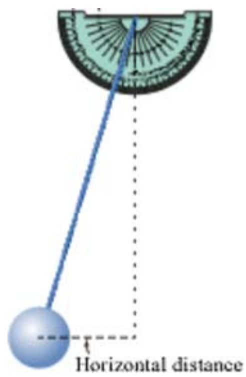
a. What are the radii of C_2 and C_3? (*Hint:* One method uses similar right triangles.)
b. What is the radius of C_n?



Project

A DAMPENED SINE CURVE

You have modeled the motion of springs and pendulums using sinusoidal curves. In each case, you assumed that the amplitude of the motion did not decrease over time. In reality, forces of friction dampen, or reduce, the amplitude of the motion over time, until the object comes to rest. In this project you will investigate this phenomenon.



Set up a pendulum with a protractor at the top. Hold the string at a 20° angle from the center, and release. Time the first few swings and determine the period of your pendulum. Begin the experiment again and record the angle of the swing after each swing, or at regular intervals of swings, until the pendulum is nearly at rest. Use these angles and the length of the pendulum to calculate the horizontal distance of the pendulum from the center at the end of each swing.

Because the period of the pendulum is constant throughout the experiment, you can assign time values for each swing. Plot the data (*time*, *distance*), and draw a smooth sinusoidal curve that peaks at each of these points. Draw a second curve that just passes through these maximum points. Find an equation that models this second curve. Then use this equation as the amplitude for the sine curve that models the entire motion. Write an equation for your dampened sine function.

Your project should include

- A description of your experimental process.
- Tables showing all of the data you collected.
- The equation you found for the amplitude.
- An equation for your final dampened sine curve.

LESSON 10.6

Fundamental Trigonometric Identities

In this lesson you will discover several equations that express relationships among trigonometric functions. When an equation is true for all values of the variables for which the expressions are defined, the equation is called an **identity.** You've already learned that $\tan A = \frac{y\text{-coordinate}}{x\text{-coordinate}}$ when a point $P(x, y)$ is rotated $A°$ counterclockwise about the origin from the positive x-axis. You've also seen that the y-coordinate is equivalent to $r \sin A$, where r is the distance between the point and the origin, and the x-coordinate is equivalent to $r \cos A$. You can use these relationships to develop an identity that relates $\tan A$, $\sin A$, and $\cos A$.

$\tan A = \frac{y}{x}$	Definition of tangent.
$\tan A = \frac{r \sin A}{r \cos A}$	Substitute $r \sin A$ for y and $r \cos A$ for x.
$\tan A = \frac{\sin A}{\cos A}$	Reduce.

The reciprocals of the tangent, sine, and cosine functions are also trigonometric functions. The reciprocal of tangent is called **cotangent,** abbreviated cot. The reciprocal of sine is called **cosecant,** abbreviated csc. The reciprocal of cosine is called **secant,** abbreviated sec. These definitions lead to six more identities.

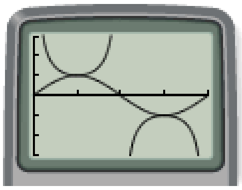
History CONNECTION

The reciprocal trigonometric functions were introduced by Muslim astronomers in the 9th and 10th centuries C.E. Before there were calculating machines, these astronomers developed remarkably precise trigonometric tables based on earlier Greek and Indian findings. They used these tables to record planetary motion, to keep time, and to locate their religious center of Mecca. Western Europeans began studying trigonometry when Arabic astronomy handbooks were translated in the 12th century.

Reciprocal Identities

$\csc A = \frac{1}{\sin A}$	or	$\sin A = \frac{1}{\csc A}$
$\sec A = \frac{1}{\cos A}$	or	$\cos A = \frac{1}{\sec A}$
$\cot A = \frac{1}{\cot A}$	or	$\tan A = \frac{1}{\cot A}$

Your calculator does not have special keys for secant, cosecant, and cotangent. Instead, you must use the reciprocal identities in order to enter them into your calculator. For example, to graph $y = \csc x$, you use $y = \frac{1}{\sin x}$. The calculator screen at right shows the graphs of the principal cycles of the parent sine and cosecant functions.[See Calculator Note 10C for more information about using secant, cosecant, and cotangent on your calculator.]



[0, 2π, π/2, – 3, 3, 1]

Once you know a few identities, you can use them to prove other identities. One strategy for proving that an equation is an identity is to verify that both sides of the equation are always equivalent. You can do this by writing equivalent expressions for one side of the equation until it is the same as the other side.

EXAMPLE Prove algebraically that $\cot A = \frac{\cos A}{\sin A}$ is an identity. (Assume $\sin A \neq 0$.)

Solution Use definitions and identities that you know in order to show that both sides of the equation are equivalent. Be sure to work on only one side of the equation.

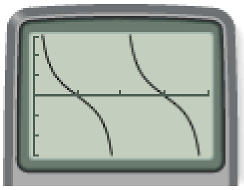
$\cos A \stackrel{?}{=} \dfrac{\cos A}{\sin A}$ — Original identity.

$\dfrac{1}{\tan A} \stackrel{?}{=} \dfrac{\cos A}{\sin A}$ — Use the reciprocal identity to replace $\cot A$ with $\frac{1}{\tan A}$.

$\dfrac{1}{\frac{\sin A}{\cos A}} \stackrel{?}{=} \dfrac{\cos A}{\sin A}$ — Replace $\tan A$ with $\frac{\sin A}{\cos A}$.

$\dfrac{\cos A}{\sin A} = \dfrac{\cos A}{\sin A}$ — $1 \div \frac{\sin A}{\cos A}$ is equivalent to $1 \cdot \frac{\cos A}{\sin A}$, or simply $\frac{\cos A}{\sin A}$.

Therefore, $\cot A = \frac{\cos A}{\sin A}$ is an identity. To verify this, you can graph the two equations $y = \cot x$ and $y = \frac{\cos x}{\sin x}$, and check that they give the same graph. Now that you have proved this identity, you can use it to prove other identities.

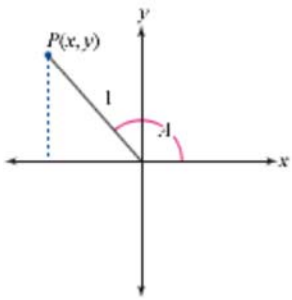


[0, 2π, π/2, – 3, 3, 1]

In the investigation you will discover a set of trigonometric identities that are collectively called the Pythagorean identities. To prove a new identity, you can use any previously proved identity.

Investigation
Pythagorean Identities

Step 1 Use your calculator to graph the equation $y = \sin^2 x + \cos^2 x$. (You'll probably have to enter this as $y = (\sin x)^2 + (\cos x)^2$.) Does this graph look familiar? Use your graph to write an identity.



Step 2 Use the definitions for sin *A*, cos *A*, and the diagram at right to prove your identity.

Step 3 Explain why you think this identity is called a Pythagorean identity.

Step 4 Solve the identity from Step 1 for $\cos^2 x$ to get another identity. Then solve for $\sin^2 x$ to get another variation.

$\cos^2 x = \underline{\ ?\ }$

$\sin^2 x = \underline{\ ?\ }$

Step 5 Divide both sides of the identity from Step 1 by $\cos^2 x$ to develop a new identity. Simplify so that there are no trigonometric functions in the denominator.

Step 6 Verify your identity from Step 5 with a graph. Name any domain values for which the identity is undefined.

Step 7 Divide both sides of the identity from Step 1 by $\sin^2 x$ to develop a new identity. Simplify so that there are no trigonometric functions in the denominator.

Step 8 Verify the identity from Step 7 with a graph. Name any domain values for which the identity is undefined.

You have verified identities by setting each side of the equation equal to *y* and graphing. If the graphs and table values match, you may have an identity. You should always use an algebraic proof to be certain. You may have used your calculator in this way to verify the Pythagorean identities in the investigation. If not, try it now on one of the three identities below.

Pythagorean Identities

$\sin^2 A + \cos^2 A = 1$

$1 + \tan^2 A = \sec^2 A$

$1 + \cot^2 A = \csc^2 A$

EXERCISES

Practice Your Skills

1. Because the cotangent function is not built into your calculator, explain how you would graph $y = \cot x$ on your calculator.
2. Use graphs to determine which of these equations may be identities.
   a. $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ **b.** $\cos x = \sin\left(x - \frac{\pi}{2}\right)$
   c. $(\csc x - \cot x)(\sec x + 1) = 1$ **d.** $\tan x(\cot x + \tan x) = \sec^2 x$
3. Prove algebraically that the equation in Exercise 2d is an identity.
4. Evaluate.
   a. $\sec\frac{\pi}{6}$ **b.** $\csc\frac{5\pi}{6}$ **c.** $\csc\frac{2\pi}{3}$
   d. $\sec\frac{3\pi}{2}$ **e.** $\cot\frac{5\pi}{3}$ **f.** $\csc\frac{4\pi}{3}$

Reason and Apply

5. In your own words, explain the difference between a trigonometric equation and a trigonometric identity.

6. A function f is **even** if $f(-x) = f(x)$ for all x-values in its domain. It is **odd** if $f(-x) = -f(x)$ for all x-values in its domain. Determine whether each function is even, odd, or neither.

a. $f(x) = \sin x$ **b.** $f(x) = \cos x$ **c.** $f(x) = \tan x$

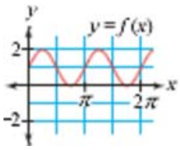
d. $f(x) = \cot x$ **e.** $f(x) = \sec x$ **f.** $f(x) = \csc x$

7. In the next lesson, you'll see that $\cos 2A = \cos^2 A - \sin^2 A$ is an identity. Use this identity and the identities from this lesson to prove that

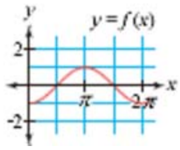
a. $\cos 2A = 1 - 2\sin^2 A$ **b.** $\cos 2A = 2\cos^2 A - 1$

8. Sketch the graph of $y = \frac{1}{f(x)}$ for each function.

a.



b.



9. Find another function that has the same graph as each function named below. More than one answer is possible.

a. $y = \cos\left(\frac{\pi}{2} - x\right)$ **b.** $y = \sin\left(\frac{\pi}{2} - x\right)$ **c.** $y = \tan\left(\frac{\pi}{2} - x\right)$

d. $y = \cos(-x)$ **e.** $y = \sin(-x)$ **f.** $y = \tan(-x)$

g. $y = \sin(x + 2\pi)$ **h.** $y = \cos\left(\frac{\pi}{2} + x\right)$ **i.** $y = \tan(x + \pi)$

10. Find the first three positive x-values that make each equation true.

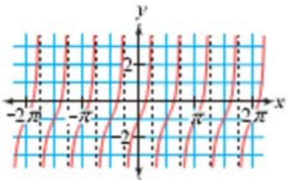
a. $\sec x = -2.5$ **b.** $\csc x = 0.4$

11. Sketch a graph for each equation with domain $0 \leq x < 4\pi$. Include any asymptotes and state the x-values at which the asymptotes occur.

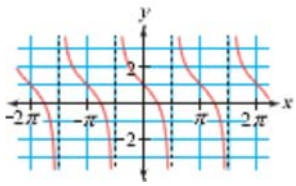
a. $y = \csc x$ **b.** $y = \sec x$ **c.** $y = \cot x$

12. Write an equation for each graph. More than one answer is possible. Use your calculator to check your work.

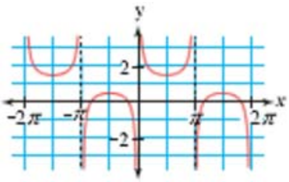
a.



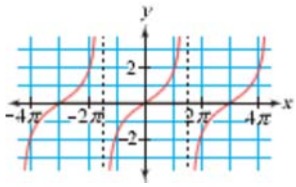
b.



c.



d.



13. Use the trigonometric identities to rewrite each expression in a simplified form that contains only one type of trigonometric function. For each expression, give values of θ for which the expression is undefined. Use domain $0 \leq \theta < 2\pi$.

a. $2\cos^2\theta + \dfrac{\sin\theta}{\csc\theta} + \sin^2\theta$

b. $(\sec\theta)(2\cos^2\theta) + (\cot\theta)(\sin\theta)$

c. $\sec^2\theta + \dfrac{1}{\cot\theta} + \dfrac{\sin^2\theta - 1}{\cos^2\theta}$

d. $\sin\theta(\cot\theta + \tan\theta)$

Review

14. **APPLICATION** Tidal changes can be modeled with a sinusoidal curve. This table gives the time and height of the low and high tides for Burntcoat Head, Nova Scotia, on June 10 and 11, 1999.

High and Low Tides for Burntcoat Head, Nova Scotia, 1999

	Low	High	Low	High
	Time (AST)/Height (m)	**Time (AST)/Height (m)**	**Time (AST)/Height (m)**	**Time (AST)/Height (m)**
June 10	04:13/1.60	10:22/13.61	16:40/1.42	22:49/14.15
June 11	05:12/1.01	11:19/14.09	17:38/0.94	23:45/14.71

(www.ldeo.columbia.edu)

a. Select one time value as the starting time and assign it time 0. Reassign the other time values in minutes relative to the starting time.

b. Let x represent time in minutes relative to the starting time, and let y represent the tide height in meters. Make a scatter plot of the eight data points.

c. Find mean values for

i. Height of the low tide.

ii. Height of the high tide.

iii. Height of "no tide," or the mean water level.

iv. Time in minutes of a tide change between high and low tide.

d. Write an equation to model these data.

e. Graph your equation with the scatter plot in order to check the fit.

f. Predict the water height at 12:00 on June 10, 1999.

g. Predict when the high tide(s) occurred on June 12, 1999.

The Bay of Fundy borders Maine and the Canadian provinces of New Brunswick and Nova Scotia. This bay experiences some of the most dramatic tides on Earth, with water depths fluctuating up to 18 m.

15. **APPLICATION** Juan's parents bought a $500 savings bond for him when he was born. Interest has compounded monthly at an annual fixed rate of 6.5%.

a. Juan just turned 17, and he is considering using the bond to pay for college. How much is his bond currently worth?

b. Juan also considers saving the bond and using it to buy a used car after he graduates. If he would need about $4000, how long would he have to wait?

16. **APPLICATION** A pharmacist has 100 mL of a liquid medication that is 60% concentrated. This means that in 100 mL of the medication, 60 mL is pure medicine and 40 mL is water. She alters the concentration when filling a specific prescription. Suppose she alters the medication by adding water.

a. Write a function that gives the concentration of the medication, $d(x)$, as a function of the amount of water added in milliliters.

b. What is the concentration if the pharmacist adds 20 mL of water?

c. How much water should she add if she needs a 30% concentration?

d. Graph $y = d(x)$. Explain the meaning of the asymptote.

17. **APPLICATION** The pharmacist in Exercise 16 could also alter the medication by increasing the amount of pure medicine.

a. Write a function that gives the concentration of the medication, $c(x)$, as a function of the amount of pure medicine added in milliliters.

b. What is the concentration if the pharmacist adds 20 mL of pure medicine?

c. How much pure medicine should she add if she needs a 90% concentration?

d. Graph $c(x)$. Explain the meaning of the asymptote.

e. Under what circumstances should the pharmacist use the function $d(x)$? When should she use the function $c(x)$?

18. Solve. Give each answer correct to the nearest 0.01.

a. $4 + 5^x = 18$

b. $\log_3 15 = \dfrac{\log x}{\log 3}$

c. $120(0.5)^{2x} = 30$

d. $\log_6 100 = x$

e. $2 \log x = 2.5$

f. $\log_5 5^3 = x$

g. $4 \log x = \log 16$

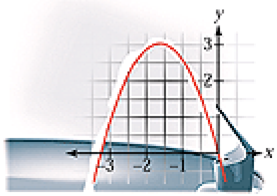
h. $\log (5 + x) - \log 5 = 2$

IMPROVING YOUR VISUAL THINKING SKILLS

An Equation is Worth a Thousand Words

You have learned to model real-world data with a variety of equations. You've also seen that many types of equations have real-world manifestations. For example the path of a fountain of water is parabolic because its projectile-motion equation is quadratic.

Look for a photo of a phenomenon that suggests the graph of an equation. Impose coordinate axes, and find the equation that models the photo. If you have geometry software, you can import an electronic version of your photo and graph your function over it.



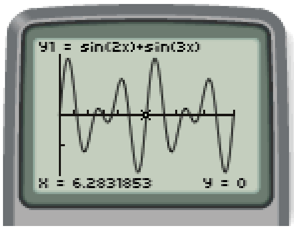
LESSON 10.7

Combining Trigonometric Functions

Music is the pleasure that the human soul experiences from counting without being aware that it is counting.

GOTTFRIED LEIBNIZ

When you add sinusoidal functions, your result is still a periodic function. Two periods of the graph of $y = \sin 2x + \sin 3x$ are shown below.

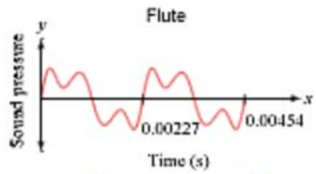


[0, 4π, π/3, –2, 2, 1]

American cellist Yo Yo Ma (b 1955) is one of the world's most renowned classical musicians. He has released more than 50 albums and won 14 Grammy awards.

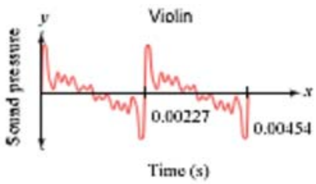
For many applications you'll need to combine several functions to get a realistic model. When a musician plays a note, the sound produced by a musical instrument is actually the sum of several different tones. Different musical instruments have different characteristic sounds. Flutes and violins sound very different, even when they play the same note. As each instrument plays an A note at a frequency of 440 cycles per second, for example, it also sounds the next higher A at 880 cycles per second and several other tones. These additional tones form the "overtone series" of the note and contribute to the distinctive sound of each instrument. Overtones are present in varying degrees of loudness for different instruments. The graphs and equations below show a flute and a violin playing the same A above middle C.

The coefficient of a term represents the loudness of the tone. The coefficients in this equation are relatively small compared to the leading coefficient of 16. This is why a flute has a pure sound.



$$y = 16\sin(440 \cdot 2\pi x) + 9\sin(880 \cdot 2\pi x) + 3\sin(1320 \cdot 2\pi x) + 2.5\sin(1760 \cdot 2\pi x) + 1.0\sin(2200 \cdot 2\pi x)$$

Some of the coefficients in this equation are quite large relative to the leading coefficient of 19. This causes its graph to be more ÒbumpyÓ reflecting a violinÕs complex sound.



$$y = 19\sin(440 \cdot 2\pi x) + 9\sin(880 \cdot 2\pi x) + 8\sin(1320 \cdot 2\pi x) + 9\sin(1760 \cdot 2\pi x) + 12.5\sin(2200 \cdot 2\pi x) + 10.5\sin(2640 \cdot 2\pi x) + 14\sin(3080 \cdot 2\pi x) + 11\sin(3520 \cdot 2\pi x) + 8\sin(3960 \cdot 2\pi x) + 7\sin(4400 \cdot 2\pi x) + 5.5\sin(4840 \cdot 2\pi x) + 1.0\sin(5280 \cdot 2\pi x) + 4.5\sin(5720 \cdot 2\pi x) + 4.0\sin(6160 \cdot 2\pi x) + 3\sin(6600 \cdot 2\pi x)$$

The periods of both graphs on the preceding page are the same, and are determined by the coefficient of the argument of the leading term, 440, which represents the primary A tone or fundamental frequency above middle C. The "bumps" in the graphs are caused by the overtones, which are described by the remaining terms in each equation. The coefficients of these terms indicate the loudness of each overtone. The numbers 440, 880, 1320, 1760, and so on, are the frequencies of the tones. Each of these frequencies is a multiple of the primary frequency, 440 cycles per second. One cycle is completed in $\frac{1}{440}$ of a second, or approximately 0.00227 s.

When the frequency increases, the period decreases, and you hear a higher sound. When frequency decreases, you hear a lower sound. The amplitude determines the loudness of a tone.

Each musical instrument has its own typical sound and its own graph for the sound of a particular note. A musician can only slightly affect the sound of the note and the shape of the graph it produces. Try entering the flute and violin equations on page 615 into your calculator to reproduce the graphs. Then change the coefficients of some of the terms and observe how the graph is affected.

Technology CONNECTION

Producing music electronically on a synthesizer involves a series of steps. First a sequencer creates the electronic equivalent of sheet music. This information is sent to the synthesizer using a digital code called MIDI (Musical Instrument Digital Interface). The synthesizer then reproduces each instrument's sound accurately by producing the correct strength of each individual overtone frequency. Some early synthesizers did this by adding together sine waves. Methods now used include adapting recordings of real instruments! Programs called wave editors let you create your own new "instruments" by specifying what their waves will look like.

Investigation
Sound Wave

You will need

- a microphone probe
- two tuning forks

In this investigation you'll explore the frequency of some tones and combinations of tones.

Procedure Note

1. Set up your calculator and microphone probe to collect sound frequency data. [► See Calculator Note 10D. ◄]
2. To ring a tuning fork, rap it sharply on a semisoft surface like a book or the heel of your shoe. Hold the fork close to the microphone and begin collecting data. When you use more than one fork, be sure to hold them equidistant from the microphone.

Step 1 Choose a tuning fork and collect data as described in the procedure note. Find an equation to fit the data. Repeat this process with a second tuning fork.

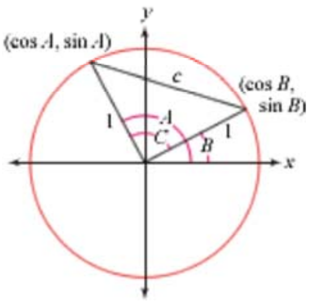
Step 2 Using the same two tuning forks that you used in Step 1, ring both forks simultaneously, and collect frequency data. You should see a combination of sinusoids, rather than a simple sinusoid. Model the data with an equation that is the sum of two simple sinusoid equations.

Step 3 Select a musical instrument, perhaps a flute, violin, piano, timpani, or your voice. Play one note (or string) and collect data. You should see a complex wave, probably too complex for you to write an equation. Identify the fundamental frequency. See if you can identify the frequencies of some of the overtones as well.

Relationships modeled by adding sinusoids are not limited to music and sound. These patterns occur in the motion of moons, planets, tides, and satellites, and in any gear-driven mechanism, from a wristwatch to a car.

You can write a horizontally translated sinusoid as a sum of two untranslated curves, for example, $y = \cos(x - 0.6435)$ is equivalent to $y = 0.8\cos x + 0.6\sin x$. (Check this on your calculator.) Here is a proof that $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

The diagram at right shows the terminal sides of $\angle A$ and $\angle B$ with $m\angle A - m\angle B = m\angle C$. Note the coordinates of the intersections of the terminal sides with the unit circle. The distance c between those points can be determined by two equations, one using the distance formula, and one using the Law of Cosines.



$$c = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}$$

and

$$c = \sqrt{1 + 1 - 2\cos C}$$

$\sqrt{2 - 2\cos C} = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}$	Set the expressions for c equal to each other.
$2 - 2\cos C = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$	Square both sides.
$2 - 2\cos C = \cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A - 2\sin A\sin B + \sin^2 B$	Expand.
$2 - 2\cos C = (\sin^2 A + \cos^2 A) + (\sin^2 B + \cos^2 B) - 2\cos A\cos B - 2\sin A\sin B$	Reorder terms.
$2 - 2\cos C = 1 + 1 - 2\cos A\cos B - 2\sin A\sin B$	Use the Pythagorean identity, $\sin^2 A + \cos^2 A = 1$.
$-2\cos C = -2\cos A\cos B - 2\sin A\sin B$	Subtract 2 from both sides.
$\cos C = \cos A\cos B + \sin A\sin B$	Divide both sides by -2.
$\cos(A - B) = \cos A\cos B + \sin A\sin B$	Substitute $(A - B)$ for C.

So, $\cos(A - B) = \cos A\cos B + \sin A\sin B$ is an identity. You can use this identity to find exact cosine values for some new angles, using values you already know.

EXAMPLE A | Find the exact value of $\cos \frac{\pi}{12}$.

Solution You know exact values of the sine and cosine of 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and π. So rewrite $\frac{\pi}{12}$ as a difference of these values.

$$\cos \frac{\pi}{12} = \cos\left(\frac{3\pi}{12} - \frac{2\pi}{12}\right) = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

Rewrite $\frac{\pi}{12}$ as a difference of two fractions, and reduce.

$$\cos \frac{\pi}{12} = \cos\frac{\pi}{4} \cdot \cos\frac{\pi}{6} + \sin\frac{\pi}{4} \cdot \sin\frac{\pi}{6}$$

Rewrite $\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$ using the identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

$$\cos \frac{\pi}{12} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

Substitute exact values for sine and cosine of $\frac{\pi}{4}$ and $\frac{\pi}{6}$.

$$\cos \frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

Combine into one rational expression.

So the exact value of $\cos \frac{\pi}{12}$ is $\frac{\sqrt{6}+\sqrt{2}}{4}$. You can check your work by evaluating $\frac{\sqrt{6}+\sqrt{2}}{4}$ and $\cos \frac{\pi}{12}$ with your calculator.

The next example shows you how to develop another identity based on identities that you already know.

EXAMPLE B | Develop an identity for $\cos(A + B)$.

Solution Notice that $\cos(A + B)$ is similar to $\cos(A - B)$, except the sign of B is changed. Start with the identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$, and replace B with $-B$.

$$\cos(A - (-B)) = \cos A \cos(-B) + \sin A \sin(-B)$$

Replace B with $-B$.

$$\cos(A + B) = \cos A \cos(-B) + \sin A \sin(-B)$$

Rewrite $A - (-B)$ as $A + B$.

Earlier in this chapter, you found that $\cos(-x) = \cos x$ and that $\sin(-x) = -\sin x$.

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

Replace $\cos(-B)$ with $\cos B$ and $\sin(-B)$ with $-\sin B$.

So, $\cos(A + B) = \cos A \cos B - \sin A \sin B$ is an identity.

There are many other trigonometric identities that can be useful in calculations or for simplifying expressions. As in Example B, you will be asked to prove new identities using existing identities. The box on the next page includes several relationships you have already seen and a few new ones.

Sum and Difference Identities

$\cos(A - B) = \cos A \cos B + \sin A \sin B$
$\cos(A + B) = \cos A \cos B - \sin A \sin B$
$\sin(A - B) = \sin A \cos B - \cos A \sin B$
$\sin(A + B) = \sin A \cos B + \cos A \sin B$
$\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$
$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Double-Angle Identities

$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$
$\sin 2A = 2\sin A \cos A$
$\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

Half-Angle Identities

$\sin\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{2}}$
$\cos\dfrac{A}{2} = \pm\sqrt{\dfrac{1+\cos A}{2}}$
$\tan\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{1+\cos A}} = \dfrac{\sin A}{1+\cos A} = \dfrac{1-\cos A}{\sin A}$

For the half-angle identities, the sign of the answer is determined by the quadrant in which the terminal side of the given angle lies.

In the exercises you will be asked to prove some of these identities. As you do so, remember to only work on one side of the equation.

EXERCISES

Practice Your Skills

1. Decide whether each expression is an identity by substituting values for A and B.

a. $\cos(A + B) = \cos A + \cos B$
b. $\sin(A + B) = \sin A + \sin B$
c. $\cos(2A) = 2\cos A$
d. $\sin(2A) = 2\sin A$

2. Prove each identity. The sum and difference identities will be helpful.

a. $\cos(2\pi - A) = \cos A$
b. $\sin\left(\dfrac{3\pi}{2} - A\right) = -\cos A$

3. Rewrite each expression with a single sine or cosine.

a. $\cos 1.5 \cos 0.4 + \sin 1.5 \sin 0.4$
b. $\cos 2.6 \cos 0.2 - \sin 2.6 \sin 0.2$
c. $\sin 3.1 \cos 1.4 - \cos 3.1 \sin 1.4$
d. $\sin 0.2 \cos 0.5 + \cos 0.2 \sin 0.5$

4. Use identities to find the exact value of each expression.

a. $\sin \frac{-11\pi}{12}$ **b.** $\sin \frac{7\pi}{12}$ **c.** $\tan \frac{\pi}{12}$ **d.** $\cos \frac{\pi}{8}$

Reason and Apply

5. Given $\pi \leq x \leq \frac{3\pi}{2}$ and $\sin x = -\frac{2}{3}$, find the exact value of $\sin 2x$.

6. Use the identity for $\cos(A - B)$ and the identities $\sin A = \cos\left(\frac{\pi}{2} - A\right)$ and $\cos A = \sin\left(\frac{\pi}{2} - A\right)$ to prove that

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

7. Use the identity for $\sin(A + B)$ from Exercise 6 to prove that

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

8. Use the identity for $\sin(A + B)$ to prove the identity $\sin 2A = 2 \sin A \cos A$.

9. Use the identity for $\cos(A + B)$ to prove the identity $\cos 2A = \cos^2 A - \sin^2 A$.

10. What is wrong with this statement?

$$\cos(\tan x) = \cos\left(\frac{\sin x}{\cos x}\right) = \sin x$$

11. Show that $\tan(A + B)$ is not equivalent to $\tan A + \tan B$. Then use the identities for $\sin(A + B)$ and $\cos(A + B)$ to develop an identity for $\tan(A + B)$.

12. Use your identity from Exercise 11 to develop an identity for $\tan 2A$.

13. You have seen that $\sin^2 A = 1 - \cos^2 A$ and $\cos^2 A = 1 - \sin^2 A$.

a. Use one of the double-angle identities to develop an expression that is equivalent to $\sin^2 A$ but does not contain the term $\cos^2 A$.

b. Use another double-angle identity to develop an expression equivalent to $\cos^2 A$ that does not contain $\sin^2 A$.

14. ***Mini-Investigation*** Set your graphing window to $0 \leq x \leq 4\pi$ and $-2 \leq y \leq 2$.

a	1	2	2	3	3	4	4	4
b	2	3	4	4	6	6	8	12
Period								

a. Graph equations in the form $y = \sin ax + \sin bx$, using the a- and b-values listed in the table. Record the period for each pair.

b. Explain how to find the period of any function in the form $y = \sin ax + \sin bx$, where a and b are whole numbers.

15. ***Mini-Investigation*** Set your graphing window to $0 \leq x \leq 24\pi$ and $-2 \leq y \leq 2$.

a	2	2	2	4	3
b	4	3	5	6	6
Period					

a. Graph equations in the form $y = \sin\frac{x}{a} + \sin\frac{x}{b}$, using the a- and b-values listed in the table. Record the period for each pair.

b. Explain how to find the period of any function in the form $y = \sin\frac{x}{a} + \sin\frac{x}{b}$, where a and b are integers.

c. Predict the period for $y = \sin\frac{x}{3} + \sin\frac{x}{4} + \sin\frac{x}{8}$. Explain your reasoning.

16. When a tuning fork for middle C is struck, the resulting sound wave has a frequency of 262 cycles per second. The equation $y = \sin(262 \cdot 2\pi x)$ is one possible model for this wave.

a. Identify the period of this wave. Make a graph showing about five complete cycles.

b. Suppose middle C on an out-of-tune piano is played and that the resulting wave has a frequency of 265 cycles per second. Write an equation to model the out-of-tune wave. Identify its period. Then make a graph using the same window as in 16a.

c. A piano tuner plays the out-of-tune C at the same time she uses the tuning fork. The two waves are added to produce a new sound wave. Write the equation that models the sum of the two waves. Graph a 0.5 s interval of this new equation.

Music CONNECTION

The sound waves from an out-of-tune piano and a tuning fork have slightly different frequencies. When they are played together, the resulting wave will vary in amplitude, getting louder and softer in cycles. These periodic variations are called beats. Because the difference between the number of cycles is three (265 – 262), there are three beats per second. The loudness will rise and fall three times per second. Musicians listen for beats to see if their instruments are out of tune. An out-of-tune piano is tuned by adjusting the tension of a string until the beats disappear.

Review

17. Solve.

a. $\sec 144° = x$ b. $\csc\frac{24\pi}{9} = x$ c. $\cot 3.92 = x$ d. $\cot 630° = x$

18. Find all values of θ that satisfy each equation. Use domain $0° \le \theta < 360°$.

a. $\tan\theta = 0.5317$ b. $\sec\theta = -3.8637$ c. $\csc\theta = 1.1126$ d. $\cot\theta = -4.3315$

19. A fishing boat rides gently up and down, 10 times per minute, on the ocean waves. The boat rises and falls 1.5 m between each wave crest and trough. Assume the boat is on a crest at time 0 min.

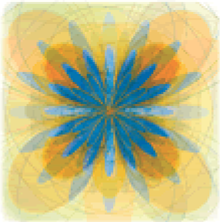
a. Sketch a graph of the boat's height above sea level over time.

b. Use a cosine function to model your graph.

c. Use a sine function to model your graph.

EXPLORATION

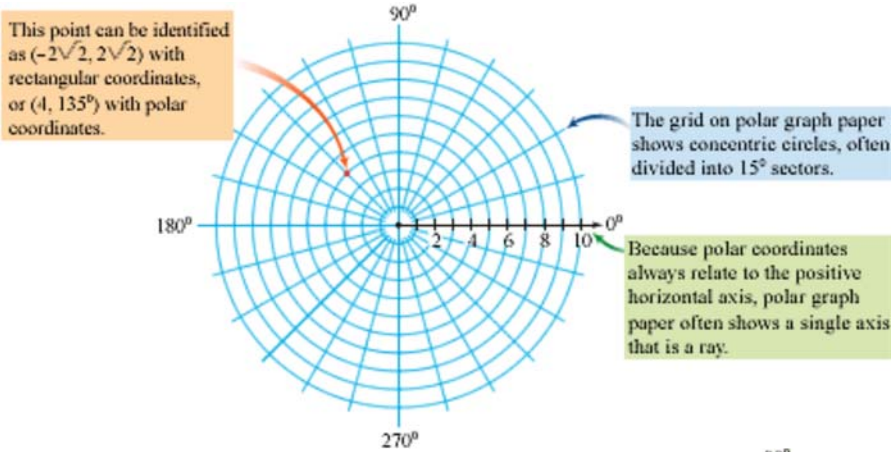
Polar Coordinates



You are very familiar with graphing points on a plane with rectangular coordinates. When you graph points in the form (x, y), you need both the x-coordinate and the y-coordinate to identify the exact location of any point. But this is not the only way to locate points on a plane.

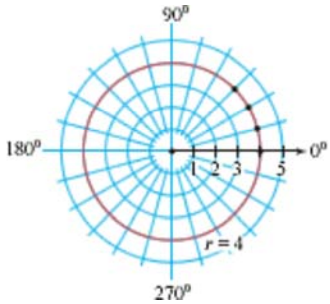
In this chapter you have worked with circles centered at the origin. As you move around a circle, you can identify any point on the circle with coordinates in the form (x, y). However, because the radius of the circle remains constant, you can also identify these points by the radius, r, and the angle of rotation from the positive x-axis, θ.

Imagine infinitely many concentric circles covering a plane, all centered at the origin. You can identify any point on the plane with coordinates in the form (r, θ), called **polar coordinates.** Polar equations in the form $r = f(\theta)$ may lead to familiar or surprising results.

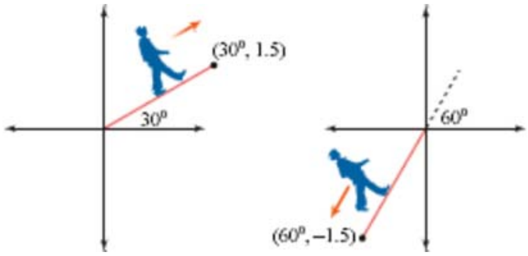


For instance, look at the graph of the polar equation $r = 4$, shown at right.

There is no θ in this equation, so r will always be 4. This is the set of all points 4 units from the origin—a circle with radius 4. Notice how much simpler this equation is than the equation of the same circle using rectangular coordinates!



Imagine standing at the origin looking in the direction of the positive horizontal axis. First, rotate counterclockwise by the necessary angle. Then, imagine walking straight out from the origin and placing a point at distance *r*. If *r* is positive, walk forward. If *r* is negative, walk backward.



In the activity and questions that follow, you'll explore some of the elegant and complicated-looking graphs that result from polar equations.

Activity

Rose Curves

Use these steps to explore polar equations in the form $r = a \cos n\theta$.

For Steps 1–3, make a table of values, plot the points, and connect them in order with a smooth curve. The results are called rose curves, and each will look like a flower with petals. You can easily check your work with your graphing calculator. [► See Calculator Note 10E to learn how to graph polar equations on your calculator. ◄]

Step 1 Graph the family of curves $r = a \cos 2\theta$ with $a = \{1, 2, 3, 4, 5, 6\}$. How does the coefficient a affect the graph?

Step 2 Graph the family of curves $r = 3 \cos n\theta$ with $n = \{1, 2, 3, 4, 5, 6\}$. Generalize the effect of the coefficient n. Write statements that describe the curves when n is even and when n is odd.

Step 3 Graph the family of curves $r = 3 \sin n\theta$ with $n = \{1, 2, 3, 4, 5, 6\}$. Generalize your results. How do these curves differ from the curves in Step 2?

Step 4 Find a way to graph a rose curve with only two petals. Explain why your method works.

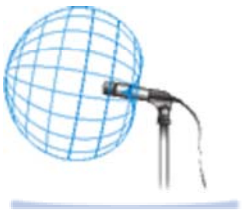
Questions

1. Find a connection between the graph of the polar equation $r = a \cos n\theta$ and the graph of the associated rectangular equation $y = a \cos nx$. Explain whether or not you can look at the graph of $y = a \cos nx$ and predict the shape and number of petals in the polar graph.

2. The graphs of polar equations in the forms $r = a(\cos\theta + 1)$, $r = a(\cos\theta - 1)$, $r = a(\sin\theta + 1)$, and $r = a(\sin\theta - 1)$ are called cardioids because they resemble hearts. Graph several curves in the cardioid family. Generalize your results by answering the questions at the top of page 624.

Technology CONNECTION

Cardioid microphones are designed to pick up sound at equal intensity levels from any point on a cardioid around the microphone. These microphones are designed so that the dimple of the curve is directed toward the base of the microphone, so that they minimize sound coming from behind. For this reason, cardioid microphones are especially useful for recording sound from a stage because they pick up little noise from the audience.



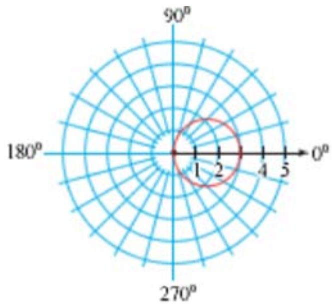
a. How do the graphs of $r = a(\cos\theta + 1)$ and $r = a(\cos\theta - 1)$ differ?

b. How do the cardioids created with sine differ from those created with cosine?

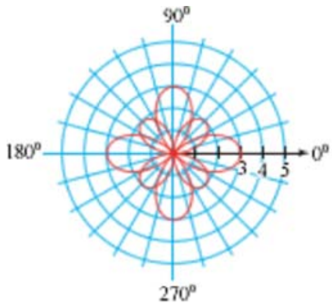
c. What is the effect of the coefficient a?

3. Write polar equations to create each graph. For 3b and d, you'll need more than one equation.

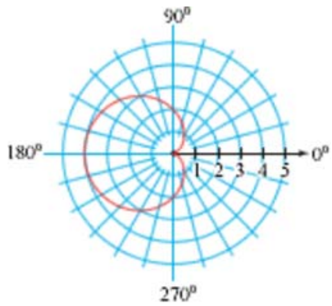
a.



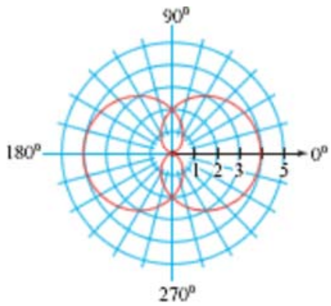
b.



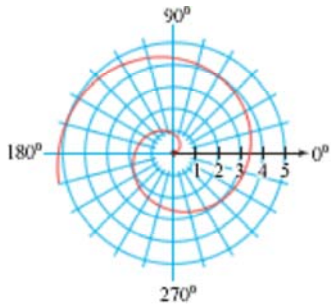
c.



d.



4. Think about what happens in a spiral and how the value of r changes as the value of θ changes.



a. Find an equation that creates a spiral. Check your work by graphing on your calculator with the domain $0° \leq \theta \leq 360°$. What is the general form of the equation that creates a spiral?

b. What happens as you extend the domain of θ to values greater than 360°?

c. What happens if you change the domain of θ to include negative values?

5. In general, are polar equations functions? Explain your reasoning.

CHAPTER 10 REVIEW

In this chapter you expanded your understanding of trigonometry to include angles with any real number measure. You learned to measure angles in **radians,** and you then identified relationships among radian measure, arc length, speed, and **angular speed.** You studied circular functions, their graphs, and their applications, and you learned to think of the sine and cosine of an angle as the *y*- and *x*-coordinates of a point on the unit circle. This allowed you to identify angles in **standard position** that are **coterminal.** That is, they share the same **terminal side.** Coterminal angles have the same trigonometric values. The remaining **trigonometric functions.**—tangent, **cotangent, secant,** and **cosecant**—are also defined either as a ratio involving an *x*-coordinate and a *y*-coordinate, or as a ratio of one of these coordinates and the distance from the origin to the point.

Graphs of the trigonometric functions model periodic behavior and have domains that extend in both the positive and negative directions. You worked with many relationships that can be modeled with **sinusoids.** You studied transformations of sinusoidal functions and defined their **amplitude, period, phase shift, vertical translation,** and **frequency.**

You learned the difference between an inverse trigonometric relation and an inverse trigonometric function, and used this distinction to solve equations involving periodic functions. You learned, for example, that $x = \cos y$ provides infinitely many values of y for a given choice of x, whereas the function $y = \cos^{-1} x$ provides exactly one value of y for each value of x, called the **principal value.**

Finally, you discovered several properties and identities involving trigonometric expressions, and you learned how to prove that an equation is an identity.

EXERCISES

1. For each angle in standard position given, identify the quadrant that the angle's terminal side lies in, and name a coterminal angle. Then convert each angle measure from radians to degrees, or vice versa.

a. 60° **b.** $\frac{4\pi}{3}$ **c.** 330° **d.** $-\frac{\pi}{4}$

2. Find exact values of the sine and cosine of each angle in Exercise 1.

3. State the period of the graph of each equation, and write one other equation that has the same graph.

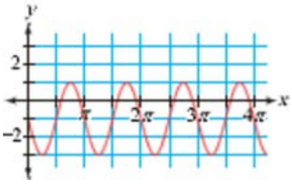
a. $y = 2\sin\left(3\left(x - \frac{\pi}{6}\right)\right)$ **b.** $y = -3\cos 4x$

c. $y = \sec 2x$ **d.** $y = \tan(-2x) + 1$

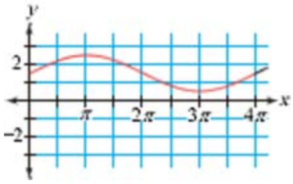
4. For the sinusoidal equations in 3a and b, state the amplitude, phase shift, vertical translation, and frequency. Then sketch a graph of one complete cycle.

5. Write an equation for each graph.

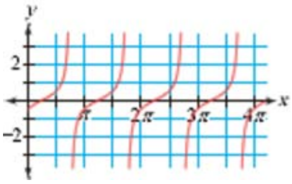
a.



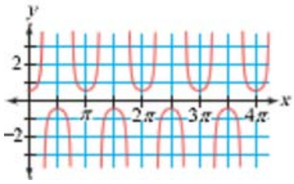
b.



c.



d.



6. Find the area and arc length of a sector of a circle that has radius 3 cm and central angle $\frac{\pi}{4}$. Give exact answers.

7. Identify the domain and range of $\cos y = x$ and $y = \cos^{-1} x$.

8. Find these values without using your calculator. Then verify your answers with your calculator.

a. $\sin\left(\tan^{-1}\frac{3}{4}\right)$

b. $\cos\left(\sin^{-1}\frac{3}{5}\right)$

c. $\sin\left(\sin^{-1}\frac{8}{17}\right)$

9. Write an equation for a transformation of $y = \sin x$ that has a reflection across the x-axis, amplitude 3, period 8π, and phase shift $\frac{\pi}{2}$.

10. Prove that each of these identities is true. You may use any of the identities that have been proved in this chapter.

a. $\sec A - \sin A \tan A = \cos A$

b. $\dfrac{1}{\sin^2 A} - \dfrac{1}{\tan^2 A} = 1$

c. $\dfrac{\sec A \cos B - \tan A \sin B}{\sec A} = \cos(A + B)$

A detail from American sculptor Ruth Asawa’s b 1926) hourglass-shaped baskets shows her focus on circular patterns as a metaphor for family and community circles. The piece is called *Completing the Circle*.

11. A mass hanging from a spring is pulled down 2 cm from its resting position and released. It makes 12 complete bounces in 10 s. At what times during the first 3 s was the mass 0.5 cm below its resting position?

12. **APPLICATION** These data give the ocean tide heights each hour on November 17, 2002, at Saint John, New Brunswick, Canada.

Time	Tide height (m)
00:00	5.56
01:00	4.12
02:00	2.71
03:00	1.77
04:00	1.56
05:00	2.09
06:00	3.21
07:00	4.70
08:00	6.18
09:00	7.20
10:00	7.50
11:00	7.09

Time	Tide height (m)
12:00	6.09
13:00	4.65
14:00	3.08
15:00	1.85
16:00	1.33
17:00	1.60
18:00	2.54
19:00	3.96
20:00	5.51
21:00	6.75
22:00	7.33
23:00	7.17

a. Create a scatter plot of the data.

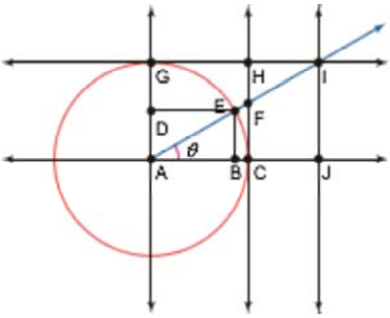
b. Write a function to model the data, and graph this function on the scatter plot from 12a.

c. What would you estimate the tide height to have been at 3:00 P.M. on November 19, 2002?

d. A ship was due to arrive at Saint John on November 20, 2002. The water had to be at least 5 m for the ship to safely enter the harbor. Between what times on November 20 could the ship have entered the harbor?

TAKE ANOTHER LOOK

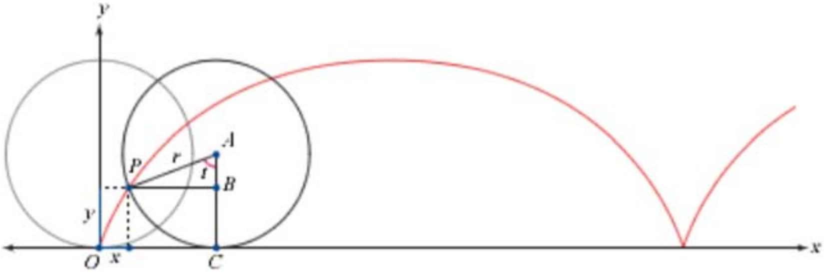
1. Consider the geometry-software diagram at right. You have seen that in a unit circle, the length *AB* has the same value as cos θ. The lengths *AF*, *GI*, *AD*, *AI*, and *CF* correspond to other trigonometric values of θ. Decide which segment length equals each of the values sin θ, tan θ, sec θ, csc θ, and cot θ. Justify your answers.



2. You have seen how to find the exact value of cos $\frac{\pi}{12}$ by rewriting the expression as cos $\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$, and using a trigonometric identity to expand and evaluate. How can you find the exact value of sin $\frac{11\pi}{12}$? (*Hint*: Write $\frac{11\pi}{12}$ as a sum or difference of three terms.) Find some other exact values of sine, cosine, or tangent using this method.

3. You are now familiar with angle measures in degrees and radians, but have you ever heard of gradians? Research the gradian angle measure. Explain how it compares to radians and degrees, when and where it was used, and any advantages it might have. Can you find any other units that measure angle or slope?

4. The path traced by a fixed point, P, on a moving wheel is called a **cycloid** and is shown below. A cycloid can be defined with parametric equations. Use the diagram to derive parametric equations for x and y, the coordinates of point P. Note that the length of arc CP equals the length of segment CO.



5. In Exercise 13b on page 620, you developed the formula $\cos^2 A = \frac{1+\cos 2A}{2}$. Use this formula to develop the half-angle formula for cosine. Begin by taking the square root of both sides, then substitute $\frac{\theta}{2}$ for A. Use the formula from Exercise 13a for $\sin^2 A$ and a similar process to develop the half-angle formula for sine. Then use the half-angle formulas for sine and cosine to develop the half-angle formula for tangent.

Assessing What You've Learned

PERFORMANCE ASSESSMENT Demonstrate for a friend or family member how to find an equation that models periodic motion data. Be sure to use the words *amplitude*, *period*, *frequency*, *phase shift*, and *vertical translation*. Describe what each of these values tells you about the data.

ORGANIZE YOUR NOTEBOOK Check that your notebook is in order. Make sure that you have definitions of all the terminology from this chapter. Include terms related to angles, like *standard position* and *terminal side*, and terms related to sinusoidal graphs, like *amplitude* and *period*. Check that all the trigonometric identities you have learned are in your notes as well.

WRITE IN YOUR JOURNAL Are your understandings of the sine, cosine, and tangent functions different now than they were when you started this chapter? Write a journal entry that describes how your understanding of the trigonometric functions has changed over time.

