CHAPTER

9

# Conic Sections and Rational Functions

*Timetable* (2000), designed by American architect and sculptor Maya Lin (b 1959), is a rotating circular clock/fountain located at Stanford University in Palo Alto, California. Made of black granite, steel, stone, and water, with the clock rings, motorized discs, and rotating parts submerged, the geometric sculpture gives the time in Pacific standard time, daylight saving time, and Universal time. Maya Lin's message, "Although we tend to think of time as an absolute, it is relative to location," is inscribed on a panel near the fountain.

## OBJECTIVES

**In this chapter you will**

- **use the distance formula to find the distance between two points on a plane and to solve distance and rate problems**
- **learn about conic sections- circles, ellipses, parabolas, and hyperbolas-which are created by intersecting a plane and a cone**
- **investigate the properties of conic sections**
- **write the equations of conic sections in different forms**
- **study rational functions and learn special properties of their graphs**
- **add, subtract, multiply, and divide rational expressions**

LESSON 9.1

# Using the Distance Formula

*It is impossible to be a mathematician without being a poet in soul.*

SOPHIA KOVALEVSKAYA

Imagine a race in which you carry an empty bucket from the starting line to the edge of a pool, fill the bucket with water, and then carry the bucket to the finish line. Luck, physical fitness, common sense, a calm attitude, and a little mathematics will make a difference in how you perform. Finding the shortest path will help minimize your effort, distance, and time involved. In the investigation you will mathematically analyze this situation.

This illustration of "Jack and Jill" was created by the English illustrator Walter Crane (1845-1915).

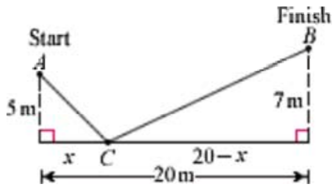
## Investigation

### Bucket Race

**You will need**

- centimeter graph paper
- a ruler

The starting line of a bucket race is 5 m from one end of a pool, the pool is 20 m long, and the finish line is 7 m from the opposite end of the pool, as shown. In this investigation you will find the shortest path from point $A$ to a point $C$ on the edge of the pool to point $B$. That is, you will find the value of $x$, the distance in meters from the end of the pool to point $C$, such that $AC + CB$ is the shortest path possible.




Step 1 Make a scale drawing of the situation on graph paper.

Step 2 Plot several different locations for point $C$. For each, measure the distance $x$ and find the total length $AC + CB$. Record your data.

Step 3 What is the best location for $C$ such that the length $AC + CB$ is minimized? What is the distance traveled? Is there more than one best location? Describe at leas two different methods for finding the best location for $C$.

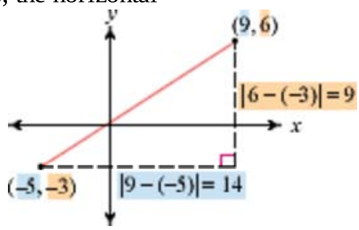
Step 4 Make a scale drawing of your solution.

Imagine that the amount of water you empty out at point $B$ is an important factor in winning the race. This means you must move carefully so as not to spill water, and you'll be able to move faster with the empty bucket than you can with the bucket full of water. Assume that you can carry an empty bucket at a rate of 1.2 m/s and that you can carry a full bucket, without spilling, at a rate of 0.4 m/s.

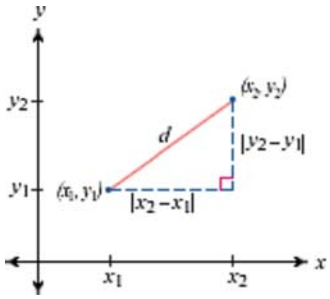
Step 5 Go back to the data collected in Step 2 and find the time needed for each $x$-value.

Step 6 Now find the best location for point $C$ so that you minimize the time from point $A$ to the pool edge, then to point $B$. What is your minimum time? What is the distance traveled? How does this compare to your answer in Step 3? Describe your solution process.

Many of the equations you will study in this chapter are based on finding the distance between two points. Consider the distance, $d$, between two points with coordinates $(-5, -3)$ and $(9, 6)$. Based on the $x$- and $y$-coordinates, the horizontal distance between the points is 14 units and the vertical distance between them is 9 units. The horizontal and vertical components of the distance create a right triangle. By the Pythagorean Theorem, the distance between the two points is $\sqrt{14^2 + 9^2}$, which is $\sqrt{277}$, or approximately 16.64 units. In general, if two points have coordinates $(x_1, y_1)$ and $(x_2, y_2)$, then the Pythagorean Theorem gives




$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2$$




Taking the square root of both sides gives you a formula for distance on a coordinate plane. Because quantities are squared in the formula, two positive numbers are being added, so absolute value signs are no longer necessary.

## Distance Formula

The distance, $d$, between two points on a coordinate plane, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In Chapter 4, you used the Pythagorean Theorem to find the equation of a circle centered at the origin. You can also use the distance formula. If $(x, y)$ is any point located on the circumference of a circle, its distance from the center, $(0, 0)$, is $\sqrt{(x-0)^2 + (y-0)^2}$. Because the distance from the center of the circle is defined as the radius, $r$, you get $r = \sqrt{x^2 + y^2}$, or $r^2 = x^2 + y^2$.

The distance formula also enables you to write equations that represent other distance situations. One use is to write an equation that describes a set of points that all meet a certain condition. A set of points that fit a given condition is called a **locus.** For example, the locus of points that are 1 unit from the point (0, 0) is the circle with the equation $x^2 + y^2 = 1$. In this chapter you will explore equations describing a variety of different loci (the plural of locus).

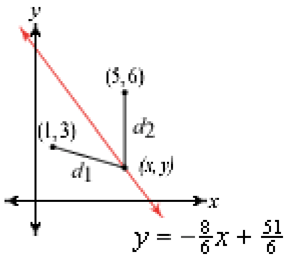
## EXAMPLE A

Find the equation of the locus of points that are equidistant from the points (1, 3) and (5, 6).

### ▸ Solution

Let $d_1$ represent the distance between (1, 3) and any point, $(x, y)$, on the locus. By the distance formula,

$$d_1 = \sqrt{(x-1)^2 + (y-3)^2}$$




Let $d_2$ represent the distance between (5, 6) and the same point on the locus, so

$$d_2 = \sqrt{(x-5)^2 + (y-6)^2}$$

The locus of points contains all points whose coordinates satisfy the equation $d_1 = d_2$, or

$$\sqrt{(x-1)^2 + (y-3)^2} = \sqrt{(x-5)^2 + (y-6)^2}$$

Use algebra to transform this equation into something more familiar.

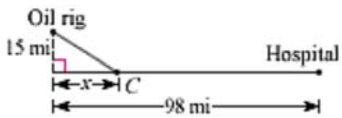
| | |
|---|---|
| $\sqrt{(x-1)^2 + (y-3)^2} = \sqrt{(x-5)^2 + (y-6)^2}$ | Original equation. |
| $(x-1)^2 + (y-3)^2 = (x-5)^2 + (y-6)^2$ | Square both sides. |
| $x^2 - 2x + 1 + y^2 - 6y + 9 = x^2 - 10x + 25 + y^2 - 12y + 36$ | Expand the binomials. |
| $-2x + 1 - 6y + 9 = -10x + 25 - 12y + 36$ | Subtract $x^2$ and $y^2$ from both sides. |
| $8x + 6y = 51$ | Rewrite in the form $ax + by = c$ by moving variables to the left side and constants to the right. |

The locus is the line with the equation $8x + 6y = 51$, or $y = -\frac{8}{6}x + \frac{51}{6}$. You may recognize this locus as the perpendicular bisector of the segment joining the two points.

Just as the Pythagorean Theorem and the distance formula are useful for finding the equation of a locus of points, they are also helpful for solving real-world problems.

## EXAMPLE B

An injured worker must be rushed from an oil rig 15 mi offshore to a hospital in the nearest town 98 mi down the coast from the oil rig.




**a.** Let $x$ represent the distance in miles from the point on the shore closest to the oil rig and another point, $C$, on the shore.

How far does the injured worker travel, in terms of $x$, if a boat takes him to $C$ and then an ambulance takes him to the hospital?

**b.** Assume the boat travels at an average rate of 23 mi/h and the ambulance travels at an average rate of 70 mi/h. What value of $x$ makes the trip 3 h?

### ▶ *Solution*

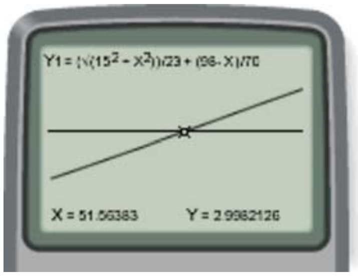
Use the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

**a.** The boat must travel $\sqrt{15^2 + x^2}$, and the ambulance must travel $98 - x$. The total distance in miles is $\sqrt{15^2 + x^2} + 98 - x$.

**b.** Distance equals rate times time, $d = rt$ or . Solving for time, $t = \frac{d}{r}$. So the boat's time is $\frac{\sqrt{15^2 + x^2}}{23}$, and the ambulance's time is $\frac{98 - x}{70}$. The total time in hours, $y$, is represented by

$$y = \frac{\sqrt{15^2 + x^2}}{23} + \frac{98 - x}{70}$$

One way to find the value of $x$ that gives a trip of 3 h is to graph the total time equation and $y = 3$, and trace to approximate the intersection. The graphs intersect when $x$ is approximately 51.6. For the trip to be 3 h, the boat and the ambulance should meet at the point on the shore 51.6 mi from the point closest to the oil rig.




[38, 63, 10, 2.3, 3.6, 1]

**History CONNECTION**

In 1790, the U.S. Coast Guard was formed to prevent smuggling and maintain customs laws. Combined with the Life Saving Service in 1915, it now oversees rescue missions, environmental protection, navigation, safety during weather hazards, port security, boat safety, and oil tanker transfers. The U.S. Coast Guard has both military and volunteer divisions.

U.S. Coast Guard ship *Acushnet*

## EXERCISES

You will need

**Geometry software** for Exercise 13

### ▶ Practice Your Skills

**1.** Find the distance between each pair of points.

**a.** (2, 5) and (8, 13)

**b.** (0, 3) and (5, 10)

**c.** (– 4, 6) and (– 2, – 3)

**d.** (3, $d$) and (– 6, 3$d$)

**2.** The distance between the points (2, 7) and (5, $y$) is 5 units. Find the possible value(s) of $y$.

3. The distance between the points (– 1, 5) and ($x$, – 2) is 47. Find the possible value(s) of $x$.

4. Which side is longest in the triangle with vertices $A(1, 2)$, $B(3, -1)$, and $C(5, 3)$?

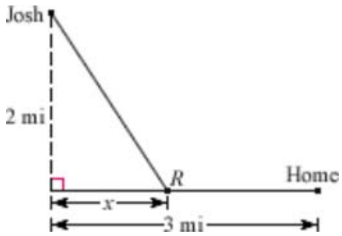
5. Find the perimeter of the triangle with vertices $A(8, -2)$, $B(1, 5)$, and $C(4, -5)$.

## Reason and Apply

6. Find the equation of the locus of points that are twice as far from the point (2, 0) as they are from (5, 0).

7. If you are too close to a radio tower, you will be unable to pick up its signal. Let the center of a town be represented by the origin of a coordinate plane. Suppose a radio tower is located 2 mi east and 3 mi north of the center of town, or at the point (2, 3). A highway runs north-south 2.5 mi east of the center of town, along the line $x = 2.5$. Where on the highway will you be less than 1 mi from the tower and therefore unable to pick up the signal?

8. Josh is riding his mountain bike when he realizes that he needs to get home quickly for dinner. He is 2 mi from the road, and home is 3 mi down that road. He can ride 9 mi/h through the field separating him from the road and can ride 22 mi/h on the road.




a. If Josh rides through the field to a point $R$ on the road, and then home along the road, how far will he ride through the field? How far on the road? Let $x$ represent the distance in miles between point $R$ and the point on the road that is closest to his current location.

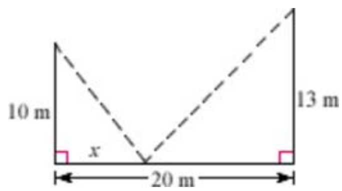
b. How much time will Josh spend riding through the field? How much time on the road?

c. What value of $x$ gets Josh home the fastest? What is the minimum time?

9. A 10 m pole and a 13 m pole are 20 m apart at their bases. A wire connects the top of each pole with a point on the ground between them.

a. Let $y$ represent the total length of the wire. Write an equation that relates $x$ and $y$.

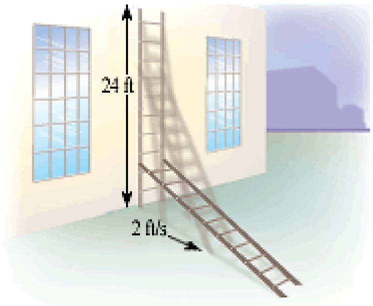
b. What domain and range make sense in this situation?



**c.** Where should the wire be fastened to the ground so that the length of wire is minimized? What is the minimum length?

This cable-stay bridge, the Fred Hartman Bridge, connects Baytown, Texas, and La Porte, Texas. Taut cables stretch from the tops of two towers to support the roadway. For more information on cable-stay bridges, see the links at www.keymath.com/DAA .

**10.** A 24 ft ladder is placed upright against a wall. Then the top of the ladder slides down the wall while the foot of the ladder is pulled outward along the ground at a steady rate of 2 ft/s.




**a.** Find the heights that the ladder reaches at 1 s intervals while the ladder slides down the wall.

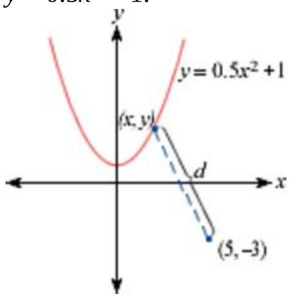
**b.** How long will it take before the ladder is lying on the ground?

**c.** Does the top of the ladder also slide down the wall at a steady rate of 2 ft/s? Explain your reasoning.

**d.** Write parametric equations that model the distance in feet of the foot of the ladder from the wall, $x$, and the height in feet that the top of the ladder reaches, $y$. Let $t$ represent time in seconds.

**e.** Write a complete explanation of the rate at which the ladder slides down the wall.

**11.** Let $d$ represent the distance between the point $(5, -3)$ and any point, $(x, y)$, on the parabola $y = 0.5x^2 + 1$.




**a.** Write an equation for $d$ in terms of $x$.

**b.** What is the minimum distance? What are the coordinates of the point on the parabola that is closest to the point $(5, -3)$?

12. **APPLICATION** The city councils of three neighboring towns-Ashton, Bradburg, and Carlville-decide to pool their resources and build a recreation center. To be fair, they decide to locate the recreation center equidistant from all three towns.

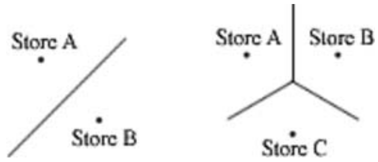
**a.** When a coordinate plane is placed on a map of the towns, Ashton is at (0, 4), Bradburg is at (3, 0), and Carlville is at (12, 8). At what point on the map should the recreation center be located?

**b.** If the three towns were collinear (along a line), could the recreation center be located equidistant from all three towns? Explain your reasoning.

**c.** What other factors might the three city councils consider while making their decision as to where to locate the recreation center?

**Career CONNECTION**

A Voronoi diagram shows regions formed by a set of points such that any point inside one of the regions is closer to that region's site than to any other site. Voronoi diagrams for two and three sites are shown below. Marketing analysts use Voronoi diagrams as well as demographic data, topological features, and traffic patterns to place stores and restaurants strategically.




*Boundary functions*, by Scott Snibbe, is an interactive art piece that divides space into a Voronoi diagram as people move on a section of floor. This piece is a commentary on personal space and the separation of people.

13. ***Technology*** Follow these steps to solve Exercise 12a with geometry software.

**a.** Open a new sketch. Define a coordinate system and plot three points, *A*, *B*, and *C*, that represent the locations of Ashton, Bradburg, and Carlville.

**b.** Connect the three points with line segments. Construct the perpendicular bisector of each segment. What happens with the perpendicular bisectors? Did you really need to construct all three bisectors?

**c.** Locate the intersection of the perpendicular bisectors. Do the coordinates of the intersection agree with your answer to Exercise 12a?

**d.** Construct a circle using the intersection of the perpendicular bisectors as the center and *A*, or *B*, or *C* as a point on the circle. Describe what happens. How does this confirm that the recreation center is equidistant from all three towns?

## Review

14. Complete the square in each equation such that the left side represents a perfect square or a sum of perfect squares.

**a.** $x^2 + 6x = 5$ **b.** $y^2 - 4y = -1$ **c.** $x^2 + 6x + y^2 - 4y = 4$

**15.** Triangle *ABC* has vertices $A(8, -2)$, $B(1, 5)$, and $C(4, -5)$.

**a.** Find the midpoint of each side.

**b.** Write the equations of the three medians of the triangle. (A median of a triangle is a segment connecting one vertex to the midpoint of the opposite side.)

**c.** Locate the point where the medians meet.

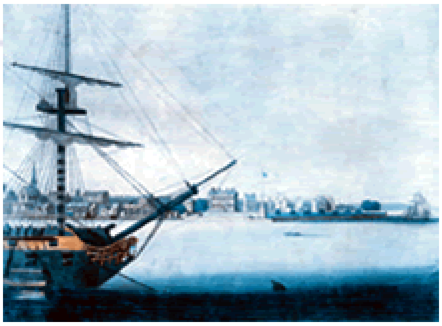
**16.** Give the domain and range of the function $f(x) = x^2 + 6x + 7$.

**17.** A ship leaves port and travels on a bearing of 205° for 2.5 h at 8 knots, and then on a bearing of 150° for 3 h at 10 knots. How far is the ship from its port? (A knot is equivalent to 1 nautical mile per hour.)

**History CONNECTION**

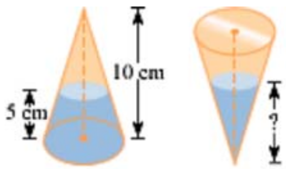
Any cross-section of Earth is close to a circle, so it can be divided into 360 degrees. Each degree can be divided into 60 minutes. The length of 1 minute of Earth's surface is called a nautical mile. However, Earth is not a perfect sphere, so the length of a nautical mile could vary depending on where you are on Earth's surface. Since 1959, all countries have agreed to define a nautical mile as 1.852 km. A knot is defined as a speed of 1 nautical mile per hour.

In the 1500s, ships would trail ropes with knots tied every 47 ft 3 in. The crew would measure the number of knots that were pulled into the water as a 28 s hourglass emptied. Counting the number of knots gave them their speed, in nautical miles per hour.

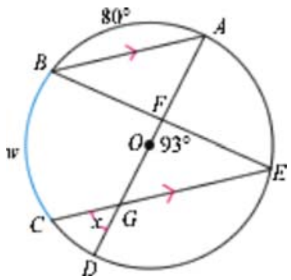


A print of an early view of New York Harbor

**18.** A sealed 10 cm tall cone resting on its base is filled to half its height with liquid. It is then turned upside down. To what height, to the nearest hundredth of a centimeter, does the liquid reach?




**19.** Find *w* and *x*.



LESSON 9.2

# Circles and Ellipses

*Before you can be eccentric you must know where the circle is.*

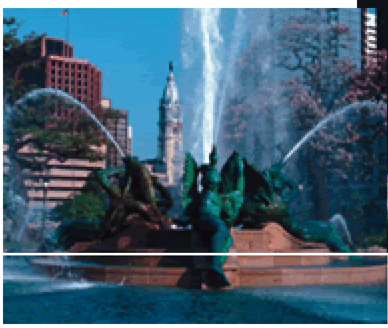
ELLEN TERRY

In the next few lessons, you will learn about circles, ellipses, parabolas, and hyperbolas. The orbital paths of the planets around the Sun are not exactly circular. These paths are examples of an important mathematical curve-the ellipse. The curved path of a stream of water from a water fountain, the path of a football kicked into the air, and the pattern of cables hanging between the towers of the Golden Gate Bridge are all examples of parabolas. The design of nuclear cooling towers, transmission gears, and the long-range navigational system known as LORAN all depend on hyperbolas. These curves all belong to a family of planar mathematical curves.

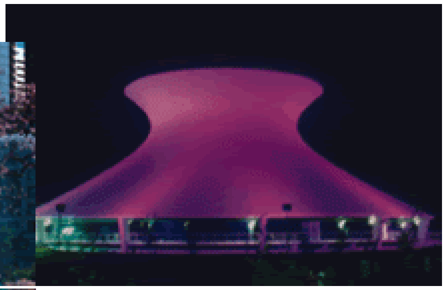
These curves-the circle, the ellipse, the parabola, and the hyperbola-are called **conic sections** because each can be created by slicing a double cone.

Sonia Delaunay's (1885-1979) oil-on-canvas painting, *Rhythm and Colors* (1939), contains many circles.

Dutch-German mathematician and cosmographer Andreas Cellarius (ca. 1595-1665) showed elliptical orbits in this celestial atlas titled *Harmonia Macrocosmica* (1660).



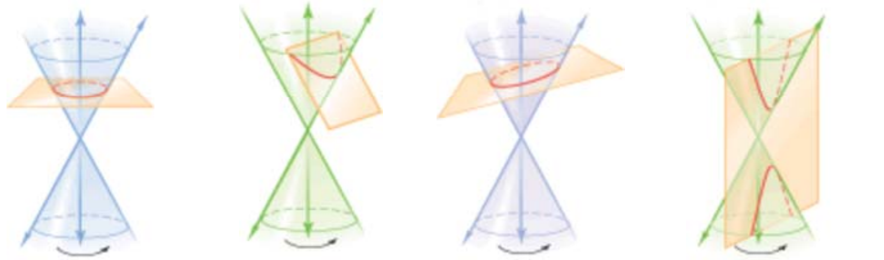
The Swann Memorial Fountain at Logan Circle in Philadelphia, Pennsylvania, was designed by American sculptor Alexander Stirling Calder (1898-1976). The jets of water form parabolas.



The McDonnell Planetarium, built in 1963 in St. Louis, Missouri, is a **hyperboloid,** a three- dimensional shape created by revolving a hyperbola.

When two lines meet at an acute angle, revolving one of the lines about the other creates a double cone. These cones do not have bases. They continue infinitely, like the original lines.

Slicing these cones with a plane at different angles produces different conic sections.



Circle | Parabola | Ellipse | Hyperbola

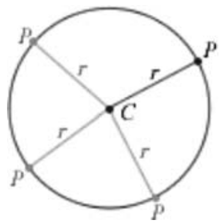
**History CONNECTION**

Students of mathematics have studied conic sections for more than 2000 years. Greek mathematician Menaechmus (ca. 380-320 B.C.E.) thought of conic sections as coming from different kinds of cones. Parabolas came from righ-tangled cones, ellipses from acute-angled cones, and hyperbolas from obtuse-angled cones. Apollonius of Perga (ca. 262-190 B.C.E.) later showed how all three kinds of curves could be obtained from the same cone. Known as the "Great Geometer," he wrote the eight-book *Treatise on Conic Sections* and named the ellipse, hyperbola, and parabola.

Conic sections have some interesting properties. Each of the shapes can also be defined as a locus of points. For example, all the points on a circle are the same distance from the center. So, you can describe a circle as a locus of points that are a fixed distance from a fixed point.

### Definition of a Circle

A **circle** is a locus of points *P* in a plane, that are located a constant distance, *r*, from a fixed point, *C*. Symbolically, $PC = r$. The fixed point is called the **center** and the constant distance is called the **radius.**




You can use the locus definition to write an equation that describes all the points on a circle.

*EXAMPLE A*

Write the equation for the locus of points $(x, y)$ that are 4 units from the point (0, 0).

**Solution**

The locus describes a circle with radius 4 and center (0, 0). The distance from each point of the circle, $(x, y)$, to the center of the circle, (0, 0), is 4. Using the distance formula, you can write

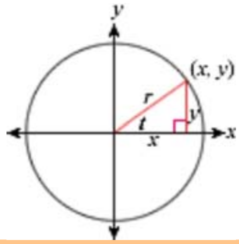
$$\sqrt{(x-0)^2+(y-0)^2}=4$$

Squaring both sides gives the equation

$$x^2+y^2=16$$

In general, the equation of a circle with center (0, 0) is $x^2 + y^2 = r^2$. In the exploration in Chapter 8, you also wrote the equation in parametric form, $x = r\cos t$ and $y = r\sin t$.

If the circle is translated horizontally and/or vertically, you can modify the equations by replacing $x$ with $(x - h)$ and replacing $y$ with $(y - k)$.




## Equation of a Circle

The **standard form** of the equation of a circle with center $(h, k)$ and radius $r$ is

$$(x - h)^2 + (y - k)^2 = r^2$$

or, in parametric form,

$$x = r\cos t + h$$
$$y = r\sin t + k$$

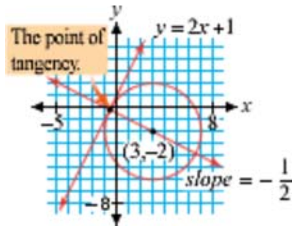
*EXAMPLE B*

A circle has center (3, – 2) and is tangent to the line $y = 2x + 1$. Write the equation of the circle.

### Solution

To write the equation of a circle, you need to know the center and the radius. You know the center, (3, – 2), but you need to find the radius.

Recall from geometry that a line tangent to a circle intersects the circle at only one point and is perpendicular to a diameter of the circle at the point of tangency. The tangent line has slope 2. A line perpendicular to this line will have slope $-\frac{1}{2}$. So, the line containing this diameter will have slope $-\frac{1}{2}$, and will pass through the center of the circle, (3, – 2).




Use this information to write the equation of the line that contains the diameter.

$$y = -2 - \frac{1}{2}(x - 3)$$

Now find the point of intersection of this line with the tangent line by solving the system of equations. The point of intersection is (– 0.6, – 0.2). You can now find the radius, which is the distance from the point of tangency to the center.

$$\sqrt{(3 + 0.6)^2 + (-2 + 0.2)^2} = \sqrt{16.2} \approx 4.025$$

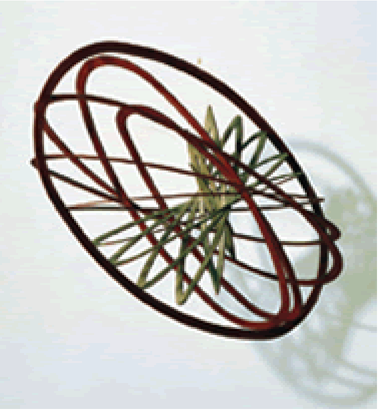
The radius of the circle is $\sqrt{16.2}$ units. Therefore, the equation of this circle is

$$(x - 3)^2 + (y + 2)^2 = 16.2$$

or, in parametric form,

$$x = \sqrt{16.2}\cos t + 3$$
$$y = \sqrt{16.2}\sin t - 2$$

In Chapter 4, you stretched a circle horizontally and vertically by different amounts to create ellipses. You can think of the equation of an ellipse as the equation of a unit circle that has been translated and stretched.



This plywood and wire elliptical sculpture by Russian artist Alexander Rodchenko (1891-1956) is titled *Oval Hanging Construction Number 12* (1920).

## Equation of an Ellipse

The standard form of the equation of an ellipse with center $(h, k)$, horizontal scale factor of $a$, and vertical scale factor of $b$ is

$$\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1$$

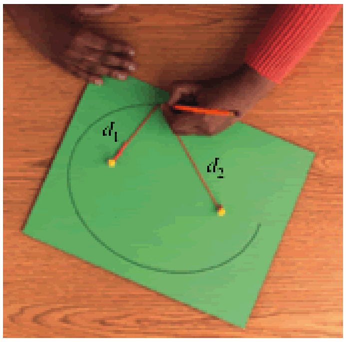
or, in parametric form,

$$x = a\cos t + h$$
$$y = b\sin t + k$$

An ellipse is like a circle, except that it involves two points called **foci** instead of just one point at the center. You can construct an ellipse by tying a string around two pins and tracing a set of points, as shown.

The sum of the distances, $d_1 + d_2$, is the same for any point on the ellipse.

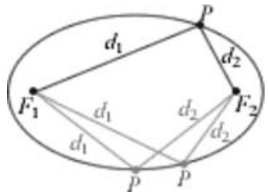
A piece of string attached to one pin helps you draw a circle. This is the same concept as using a compass to construct a circle.



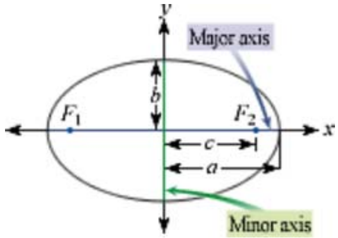

A piece of string attached to two pins helps you draw an ellipse.

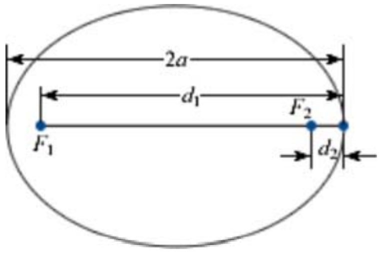
## Definition of an Ellipse

An **ellipse** is a locus of points $P$ in a plane, the sum of whose distances, $d_1$ and $d_2$, from two fixed points, $F_1$ and $F_2$, is always a constant, $d$. That is, $d_1 + d_2 = d$, or $F_1P + F_2P = d$. The two fixed points, $F_1$ and $F_2$, are called **foci** (the plural of focus).

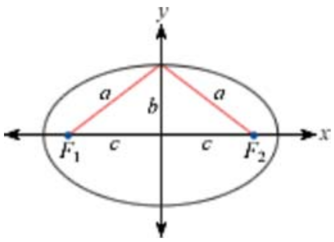


The segment that forms the longer dimension of an ellipse, and contains the foci, is the **major axis.** The shorter dimension is the **minor axis.** Recall from Chapter 4 that the length of half the horizontal axis of an ellipse corresponds to the horizontal scale factor, so the horizontal axis has length $2a$, and half the horizontal axis has length $a$. Similarly, half the vertical axis has length $b$. When the major axis is horizontal, the length of the major axis, $2a$, is equal to $d_1 + d_2$, as shown in the diagram below at right. So the sum of the distances between any point on the ellipse and the two foci is $2a$. What would be true if the major axis were vertical?

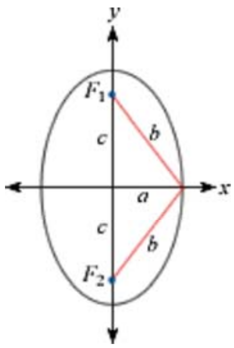



If you connect an endpoint of the minor axis to the foci, you form two congruent right triangles. For a horizontally oriented ellipse, the sum of the lengths of the hypotenuses is the same as the length of the major axis, so each hypotenuse has length $a$. Half the length of the minor axis is $b$. To locate the foci of the ellipse, call the distance from the center to each focus $c$, and write the equation $b^2 + c^2 = a^2$.




When the major axis is vertical, the hypotenuse of each triangle is equivalent to $b$, half the length of the major axis, so $a^2 + c^2 = b^2$.

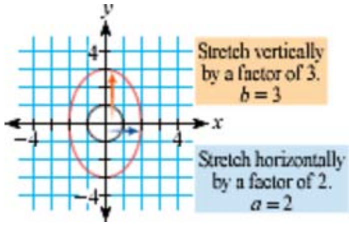


## EXAMPLE C

Graph an ellipse that is centered at the origin, with a vertical major axis of 6 units and a minor axis of 4 units. Where are the foci?

## ▶ Solution

Start with a unit circle, $x^2 + y^2 = 1$. The radius is 1 unit and the diameter is 2 units. You can stretch this circle vertically by a factor of 3 to make it 6 units tall. To make it 4 units wide, you must stretch it horizontally by a factor of 2.




Replace $y$ with $\frac{y}{3}$ and replace $x$ with $\frac{x}{2}$. The equation is

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \text{ or } \frac{x^2}{4} + \frac{y^2}{9} = 1$$


To sketch $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ by hand, plot the center, the endpoints of the major axis that are vertically 3 units from the center, and the endpoints of the minor axis that are horizontally 2 units from the center. Then connect the endpoints with a smooth curve.

To graph the ellipse on your calculator, you will need to solve for $y$. It then takes two equations to graph the entire shape.

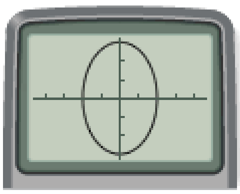
$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$ The equation in standard form.

$$\left(\frac{y}{3}\right)^2 = 1 - \left(\frac{x}{2}\right)^2$$ Subtract $\left(\frac{x}{2}\right)^2$ from both sides.

$$\frac{y}{3} = \pm\sqrt{1 - \left(\frac{x}{2}\right)^2}$$ Take the square root of both sides.

$$y = \pm 3\sqrt{1 - \left(\frac{x}{2}\right)^2}$$ Multiply both sides by 3.

Use your calculator to check the graph.

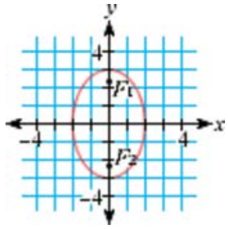


[– 4.7, 4.7, 1, –3.1, 3.1, 1]

To locate the foci, recall the relationship $a^2 + c^2 = b^2$ for an ellipse with a vertical major axis:

$$a^2 + c^2 = b^2$$
$$2^2 + c^2 = 3^2$$
$$4 + c^2 = 9$$
$$c^2 = 5$$
$$c = \pm\sqrt{5}$$

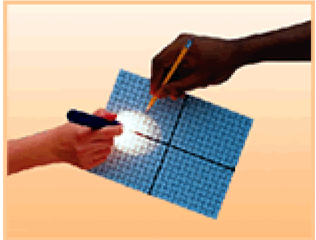
So, the foci are $\sqrt{5}$ units above and below the center, (0, 0). The coordinates of the foci are $(0, \sqrt{5})$ and $(0, -\sqrt{5})$, or approximately (0, 2.24) and (0, – 2.24).




In the investigation you will find the equation of an ellipse that you create yourself.

## Investigation
### A Slice of Light

**You will need**
- graph paper
- a flashlight
- a relatively dark classroom



The beam of a flashlight is close to the shape of a cone. A sheet of paper held in front of the flashlight shows different slices, or sections, of the cone of light.

Work with a partner, then share results with your group.

**Procedure Note**

1. Shine a flashlight on the graph paper at an angle.
2. Align the major axis of the ellipse formed by the beam with one axis of the paper. You might start by placing four points on the paper to help the person holding the flashlight stay on target.
3. Carefully trace the edge of the beam as your partner holds the light steady.

Step 1 Draw a pair of coordinate axes at the center of your graph paper. Follow the procedure note and trace an ellipse.

Step 2 Write an equation that fits the data as closely as possible. Find the lengths of both the major and minor axes. Use the values in your equation to locate the foci. Finally, verify your equation by selecting any two pairs of points on the ellipse and checking that the sum of the distances to the foci is constant.

**Eccentricity** is a measure of how elongated an ellipse is. Eccentricity is defined as the ratio $\frac{c}{a}$, for an ellipse with a horizontal major axis, or $\frac{c}{b}$, for an ellipse with a vertical major axis. If the eccentricity is close to 0, then the ellipse looks almost like a circle. The higher the ratio, the more elongated the ellipse.

Step 3 Use your flashlight to make ellipses with different eccentricities. Trace three different ellipses. Calculate the eccentricity of each one and label it on your paper. What is the range of possible values for the eccentricity of an ellipse?

Step 4 Continue to tilt your flashlight until the eccentricity becomes too large and you no longer have an ellipse. What shape can you trace now?

# EXERCISES

## Practice Your Skills

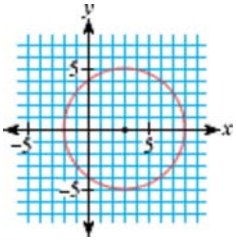
**1.** Sketch each circle on your paper, and label the center and the radius. For a-d, rewrite the equation as two functions. Use your calculator to check your work.

**a.** $x^2 + y^2 = 4$ **b.** $(x - 3)^2 + y^2 = 1$ **c.** $(x + 1)^2 + (y - 2)^2 = 9$

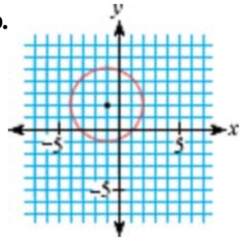
**d.** $x^2 + (y - 1.5)^2 = 0.25$ **e.** $x = 2\cos t + 1$, $y = 2\sin t + 2$ **f.** $x = 4\cos t - 3$, $y = 4\sin t$

**2.** Write an equation in standard form for each graph.

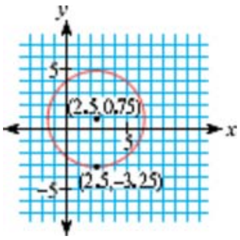
**a.**



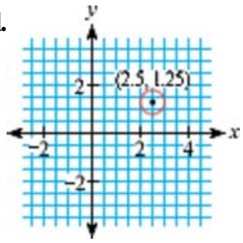

**b.**




**c.**




**d.**




**3.** Write parametric equations for each graph in Exercise 2.

**4.** Sketch a graph of each equation. Label the coordinates of the endpoints of the major and minor axes.

**a.** $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$

**b.** $\left(\frac{x-2}{3}\right)^2 + \left(\frac{y+2}{1}\right)^2 = 1$

**c.** $\left(\frac{x-4}{3}\right)^2 + \left(\frac{y-1}{3}\right)^2 = 1$

**d.** $y = \pm 2\sqrt{1 - \left(\frac{x+2}{3}\right)^2} - 1$

**e.** $x = 4\cos t - 1$
$y = 2\sin t + 3$

**f.** $x = 3\cos t + 3$
$y = 5\sin t$

The Meeting Center (1973), nicknamed "The Egg," at the Governor Nelson A. Rockefeller Empire State Plaza in Albany, New York, contains two interior auditoriums. It appears to be the lower half of an **ellipsoid,** a three-dimensional shape created by revolving an ellipse about one of its axes.

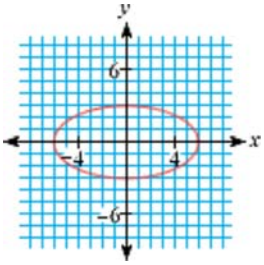
**5.** Write parametric equations for each circle described.

**a.** radius 2, center (0, 3)

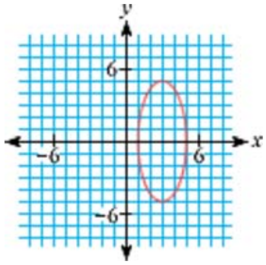
**b.** radius 6, center (– 1, 2)

**6.** Write an equation in standard form for each graph.

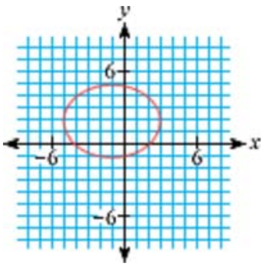
**a.**



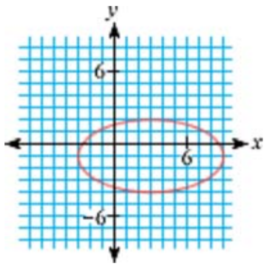

**b.**




**c.**




**d.**

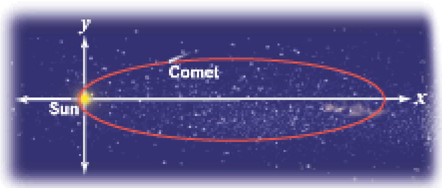



Keymath.com More Practice Your Skills

**7.** Find the exact coordinates of the foci for each ellipse in Exercise 6.

## Reason and Apply

**8.** Suppose you placed a grid on the plane of a comet's orbit, with the origin at the sun and the *x*-axis running through the longer axis of the orbit, as shown in the diagram. The table gives the approximate coordinates of the comet as it orbits the Sun. Both *x* and *y* are measured in astronomical units (AU).




| *x* | – 2.1 | 12.9 | 62.6 | 244.5 | 579.3 | 778.1 | 900.1 | 982.4 | 923.4 | 663.0 | 450.0 | 141.6 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *y* | 5.5 | 16.3 | 31.5 | 54.6 | 62.0 | 51.6 | 36.1 | 10.9 | – 31.5 | – 59.2 | – 62.8 | – 44.5 |

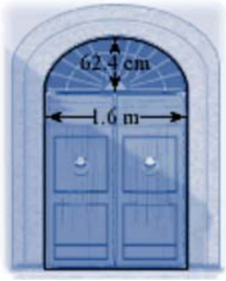
**a.** Find an equation to fit these data.

**b.** Find the *y*-coordinate when the *x*-coordinate is 493.0 AU.

**c.** What is the greatest distance of the comet from the Sun?

**d.** What are the coordinates of the foci?

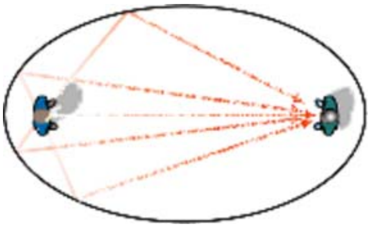
**9.** **APPLICATION** The top of a doorway is designed to be half an ellipse. The width of the doorway is 1.6 m, and the height of the half-ellipse is designed to be 62.4 cm. The crew have nails and string available. They want to trace the half-ellipse with a pencil before they cut the plywood to go over the doorway.




**a.** How far apart should they place the nails?

**b.** How long should the string be?

**10.** Read the connection below about the reflection property of an ellipse. If a room is constructed in the shape of an ellipse, and you stand at one focus and speak softly, a person standing at the other focus will hear you clearly. Such rooms are often called whispering chambers. Consider a whispering chamber that is 12 m long and 6 m wide.



The Whispering Gallery (1937) at Chicago's Museum of Science and Industry is constructed in an ellipsoid shape with two parabolic dishes that reflect the quietest sounds in perfect clarity from one dish's focus to the other's.

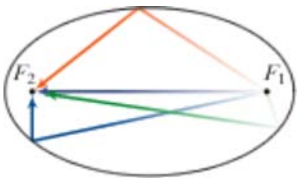
**a.** Where should two whisperers stand to talk to each other?

**b.** How far does the sound travel from one person to the other, bouncing off the wall in between?

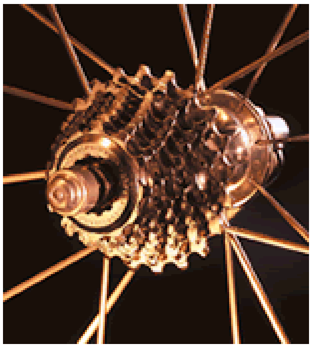
**Science CONNECTION**

A signal from one focus of an ellipse will always bounce off the ellipse in such a way that it will travel to the other focus.

This is called the reflection property of an ellipse.




**11.** **APPLICATION** One possible gear ratio on Matthew's mountain bike is 4 to 1. This means that the front gear has four times as many teeth as the gear on the back wheel. So each revolution of the pedal causes the rear wheel to make four revolutions.



**a.** If Matthew is pedaling 60 revolutions per minute (r/min), how many revolutions per minute is the rear tire making?

**b.** If the diameter of the rear tire is 26 in., what speed in miles per hour will Matthew attain?

**c.** Matt downshifts to a front gear that has 22 teeth and a rear gear that has 30 teeth. If he keeps pedaling 60 r/min, what will his new speed be?

**12.** The Moon's greatest distance from Earth is 252,710 mi, and its smallest distance is 221,643 mi. Write an equation that describes the Moon's orbit around Earth. Earth is at one focus of the Moon's elliptical orbit.

## Review

**13.** Write the equation of a parabola congruent to $y = x^2$ that has been reflected across the $x$-axis and translated left 3 units and up 2 units.

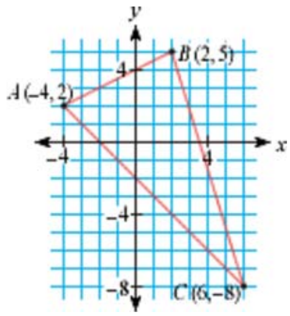
**14.** Solve for $y$.

$$\frac{y-4}{0.5} = \left(\frac{x+2}{3}\right)^2$$

**15.** Solve a system of equations and find the quadratic equation, $y = ax^2 + bx + c$, that fits these data points.

| **x** | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| **y** | 117 | 95 | 77 | 63 | 53 | 47 |

**16.** Find the perimeter of this triangle.




### IMPROVING YOUR REASONING SKILLS

***Elliptical Pool***

You could use the reflection property of an ellipse to design an unusual pool table. On an elliptical pool table, if you start with a ball at one focus and hit it in any direction, it will always rebound off the side and roll toward the other focus. Suppose a pool table is designed in the shape of an ellipse with a pocket at one focus. Describe how you will hit ball 1 to land in the pocket, even though ball 2 is in the way.

LESSON 9.3

# Parabolas

*There are two ways of spreading light: to be the candle or the mirror that reflects it.*

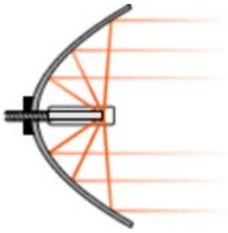
EDITH WHARTON

You have studied parabolas in several different lessons, and you have used parabolic equations to model a variety of situations. In this lesson you will study the parabola from a different perspective. In previous chapters you worked with parabolas as variations on the equation $y = x^2$ or the parametric equations $x = t$ and $y = t^2$. However, there is also a locus definition of a parabola.

Built in 1969, one side of this building in Odeillo, France, is a large parabolic mirror that focuses sunlight on the building's solar-powered furnace.

**Technology CONNECTION**

A reflecting telescope is a type of optical telescope that uses a curved, mirrored lens to magnify objects. The most powerful reflecting telescope uses a parabolic or hyperbolic mirror and can bring the faintest light rays into clear view. The larger the mirror, the more distant objects a telescope can detect. To avoid the expense and weight of producing one massive lens, today's most sophisticated telescopes have a tile-like combination of hexagonal mirrors that produce the same effect as one concave mirror.



The designs of telescope lenses, spotlights, satellite dishes, and other parabolic reflecting surfaces are based on a remarkable property of parabolas: A ray that travels parallel to the axis of symmetry will strike the surface of the parabola or paraboloid and reflect toward the **focus.** Likewise, when a ray from the focus strikes the curve, it will reflect in a ray that is parallel to the axis of symmetry. A **paraboloid** is a three-dimensional parabola, formed when a parabola is rotated about its line of symmetry.

**Science CONNECTION**

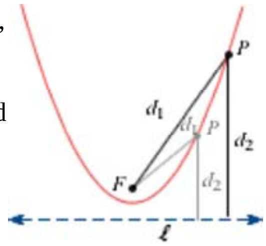
Satellite dishes, used for television, radio, and other communications, are always parabolic. A satellite dish is set up to aim directly at a satellite. As the satellite transmits signals to a dish, the signals are reflected off the dish surface and toward the receiver, which is located at the focus of the paraboloid. In this way, every signal that hits a parabolic dish can be directed into the receiver.

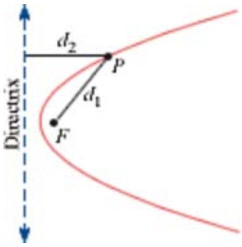
A large satellite dish

This reflective property of parabolas can be proved based on the locus definition of a parabola. You'll see how in the exercises. Compare this locus definition of a parabola to the locus definition of an ellipse.

## Definition of a Parabola

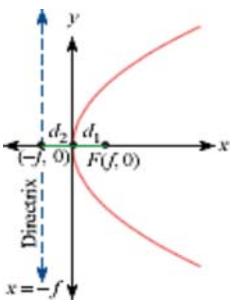
A **parabola** is a locus of points $P$ in a plane, whose distance from a fixed point, $F$, is the same as the distance from a fixed line, $\ell$. That is, $d_1 = d_2$. The fixed point, $F$, is called the **focus.** The line, $\ell$, is called the **directrix.**

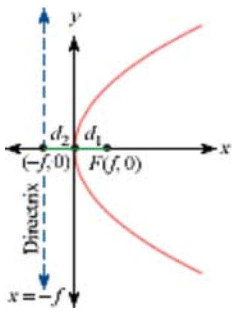


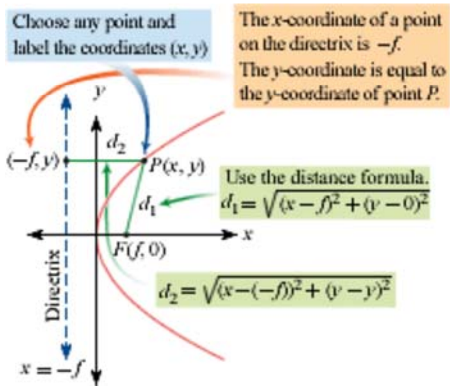




A parabola is the set of points for which the distances $d_1$ and $d_2$ are equal. If the directrix is a horizontal line, the parabola is vertically oriented, like the one in the definition box above. If the directrix is a vertical line, the parabola is horizontally oriented, like the one at left. The directrix can also be neither horizontal nor vertical, creating a parabola that is rotated at an angle.

How can you locate the focus of a given parabola? Suppose the parabola is horizontally oriented, with vertex (0, 0). It has a focus inside the curve at a point, $(f, 0)$, as shown in the first diagram below. The vertex is on the curve and will be the same distance from the focus as it is from the directrix, as shown in the second diagram. This means the equation of the directrix is $x = -f$. You can use this information, and the distance formula, to find the value of $f$ when the vertex is the origin, as shown in the third diagram.



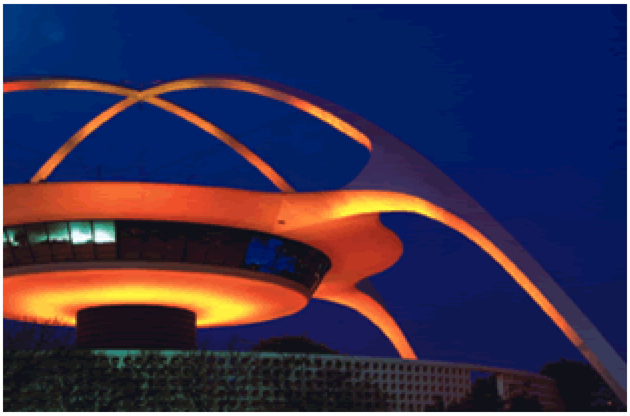




$$
\begin{array}{rcll}
\sqrt{(x-f)^2+(y-0)^2} &=& \sqrt{(x+f)^2+(y-y)^2} & \text{Definition of parabola states that } d_1 = d_2. \\
\sqrt{(x-f)^2+y^2} &=& \sqrt{(x+f)^2+(0)^2} & \text{Subtract.} \\
(x-f)^2+y^2 &=& (x+f)^2 & \text{Square both sides.} \\
x^2-2fx+f^2+y^2 &=& x^2+2fx+f^2 & \text{Expand.} \\
y^2 &=& 4fx & \text{Combine like terms.}
\end{array}
$$

This result means that the coefficient of the variable $x$ is $4f$, where $f$ is the distance from the vertex to the focus. What do you think it means if $f$ is negative?

If the parabola is vertically oriented, the $x$- and $y$-coordinates are exchanged, for a final equation of $x^2 = 4fy$, or $y = \frac{1}{4f}x^2$.



Designed in 1960, the Theme Building at Los Angeles International Airport in California uses double parabolic arches.

## EXAMPLE

Consider the parent equation of a horizontally oriented parabola, $y^2 = x$.

**a.** Write the equation of the image of this graph after the following transformations have been performed, in order: a vertical stretch by a factor of 3, a translation right 2 units, and then a translation down 1 unit. Graph the new equation.

**b.** Where is the focus of $y^2 = x$? Where is the directrix?

**c.** Where is the focus of the transformed parabola? Where is its directrix?

## Solution

Recall the transformations of functions that you studied in Chapter 4.

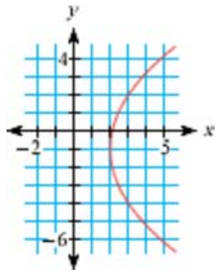
**a.** Begin with the parent equation, and perform the specified transformations.

$$y^2 = x$$ Original equation.

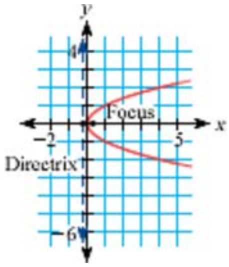
$$\left(\frac{y}{3}\right)^2 = x$$ Stretch vertically by a factor of 3.

$$\left(\frac{y+1}{3}\right)^2 = x - 2$$ Translate right 2 units and down 1 unit.

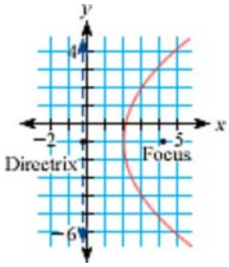
Graph the transformed parabola.



**b.** Use the general form, $y^2 = 4fx$, to locate the focus and the directrix of the graph of the equation $y^2 = x$. The coefficient of $x$ is $4f$ in the general form, and 1 in the equation $y^2 = x$. So, $4f = 1$, or $f = \frac{1}{4}$. Recall that $f$ is the distance from the vertex to the focus and from the vertex to the directrix. The vertex is $(0, 0)$, so the focus is $(\frac{1}{4}, 0)$ and the directrix is the line $x = -\frac{1}{4}$.




**c.** To locate the focus and the directrix of $\left(\frac{y+1}{3}\right)^2 = x - 2$, first rewrite the equation as $(y + 1)^2 = 9(x - 2)$. The coefficient of $x$ in this equation is 9, so $4f = 9$, or $f = 2.25$. The focus and the directrix will both be 2.25 units from the vertex in the horizontal direction. The vertex is $(2, -1)$, so the focus is $(4.25, -1)$ and the directrix is the line $x = -0.25$.




## Equation of a Parabola

The standard form of the equation of a vertically oriented parabola with vertex $(h, k)$, horizontal scale factor of $a$, and vertical scale factor of $b$ is

$$\frac{y-k}{b} = \left(\frac{x-h}{a}\right)^2$$

or, in parametric form,

$$x = at + h$$

$$y = bt^2 + k$$

The focus of a vertically oriented parabola is $(h, k + f)$, where $\frac{a^2}{b} = 4f$, and the directrix is the line $y = k - f$.

The standard form of the equation of a horizontally oriented parabola with vertex $(h, k)$, horizontal scale factor of $a$, and vertical scale factor of $b$ is

$$\left(\frac{y-k}{b}\right)^2 = \frac{x-h}{a}$$

or, in parametric form,

$$x = at^2 + h$$
$$y = bt + k$$

The focus of a horizontally oriented parabola is $(h + f, k)$, where $\frac{b^2}{a} = 4f$, and the directrix is the line $x = h - f$.

In the investigation you will construct a parabola. As you create your model, think about how your process relates to the locus definition of a parabola.

## Investigation
### Fold a Parabola

**You will need**
- patty paper
- graph paper

Fold the patty paper parallel to one edge to form the directrix for a parabola. Mark a point on the larger portion of the paper to serve as the focus for your parabola. Fold the paper so that the focus lies on the directrix. Unfold, and then fold again, so that the focus is at another point on the directrix. Repeat this many times. The creases from these folds should create a parabola. Lay the patty paper on top of a sheet of graph paper. Identify the coordinates of the focus and the equation of the directrix, and write an equation for your parabola.

## EXERCISES

### Practice Your Skills

**1.** For each parabola described, use the information given to find the location of the missing feature. It may help to draw a sketch.

**a.** If the focus is (1, 4), and the directrix is $y = -3$, where is the vertex?

**b.** If the vertex is (– 2, 2), and the focus is (– 2, – 4), what is the equation of the directrix?

**c.** If the directrix is $x = 3$, and the vertex is (6, 2), where is the focus?

**2.** Sketch each parabola, and label the vertex and line of symmetry.

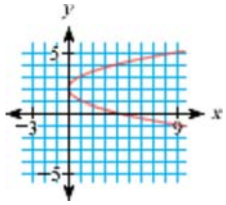
**a.** $\left(\frac{x}{2}\right)^2 + 5 = y$ **b.** $(y + 2)^2 - 2 = x$ **c.** $-(x + 3)^2 + 1 = 2y$

**d.** $2y^2 = -x + 4$ **e.** $x = 4t - 1$, $y = 2t^2 + 3$ **f.** $x = 3t^2 + 2$, $y = 5t$

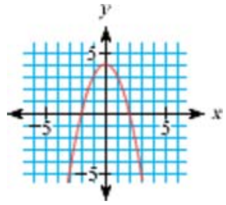
**3.** Locate the focus and directrix for each graph in Exercise 2.

**4.** Write an equation in standard form for each parabola.

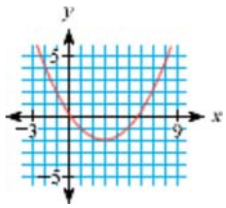
**a.**



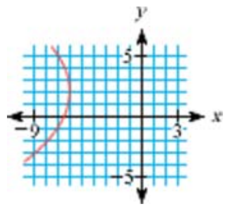

**b.**




**c.**




**d.**

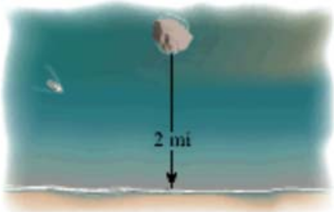


5. Write parametric equations for each parabola in Exercise 4.

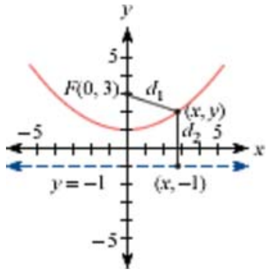
## Reason and Apply

6. Find the equation of the parabola with directrix $x = 3$ and vertex (0, 0).

7. The pilot of a small boat charts a course such that the boat will always be equidistant from an upcoming rock and the shoreline. Describe the path of the boat. If the rock is 2 miles offshore, write an equation for the path of the boat.




8. Consider the graph at right.




**a.** Because $d_1 = d_2$, you can write the equation

$$\sqrt{(x-0)^2+(y-3)^2}=\sqrt{(x-x)^2+(y+1)^2}$$

Rewrite this equation by solving for $y$.

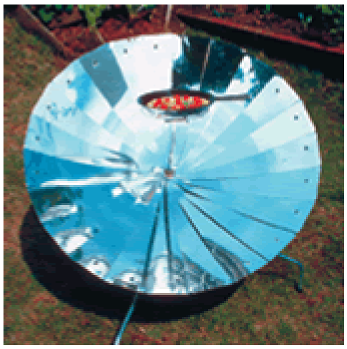
**b.** Describe the graph represented by your equation from 8a.

9. Write the equation of the parabola with focus (1, 3) and directrix $y = -1$.

10. **APPLICATION** Sheila is designing a parabolic dish to use for cooking on a camping trip. She plans to make the dish 40 cm wide and 20 cm deep. Where should she locate the cooking grill so that all of the light that enters the parabolic dish will be reflected toward the food?

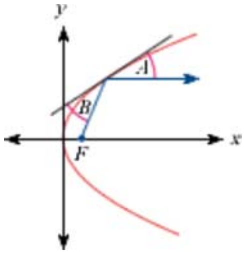
**Engineering CONNECTION**

Solar cookers focus the heat of the sun into a single spot, in order to boil water or cook food. A well-designed cooker can create heat up to 400°C. Solar cookers can be created with minimal materials, and can help save natural resources, particularly firewood. Inexpensive solar cookers are now being designed and distributed for use in developing countries. For more information on solar cookers, see the links at www.keymath.com/DAA .



A solar cooker fries an omelette.

11. The diagram at right shows the reflection of a ray of light in a parabolic reflector. The angles $A$ and $B$ are equal. Follow these steps to verify this property of parabolas.




a. Sketch the parabola $y^2 = 8x$.

b. What are the coordinates of the focus of this parabola?

c. On the same graph, sketch the line $y = 2x + 1$, tangent to the parabola. Find the coordinates of the point of tangency. What is the slope of this line?

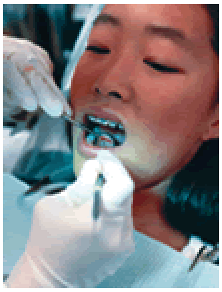
d. Sketch a ray through the point of tangency parallel to the axis of symmetry. What is the slope of this line?

e. Draw the segment from the point of intersection to the focus. What is the slope of this segment?

f. The formula $\tan A = \frac{m_2 - m_1}{1 + m_2 m_1}$ applies when $\angle A$ is the angle between two lines with slopes $m_1$ and $m_2$. Use this formula to find the angle between the tangent line and the horizontal line. Then find the angle between the tangent line and the segment joining the focus and the point of tangency. What do you notice about the angles?

**Technology CONNECTION**

The reflective backing in a flashlight is in the shape of a paraboloid. The light source is at the focus, and the light rays reflect off the backing and travel outward parallel to the axis of symmetry. But not all light backings are parabolic. Next time you're sitting in the dentist's chair, see if you can determine the shape of the reflective surface of the light used. It may be a parabola, but it could also be an ellipse. A light source at the focus of an elliptical reflector travels to the other focus of the ellipse, illuminating an area smaller than the area that would be illuminated by a parabolic reflector.



## Review

12. Find the equation that describes a parabola containing the points (3.6, 0.764), (5, 1.436), and (5.8, – 2.404).

13. Find the minimum distance from the origin to the parabola $y = -x^2 + 1$. What point(s) on the parabola is closest to the origin?

14. Find the equation of the ellipse with foci (– 6, 1) and (10, 1) that passes through the point (10, 13).

15. Consider the polynomial function $f(x) = 2x^3 - 5x^2 + 22x - 10$.

a. What are the possible rational roots of $f(x)$?

b. Find all rational roots.

c. Write the equation in factored form.

16. On a three-dimensional coordinate system with variables $x$, $y$, and $z$, the standard equation of a plane is in the form $ax + by + cz = d$. Find the intersection of the three planes described by $3x + y + 2z = -11$, $-4x + 3y + 3z = -2$, and $x - 2y - z = -3$.

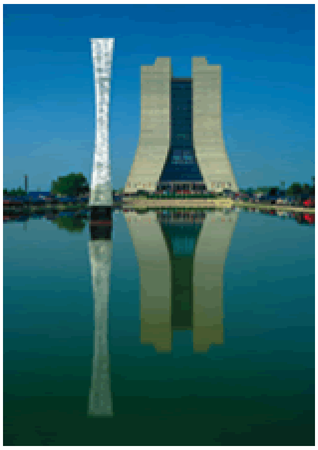
LESSON 9.4

# Hyperbolas

*The best paths usually lead to the most remote places.*

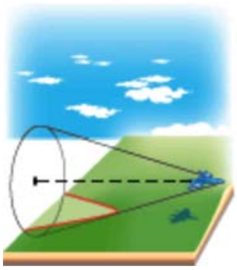
SUSAN ALLEN TOTH

The fourth, and final, conic section is the hyperbola. Comets travel in orbits that are parabolic, elliptical, or hyperbolic. A comet that comes close to another object, but never returns, is on a hyperbolic path. The two light shadows on a wall next to a cylindrical lampshade form two branches of a hyperbola. And a sonic-boom shock wave formed along the ground by a plane traveling faster than sound is one branch of a hyperbola.



*Acqua Alle Funi* stands before Wilson Hall at the Fermi National Accelerator Laboratory in Batavia, Illinois. The 32 ft sculpture is a hyperbolic obelisk designed by physicist and sculptor Robert Wilson (1914-2000), founder of the laboratory.

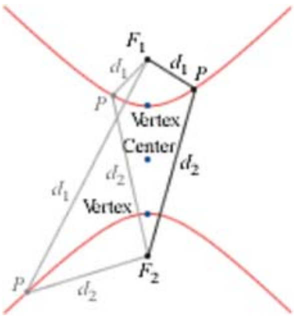
**Science CONNECTION**



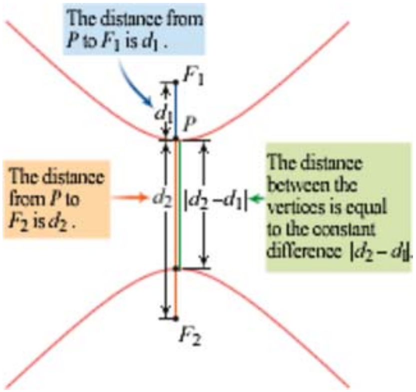
When an aircraft reaches the speed of its own sound, the airflow around the craft changes significantly, causing a disturbance in the air particles. If an aircraft catches up with its own noise, which travels ahead of it at a limited speed, the sound then compresses the air and piles up at the nose of the aircraft. This air causes a shock wave against the craft, which may cause it to vibrate or lock its controls. If the aircraft moves past the speed of sound, the shock waves fall behind the vehicle and cause a sonic boom. Because sonic booms are powerful enough to damage property on land and cause noise pollution, most supersonic flights take place above ocean waters. For more information on sonic booms, see the weblinks at www.keymath.com/DAA .

## Definition of a Hyperbola

A **hyperbola** is a locus of points $P$ in a plane, the difference of whose distances, $d_1$ and $d_2$, from two fixed points, $F_1$ and $F_2$, is always a constant, $d$. That is, $|d_2 - d_1| = d$, or $|F_2P - F_1P| = d$. The two fixed points, $F_1$ and $F_2$, are called **foci.** The points where the two branches of a hyperbola are closest to each other are called **vertices.** The **center** of a hyperbola is midway between the vertices.

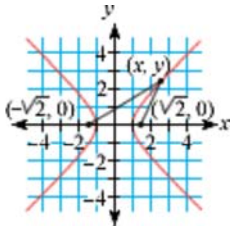


Regardless of where a point is on a hyperbola, the difference in the distances from the point to the two foci is constant. Notice that this constant is equal to the distance between the two vertices of the hyperbola. The hyperbola shown is oriented vertically. In Example A, you will see a hyperbola that is oriented horizontally.




EXAMPLE A

Just as the parent equation of any circle is a unit circle, $x^2 + y^2 = 1$, the parent equation of a hyperbola is called a **unit hyperbola.** The horizontally oriented unit hyperbola. has vertices (1, 0) and (–1, 0), and foci ($\sqrt{2}$, 0 ) and (–$\sqrt{2}$ , 0 ). The distance between the vertices is 2, so the difference in the distances from any point on the hyperbola to the two foci is 2. Find the equation of a unit hyperbola.




**Solution**

Label a point on the hyperbola $(x, y)$. Then use the definition of a hyperbola.

$$\left|\sqrt{(x+\sqrt{2})^2+y^2}-\sqrt{(x-\sqrt{2})^2+y^2}\right| = 2$$ Definition of hyperbola states that $|d_2 - d_1|$ is a constant, in this case 2.

$$\sqrt{(x+\sqrt{2})^2+y^2}-\sqrt{(x-\sqrt{2})^2+y^2} = 2$$ Consider the case $d_2 > d_1$.

$$\sqrt{(x+\sqrt{2})^2+y^2} = \sqrt{(x-\sqrt{2})^2+y^2} + 2$$ Add $\sqrt{(x-\sqrt{2})^2+y^2}$ to both sides.

$$(x+\sqrt{2})^2+y^2 = (x-\sqrt{2})^2 + y^2 + 4\sqrt{(x-\sqrt{2})^2+y^2} + 4$$ Square both sides.

$$x^2 + 2x\sqrt{2} + 2 + y^2 = x^2 - 2x\sqrt{2} + 2 + y^2 + 4\sqrt{(x-\sqrt{2})^2+y^2} + 4$$ Expand.

$$2x\sqrt{2} = -2x\sqrt{2} + 4\sqrt{(x-\sqrt{2})^2+y^2} + 4$$ Subtract $x^2$, $y^2$, and 2 from both sides.

$$4x\sqrt{2} - 4 = 4\sqrt{(x-\sqrt{2})^2+y^2}$$ Isolate the radical.

$$x\sqrt{2} - 1 = \sqrt{(x-\sqrt{2})^2+y^2}$$ Divide by 4.

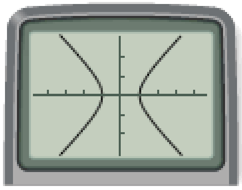
$$2x^2 - 2x\sqrt{2} + 1 = x^2 - 2x\sqrt{2} + 2 + y^2$$ Square both sides and expand.

$$x^2 - y^2 = 1$$ Combine like terms, and collect variables on one side of the equation.

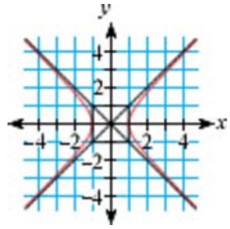
If you consider the case $d_1 > d_2$ you find the same equation for the unit hyperbola.

Check your answer by graphing on a calculator. First you must solve for $y$.

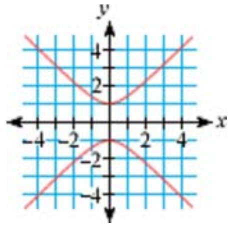
$$x^2 - y^2 = 1$$
$$-y^2 = 1 - x^2$$
$$y^2 = x^2 - 1$$
$$y = \pm\sqrt{x^2 - 1}$$



[-4.7, 4.7, 1, -3.1, 3.1, 1]




One special feature of a hyperbola is that each branch approaches two lines called **asymptotes.** Asymptotes are lines that a graph approaches as $x$- or $y$-values increase in the positive or negative direction. If you include the graphs of $y = x$ and $y = -x$ on the same coordinate axes, you will notice that they pass through the vertices of a square with corners at (1, 1), (1, –1), (–1, –1), and (–1, 1). The asymptotes are not a part of the hyperbola, but sometimes they are shown to help you see the behavior of the curve. Sketching asymptotes will help you graph hyperbolas more accurately.




The equation $y^2 - x^2 = 1$ also defines a hyperbola, shown at right. Look at the similarities to, and the differences from, the graph of $x^2 - y^2 = 1$. The features are similar, but the hyperbola is oriented vertically.

The equation of a hyperbola is similar to the equation of an ellipse, except that the terms are subtracted, rather than added. For example, the equation $\left(\frac{y}{4}\right)^2 - \left(\frac{x}{3}\right)^2 = 1$ describes a hyperbola, whereas $\left(\frac{y}{4}\right)^2 + \left(\frac{x}{3}\right)^2 = 1$ describes an ellipse. The standard form of the equation of a hyperbola centered at the origin is

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1 \text{ or } \left(\frac{y}{b}\right)^2 - \left(\frac{x}{a}\right)^2 = 1$$

where $a$ is the horizontal scale factor and $b$ is the vertical scale factor.

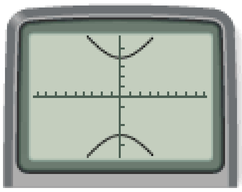
## EXAMPLE B

Graph $\left(\frac{y}{4}\right)^2 - \left(\frac{x}{3}\right)^2 = 1$.

## ▶ Solution

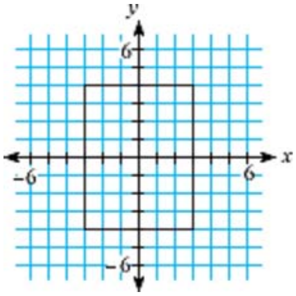
From the equation, you can tell that this is a vertically oriented hyperbola with a vertical scale factor of 4 and a horizontal scale factor of 3. The hyperbola is not translated, so its center is at the origin. To graph it on your calculator, you must solve for $y$.

$$\left(\frac{y}{4}\right)^2 - \left(\frac{x}{3}\right)^2 = 1$$
$$\left(\frac{y}{4}\right)^2 = 1 + \left(\frac{x}{3}\right)^2$$
$$\left(\frac{y}{4}\right) = \pm\sqrt{1 + \left(\frac{x}{3}\right)^2}$$
$$y = \pm 4\sqrt{1 + \left(\frac{x}{3}\right)^2}$$

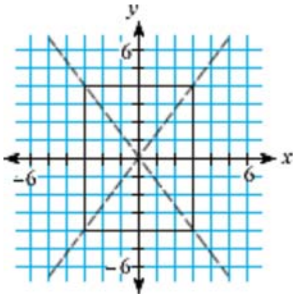


[-9.4, 9.4, 1, -6.2, 6.2, 1]

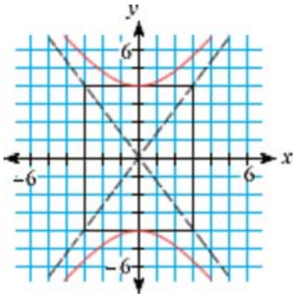
To sketch a hyperbola by hand, it is easiest if you begin by sketching the asymptotes. Start by drawing a rectangle centered at the origin that measures $2a$, or 6, units horizontally and $2b$, or 8, units vertically. The unit hyperbola begins with a 2-by-2 rectangle. Because this hyperbola is stretched horizontally and vertically, it begins with a 6-by-8 rectangle.

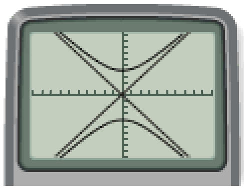



Draw the diagonals of this rectangle, and extend them outside the rectangle. These lines, with equations $y = \pm\frac{4}{3}x$, are the asymptotes of the hyperbola. In general, the slopes of the asymptotes of a hyperbola are $\pm\frac{b}{a}$.




Because this is a vertically oriented hyperbola, the vertices will lie on the top and bottom sides of the rectangle, at (0, 4) and (0, – 4). Add two curves such that each one touches a vertex and extends outward, approaching the asymptotes. You can graph the two asymptotes on your calculator to confirm that the hyperbola does approach them asymptotically.

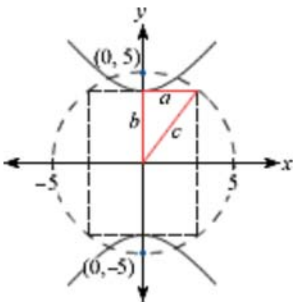


[– 10, 10, 1, – 10, 10, 1]

The location of foci in a hyperbola is related to a circle that can be drawn through the four corners of the asymptote rectangle. The distance from the center of the hyperbola to the foci is equal to the radius of the circle.

To locate the foci in a hyperbola, you can use the relationship $a^2 + b^2 = c^2$, where $a$ and $b$ are the horizontal and vertical scale factors. In the hyperbola from Example B, shown at right, $3^2 + 4^2 = c^2$, so $c = 5$, and the foci are 5 units above and below the center of the hyperbola at (0, 5) and (0, – 5).




In the investigation you will explore a situation that produces hyperbolic data and find a curve to fit your data.

# Investigation

## Passing By

**You will need**

- a motion sensor

**Procedure Note**

1. One member of your group will use a motion sensor to measure the distance to the walker for 10 s. The motion sensor must be kept pointed at the walker.
2. The walker should start about 5 m to the left of the sensor holder. He or she should walk at a steady pace, continuing past the sensor holder, and stop about 5 m to the right of the sensor holder. Be sure the walker stays at least 0.5 meter in front of the sensor holder.

Step 1 Collect data as described in the procedure note. Transfer these data from the motion sensor to each calculator in the group, and graph your data. They should form one branch of a hyperbola.

Step 2 Assume the sensor was held at the center of the hyperbola, and find an equation to fit your data. You may want to try to graph the asymptotes first.

Step 3 Transfer your graph to paper, and add the foci and the other branch of the hyperbola. To verify your equation, choose at least two points on the curve and measure their distances from the foci. Calculate the differences between the distances from each focus. What do you notice? Why?

### Equation of a Hyperbola

The standard form of the equation of a horizontally oriented hyperbola with center $(h, k)$, horizontal scale factor of $a$, and vertical scale factor of $b$ is

$$\left(\frac{x-h}{a}\right)^2-\left(\frac{y-k}{b}\right)^2=1$$

or, in parametric form,

$$x=\frac{a}{\cos t}+h$$
$$y=b\tan t+k$$

The equation of a vertically oriented hyperbola under the same conditions is

$$\left(\frac{y-k}{b}\right)^2-\left(\frac{x-h}{a}\right)^2=1$$

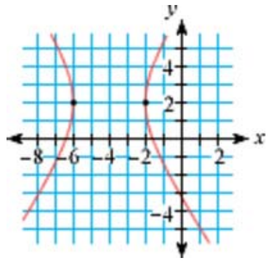
or, in parametric form,

$$x=a\tan t+h$$
$$y=\frac{b}{\cos t}+k$$

The foci are located $c$ units from the center, where $a^2+b^2=c^2$.
The asymptotes pass through the center and have slope $\pm\frac{b}{a}$.

## *EXAMPLE C*

Write the equation of this hyperbola in standard form, and find the foci.




## ▸ Solution

The center is halfway between the vertices, at the point (– 4, 2). The horizontal distance from the center to the vertex, *a*, is 2. If you knew the location of the asymptotes, you could find the value of *b* using the fact that the slopes of asymptotes of a hyperbola are $\pm\frac{b}{a}$. In this case, the value of *b* is not as easy to find. Write the equation, substituting the values you know.

$$\left(\frac{x+4}{2}\right)^2 - \left(\frac{y-2}{b}\right)^2 = 1$$

To solve for *b*, choose another point on the curve and substitute. It appears that the point (0, –3.2) is on the curve. Because this is an estimate, your value of *b* will be an approximation.

| | |
|---|---|
| $\left(\frac{0+4}{2}\right)^2 - \left(\frac{-3.2-2}{b}\right)^2 = 1$ | Substitute 0 for *x* and –3.2 for *y*. |
| $\left(\frac{x+4}{2}\right)^2 - \left(\frac{y-2}{b}\right)^2 = 1$ | Add and divide. |
| $4 - \frac{27.04}{b^2} = 1$ | Square. |
| $-\frac{27.04}{b^2} = -3$ | Subtract 4 from both sides. |
| $9.013 = b^2$ | Multiply by $b_2$ and divide by –3 on both sides. |
| $3.002 \approx b$ | Take the square root of both side |

The value of *b* is approximately 3, so the equation of the hyperbola is close to

$$\left(\frac{x+4}{2}\right)^2 - \left(\frac{y-2}{3}\right)^2 = 1$$

To write the equation in parametric form, you can simply substitute the values of *a*, *b*, *h*, and *k*, to get $x = \frac{2}{\cos t} - 4$ and $y = 3\tan t + 2$.

You can find the distance to the foci by using the equation $a^2 + b^2 = c^2$.

$$\begin{aligned} 2^2 + 3^2 &= c^2 \\ 13 &= c^2 \\ 13 &= c^2 \\ \pm\sqrt{13} &= c \end{aligned}$$

So, the foci are $\sqrt{13}$ to the right and left of the center at $(-4 + \sqrt{13}, 2)$ and $(-4 - \sqrt{13}, 2)$, or approximately (–0.39, 2) and (–7.6, 2).

You will continue to explore the relationship between the equation and graph of a hyperbola in the exercises.

# EXERCISES

## Practice Your Skills

**1.** Sketch each hyperbola on your paper. Write the coordinates of each vertex and the equation of each asymptote.

**a.** $\left(\frac{x}{2}\right)^2 - \left(\frac{y}{4}\right)^2 = 1$

**b.** $\left(\frac{y+2}{1}\right)^2 - \left(\frac{x-2}{3}\right)^2 = 1$

**c.** $\left(\frac{x-4}{3}\right)^2 - \left(\frac{y-1}{3}\right)^2 = 1$

**d.** $y = \pm 2\sqrt{1+\left(\frac{x+2}{3}\right)^2} - 1$

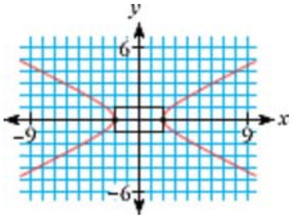
**e.** $x = \frac{4}{\cos t} - 1$
$y = 2\tan t + 3$

**f.** $x = 3\tan t + 3$
$y = \frac{5}{\cos t}$

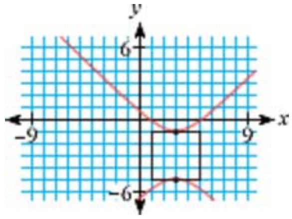
**2.** What are the coordinates of the foci of each hyperbola in Exercise 1?

**3.** Write an equation in standard form for each graph.

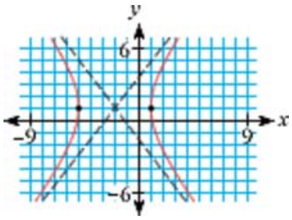
**a.**



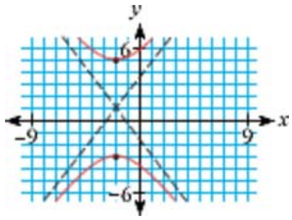

**b.**




**c.**




**d.**



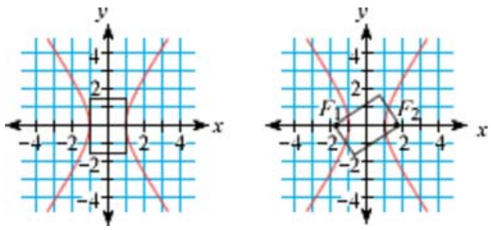

**4.** Write parametric equations for each graph in Exercise 3.

Keymath.com More Practice Your Skills

**5.** Write the equations of the asymptotes for each hyperbola in Exercise 3.

## Reason and Apply

**6.** Another way to locate the foci of a hyperbola is by rotating the asymptote rectangle about its center so that opposite corners lie on the line of symmetry that contains the vertices of the hyperbola. From the diagram, you can see that the distance from the origin to a focus is one-half the length of the diagonal of the rectangle.




**a.** Show that this distance is $\sqrt{a^2+b^2}$.

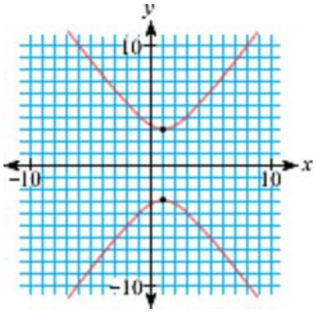
**b.** Find the coordinates of the foci for $\left(\frac{y+2}{1}\right)^2 - \left(\frac{x-2}{3}\right)^2 = 1$.

7. A point moves in a plane so that the difference of its distances from (– 5, 1) and (7, 1) is always 10 units. What is the equation of the path of this point?

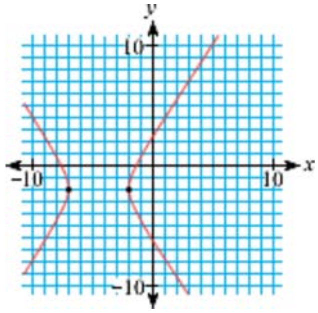
8. Graph and write the equation of a hyperbola that has an upper vertex at (– 2.35, 1.46) and has an asymptote of $y = 1.5x + 1.035$.

9. Approximate the equation of each hyperbola shown.

a.




a.




10. **APPLICATION** A receiver can determine the distance to a homing transmitter by its signal strength. These signal strengths were measured using a receiver in a car traveling due north.

| **Distance (mi)** | 0.0 | 2.0 | 4.0 | 6.0 | 8.0 | 10.0 | 12.0 | 14.0 | 16.0 |
|---|---|---|---|---|---|---|---|---|---|
| **Signal strength (W/m$^2$)** | 9.82 | 7.91 | 6.04 | 4.30 | 2.92 | 2.55 | 3.54 | 5.15 | 6.96 |

a. Find the equation of the hyperbola that best fits the data.

b. Name the center of this hyperbola. What does this point tell you?

c. What are the possible locations of the homing transmitter?

11. Sketch the graphs of the conic sections in 11a-d.

a. $y = x^2$ b. $x^2 + y^2 = 9$ c. $\frac{x^2}{9} + \frac{y^2}{16} = 1$ d. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

e. If each of the curves in 11a–d is rotated about the y-axis, describe the shape that is formed. Include a sketch.

**Mathematics CONNECTION**

One area in the study of calculus is the analysis of three-dimensional solids formed by revolving a curve about an axis. Revolving a hyperbola about the line through its foci or about the perpendicular bisector of the segment connecting the foci produces a hyperboloid. The hyperboloid is used in the design of cooling towers because the concrete shell can be relatively thin for its large size. Also, the structure of the hyperboloid allows cooling towers to use a natural draft design to bring air into the cooling process.

Cooling towers in Middletown, Pennsylvania

**12.** Find the vertical distance between a point on the hyperbola $\left(\frac{y+1}{2}\right)^2 - \left(\frac{x-2}{3}\right)^2 = 1$ and its nearest asymptote for each $x$-value shown at right.

| *x*-value | 5 | 10 | 20 | 40 |
|---|---|---|---|---|
| Distance | | | | |

## Review

**13.** Solve the quadratic equation $0 = -x^2 + 6x - 5$ by completing the square.

**14.** Mercury's orbit is an ellipse with the Sun at one focus, eccentricity 0.206, and major axis approximately $1.158 \times 10^8$ km. If you consider Mercury's orbit with the Sun at the origin and the other focus on the positive $x$-axis, what equation models the orbit?

**15.** The setter on a volleyball team makes contact with the ball at a height of 5 ft. The parabolic path of the ball reaches a maximum height of 17.5 ft when the ball is 10 ft from the setter.

**a.** Find an equation that models the ball's path.

**b.** A hitter can spike the ball when it is 8.5 feet off the floor. How far from the setter is the hitter when she makes contact?

**16.** Sketch the graph of each parabola. Give the coordinates of each vertex and focus, and the equation of each directrix.

**a.** $y = -(x+1)^2 - 2$ **b.** $y = \frac{1}{2}x^2 - 3x + 5$ **c.** $x = \frac{1}{2}t^2 - 6$, $y = t$

**17.** The half-life of radium-226 is 1620 yr.

**a.** Write a function that relates the amount $s$ of a sample of radium-226 remaining after $t$ years.

**b.** After 1000 yr, how much of a 500 g sample of radium-226 will remain?

**c.** How long will it take for a 3 kg sample of radium-226 to decay so that only 10 g remains?

**18.** Use finite differences to write an equation for the $n$th term of the sequence $-20, -14, -6, 4, 16, \ldots$.

### IMPROVING YOUR VISUAL THINKING SKILLS

***Slicing a Cone***

Describe how to slice a double cone to produce each of these geometric shapes: a circle, an ellipse, a parabola, a hyperbola, a point, one line, and two lines. Be sure to describe at what angle and where the plane must slice the cone. Sketch a diagram of each slicing.
(*Hint*: Look back at the illustrations on page 497 for help.)

EXPLORATION

# Constructing the Conic Sections

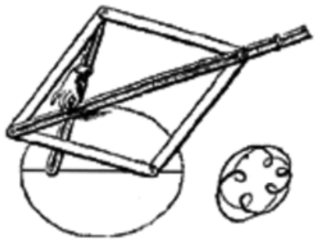
In geometry class you probably used a compass and straightedge to construct polygons, such as triangles and squares. You also know that a compass can easily construct a circle, one of the conic sections.

What about the other conic sections-the ellipse, the parabola, and the hyperbola? How could you possibly construct these complex curves? In this chapter you learned locus definitions for these shapes, but they may seem impossible to construct geometrically.

Actually, there are many different ways to construct the conic sections. Geometry software, such as The Geometer's Sketchpad, makes these constructions even easier. In the activity you will learn one way to construct an ellipse. The follow-up questions then challenge you to construct a parabola and a hyperbola.

**History CONNECTION**

In 1646, Dutch mathematician Frans van Schooten (1615-1660) wrote *Sive de Organica Conicarum Sectionum in Plano Descriptione, Tractatus*, which translates to A *Treatise on Devices for Drawing Conic Sections*. This book describes several different ways to construct each of the conic sections. Some of the constructions used unique mechanical devices.



These illustrations from van Schooten's *Sive de Organica Conicarum Sectionum in Plano Descriptione, Tractatus* show two ways of constructing an ellipse.

## Activity

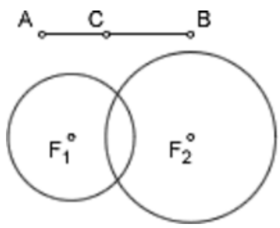
### From Circles to the Ellipse

Step 1 In a new sketch, construct a segment and label the endpoints $A$ and $B$.

Step 2 Construct a point on $\overline{AB}$. Label this point $C$.

Step 3 Construct segments $AC$ and $CB$. What is true about $AC + CB$?

Step 4 Construct and label points $F_1$ and $F_2$, not on $\overline{AB}$. These will be the foci of your ellipse.



Step 5 Select $F_1$ and $\overline{AC}$, and choose **Circle By Center+Radius** from the Construct menu. Construct another circle with $F_2$ and $\overline{CB}$.

Step 6 If necessary, adjust your model so that the circles intersect. Select the two circles and choose **Intersections** from the Construct menu. What is the distance from either intersection to $F_1$? What is the distance from either intersection to $F_2$? What is true about the sum of these distances?

Step 7 Select the two intersections and choose **Trace Intersections** from the Display menu. Slowly drag point $C$ back and forth along $\overline{AB}$. Describe the shape of the trace. Explain why this happens.

In this chapter you learned about loci. In Step 7 you traced a locus of points. Depending on how slowly you dragged point $C$, your ellipse may be smooth or it may have gaps. Sketchpad also allows you to see the locus as a smooth curve.

Step 8 Choose **Erase Traces** from the Display menu. Then select the two intersections and deselect **Trace Intersections** from the Display menu.

Step 9 Select point $C$ and one of the intersections, and choose **Locus** from the Construct menu. You should see one half of your ellipse. Repeat this process with the other intersection to see both halves.

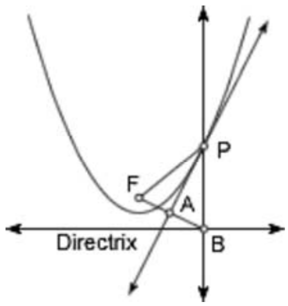
When you construct a locus, you can experiment by dragging objects and seeing how the locus changes.

## Questions

**1.** Experiment by dragging one focus. How does the ellipse change? How far apart can the foci be before you no longer have an ellipse? What happens when both foci are at the same point?

**2.** Experiment by dragging point $B$. How does the ellipse change? Based on the locus definition of an ellipse, what is changing?

**3.** The illustration at right shows one method of constructing a parabola based on the geometric definition. Use Sketchpad to construct this locus. Explain how the construction satisfies the definition. (Note that $\overleftrightarrow{PA} \perp \overline{FB}$.)




**4.** Find a way to construct a hyperbola with Sketchpad. (*Hint:* One possible construction is similar to the ellipse construction, except you want the difference $AC - BC$ to have a constant value.)

LESSON

9.5

*A complex system that works is invariably found to have evolved from a simple system that works.*

JOHN GAULE

# The General Quadratic

Circles, parabolas, ellipses, and hyperbolas are called **quadratic curves,** or seconddegree curves, because the highest power of any of the variables is 2. Although the curves look very different, all conic sections are closely related. All of the standard forms you have seen can be converted into one general equation.

## General Quadratic Equation

The **general quadratic equation** in two variables, $x$ and $y$, is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

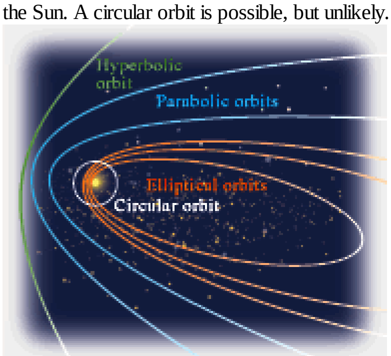
where $A$, $B$, and $C$ are not all zero.

In this lesson you will learn how to convert the general quadratic equation into standard form. You will also learn how to solve for $y$ so that you can use your calculator to graph the curves.

In all the relationships you have seen so far in this chapter, $B$ is equal to zero. When $B$ does not equal zero, the graph of the equation will be a rotated conic section. You can explore these rotated curves in the exploration that follows this lesson.

**Astronomy CONNECTION**

When a comet passes through the solar system, its motion is influenced by the Sun's gravity. Some comets swing around the Sun and leave the solar system, never to return. The paths are described by one branch of a hyperbola with the Sun at one focus. However, if a comet is moving more slowly, it will be captured into an elliptical orbit with the Sun at one focus. The outcome depends on the speed of the comet and the angle at which it approaches the Sun. Some orbits appear parabolic, but are probably very long ellipses. To have a parabolic orbit, a comet would have to start with a velocity of 0, at an infinite distance from the Sun. A circular orbit is possible, but unlikely.




In 1997, the comet Hale-Bopp passed by Earth and was one of the brightest comets seen in the 20th century. Hale-Bopp has an elliptical orbit.

## EXAMPLE A

Convert the equation $4x^2 - 9y^2 + 144 = 0$ to standard form. Then name the shape and graph it. Solve for $y$ and graph on your calculator to confirm your answer.

### ▶ Solution

Put the equation in standard form:

$$4x^2 - 9y^2 = -144$$ Subtract 144 from both sides.

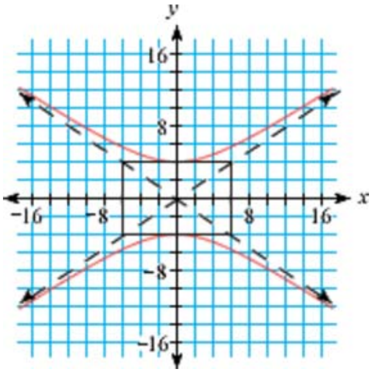
$$\frac{4x^2 - 9y^2}{-144} = 1$$ Divide by − 144.

$$-\frac{x^2}{36} + \frac{y^2}{16} = 1$$ Divide each term by − 144 and reduce the fractions.

$$\frac{y^2}{16} - \frac{x^2}{36} = 1$$ Reorder to put in standard form.

*Tractricious*, another Robert Wilson-designed sculpture at the Fermi National Accelerator Laboratory, is a free-standing hyperboloid of stainless steel tubes.

This equation is the standard form of a vertically oriented hyperbola. It is centered at the origin and has values $a = \sqrt{36} = 6$ and $b = \sqrt{16} = 4$. Sketch the asymptote rectangle and the asymptotes, then plot the vertices and draw the curve.




Now solve for $y$:

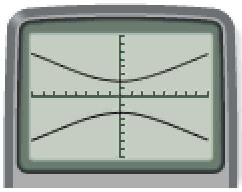
$$\frac{y^2}{16} - \frac{x^2}{36} = 1$$ The equation in standard form.

$$\frac{y^2}{16} = 1 + \frac{x^2}{36}$$ Add $\frac{x^2}{36}$ to both sides.

$$y^2 = 16\left(1 + \frac{x^2}{36}\right)$$ Multiply by 16.

$$y = \pm 4\sqrt{1 + \frac{x^2}{36}}$$ Take the square root of both sides.

Graph this equation on your calculator to confirm the sketch.



[- 16, 16, 2, - 16, 16, 2]

The equation in Example A was relatively easy to work with because the values of $B$, $D$, and $E$ in the general equation were zero. When $D$ or $E$ is nonzero, you must use the process of completing the square to convert the equation to standard form.

**EXAMPLE B** Describe the shape determined by the equation $x^2 + 4y^2 - 14x + 33 = 0$.

**Solution** Complete the square to convert from general form to standard form.

| | |
|---|---|
| $x^2 + 4y^2 - 14x + 33 = 0$ | Original equation. |
| $(x^2 - 14x) + (4y^2) = -33$ | Group $x$-terms and $y$-terms and isolate constants on the other side. |
| $(x^2 - 14x + 49) + (4y^2) = -33 + 49$ | To complete the square for $x$, add $\left(-\frac{14}{2}\right)^2$, or 49, to both sides. |
| $(x - 7)^2 + 4y^2 = 16$ | Write the equation in perfect-square form. |
| $\frac{(x-7)^2}{16} + \frac{y^2}{4} = 1$ | Divide by 16 and reduce. |
| $\left(\frac{x-7}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ | Write the equation in standard form. |

In this form, it is clear that this is the equation of an ellipse. The center is (7, 0), the horizontal stretch factor is 4, and the vertical stretch factor is 2.

Look back at the original equation in Example B. How might you have known that this equation described an ellipse?

**EXAMPLE C** Graph the equation $y^2 - 4x + 6y + 1 = 0$.

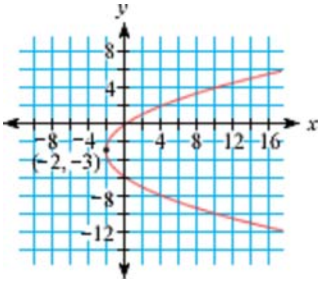
**Solution** Begin by completing the square.

| | |
|---|---|
| $(y^2 + 6y) - (4x) = -1$ | Group $x$-terms and $y$-terms and isolate constants on the other side. |
| $(y^2 + 6y + 9) - (4x) = -1 + 9$ | To complete the square for $y$, add 9 to both sides. |
| $(y + 3)^2 - 4x = 8$ | Write in perfect-square form. |

This equation describes a parabola, because only one of the variables has an exponent of 2. You can now choose whether to convert this equation to standard form to graph it, or solve for $y$. Let's solve for $y$.

| | |
|---|---|
| $(y + 3)^2 = 4x + 8$ | Add $4x$ to both sides. |
| $(y + 3)^2 = 4(x + 2)$ | Factor. |
| $y + 3 = \pm\sqrt{4(x + 2)}$ | Take the square root of both sides. |
| $y = \pm 2\sqrt{x + 2} - 3$ | Subtract 3 from both sides. |

This equation indicates a horizontally oriented parabola with vertex $(-2, -3)$ and a vertical scale factor of 2.



You could tell that the original equation in Example C described a parabola, because only one of the variables has an exponent of 2. The standard form of a parabola gives you information about the focus and directrix. However, if you only want to graph a parabola, you can solve for $y$ quickly by using the quadratic formula. Recall that the quadratic formula states that the solutions of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

**EXAMPLE D**

Use the quadratic formula to solve the equation $y^2 - 4x + 6y + 1 = 0$ for $y$.

**Solution**

In this case the variable $y$ is the quadratic term, so rewrite the equation in the form $ay^2 + by + c = 0$.

| | |
|---|---|
| $y^2 - 4x + 6y + 1 = 0$ | Original equation. |
| $y^2 + 6y - 4x + 1 = 0$ | Write the equation as $ay^2 + by + c = 0$. |

So $a = 1$, $b = 6$, and $c = -4x + 1$. Substitute these values into the quadratic formula.

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-6 \pm \sqrt{6^2 - 4(1)(-4x+1)}}{(2)(1)}$$

$$= \frac{-6 \pm \sqrt{36 + 16x - 4}}{2}$$

$$= \frac{-6 \pm \sqrt{32 + 16x}}{2}$$

$$= \frac{-6 \pm \sqrt{16(2+x)}}{2}$$

$$= \frac{-6 \pm 4\sqrt{2+x}}{2}$$

$$y = -3 \pm 2\sqrt{2+x}$$

The solution gives the equation $y = -3 \pm 2\sqrt{2+x}$.

This equation is equivalent to the equation you found in Example C by completing the square.

How can you find the points of intersection of two conic sections? In this investigation you'll begin by exploring how many intersection points are possible for various combinations of curves.

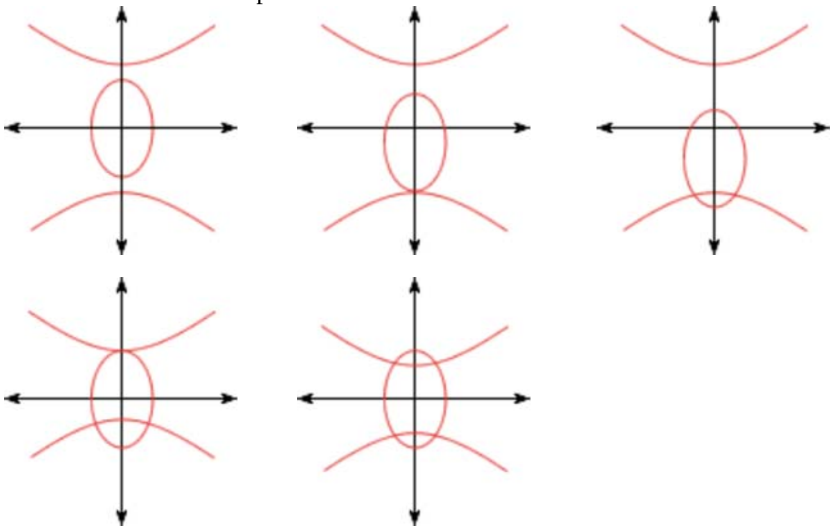
*Stringed Figure (Curlew)* (1956) was designed by British abstract sculptor Barbara Hepworth (1903-1975). What conic sections appear to be created by the intersecting strings?

## Investigation

### Systems of Conic Equations

If you graph two conic sections on the same graph, in how many ways could they intersect?

There are four conic sections: circles, ellipses, parabolas, and hyperbolas. Among the members of your group, investigate the possible numbers of intersection points for all ten pairs of shapes. For example, an ellipse and a hyperbola could intersect in 0, 1, 2, 3, or 4 points, as shown below. For each pair of conic sections, list the possible numbers of intersection points.



**History CONNECTION**

A method of constructing an ellipse was discovered by Abū Ali Al-Hasan ibn al-Haytham (or Alhazen), who lived from about 965 to 1041 C.E. in Basra, Iraq, and Alexandria, Egypt. His book *Optics* explained light and vision using geometry. He posed a problem, still famous today as "Alhazen's problem," about the reflection of light rays on a spherical surface. The solution can be found geometrically as the intersection of a great circle of the sphere and a hyperbola that passes through the center of the circle.

**EXAMPLE E** Find the points of intersection of $\frac{(x-5)^2}{4} + y^2 = 1$ and $x = y^2 + 5$.

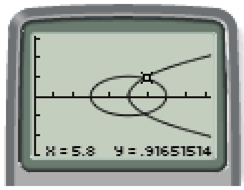
**Solution** First, graph the curves and estimate the points of intersection. Then you can solve the system algebraically.

You can graph these equations using your knowledge of transformations, or by solving for $y$ and graphing on your calculator. Graphing on the calculator provides the advantage that you can trace the curves to approximate the points of intersection. First, solve for $y$ in both equations.

| | |
|---|---|
| $\frac{(x-5)^2}{4} + y^2 = 1$ | The first equation. |
| $y^2 = 1 - \frac{(x-5)^2}{4}$ | Solve for $y^2$. |
| $y = \pm\sqrt{1 - \frac{(x-5)^2}{4}}$ | Solve for $y$. |
| $x = y^2 + 5$ | The second equation. |
| $y^2 = x - 5$ | Solve for $y^2$. |
| $y = \pm\sqrt{x-5}$ | Solve for $y$. |

Graph the two equations, and trace to approximate the points of intersection.

There are two points of intersection, approximately (5.8, 0.9) and (5.8, – 0.9). You can use algebraic methods to find the intersection points more accurately. To solve algebraically, you can use the two equations that are solved for $y$ and the substitution method. Or, in this case, you can use the two original equations and the



[0, 9.4, 1, -3.1, 3.1, 1]

substitution or elimination method. Notice that both equations have a $y^2$ term. Solve for $y^2$ in the second equation, and substitute.

| | |
|---|---|
| $\frac{(x-5)^2}{4} + y^2 = 1$ and $x = y^2 + 5$ | Original equations. |
| $y^2 = x - 5$ | Solve the second equation for $y^2$. |
| $\frac{(x-5)^2}{4} + x - 5 = 1$ | Substitute $(x - 5)$ for $y^2$ in the first equation. |
| $(x-5)^2 + 4x - 20 = 4$ | Multiply both sides by 4 to eliminate the denominator. |
| $x^2 - 10x + 25 + 4x - 20 = 4$ | Square. |
| $x^2 - 6x + 1 = 0$ | Combine like terms. |
| $x = \frac{6 \pm \sqrt{36 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \approx 5.828$ and $0.172$ | Use the quadratic equation to solve for $x$. |

Now substitute these two values into one of the equations relating $x$ and $y$, and solve for $y$.

$$y = \pm\sqrt{x-5} = \pm\sqrt{5.828 - 5} = \pm 0.910$$
$$y = \pm\sqrt{x-5} = \pm\sqrt{0.172 - 5} = \pm\sqrt{-4.828} = \pm 2.197i$$

The points of intersection are (5.828, 0.910) and (5.828, – 0.910). When you substitute 0.172 for $x$, you find two imaginary values for $y$. These nonreal numbers are solutions, but they are not points of intersection. The intersection points you estimated by graphing are close to the two real solutions.

You can always use graphing to estimate real solutions. Graphing is valuable even if you are finding solutions algebraically, because a graph will tell you how many intersection points to look for. It can also help confirm whether your algebraic solutions are correct.

## EXERCISES

You will need

**Geometry software** for Exercise 9

### Practice Your Skills

1. Rewrite each equation in the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

   **a.** $(x + 7)^2 = 9(y - 11)$ **b.** $\frac{(x-7)^2}{9} + \frac{(y+11)^2}{1} = 1$

   **c.** $(x - 1)^2 + (y + 3)^2 = 5$ **d.** $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$

2. Find the values for $a$, $b$, $c$, and $d$ as you follow these steps to complete the square for $15x^2 + 21x$.

   $15x^2 + 21x$

   $15(x^2 + ax)$

   $15(x^2 + ax + b) - 15b$

   $15(x^2 + ax + b) - c$

   $15(x + d)^2 - c$

3. Convert each equation to the standard form of a conic section. Name the shape described by each equation.

   **a.** $x^2 - y^2 + 8x + 10y + 2 = 0$ **b.** $2x^2 + y^2 - 12x - 16y + 10 = 0$

   **c.** $3x^2 + 30x + 5y - 4 = 0$ **d.** $5x^2 + 5y^2 + 20x - 6 = 0$

4. Identify each equation as true or false. If it is false, correct it to make it true.

   **a.** $y^2 + 11y + 121 = (y + 11)^2$ **b.** $x^2 - 18x + 81 = (x - 9)^2$

   **c.** $5y^2 + 10y + 5 = 5(y + 1)^2$ **d.** $4x^2 + 24x + 36 = 4(x + 6)^2$

keymath.com More Practice Your Skills

### Reason and Apply

5. Use the quadratic formula to solve each equation for $y$. Graph each curve.

   **a.** $25x^2 - 4y^2 + 100 = 0$ **b.** $4y^2 - 10x + 16y + 36 = 0$

   **c.** $4x^2 + 4y^2 + 24x - 8y + 39 = 0$ **d.** $3x^2 + 5y^2 - 12x + 20y + 8 = 0$

6. Solve each system of equations algebraically, using the substitution or elimination method.

   **a.** $\begin{cases} y = x^2 + 4 \\ y = (x-2)^2 + 3 \end{cases}$ **b.** $\begin{cases} 3x^2 + 9y^2 = 9 \\ 3x^2 + 5y^2 = 8 \end{cases}$ **c.** $\begin{cases} x^2 - \frac{y^2}{4} = 1 \\ x^2 + (y+4)^2 = 9 \end{cases}$

7. **APPLICATION** Two seismic monitoring stations recorded the vibrations of an earthquake. The second monitoring station is 50 mi due east of the first. The epicenter was determined to be 30 mi from the first station and 27 mi from the second station. Where could the epicenter of the earthquake be located?

**8.** Match each equation to one of the graphs.

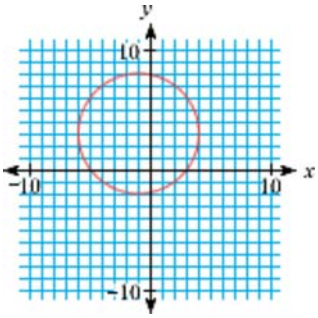
**a.** $9x^2 + 4y^2 - 36 = 0$

**b.** $x^2 - 4y^2 - 8x = 0$

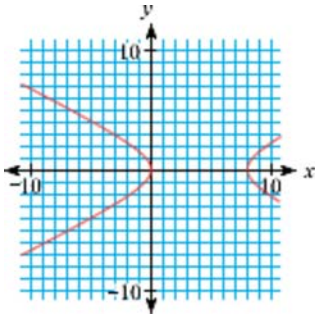
**c.** $3x^2 - 30x + 5y + 55 = 0$

**d.** $x^2 + y^2 + 2x - 6y - 15 = 0$

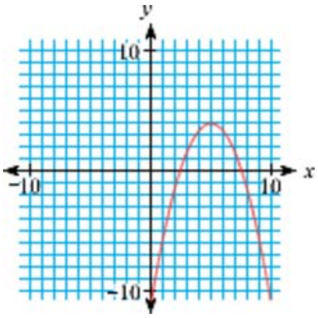
**A.**



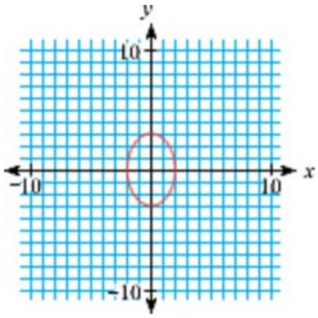

**B.**




**C.**




**D.**




**9.** ***Technology*** Use geometry software to construct and explore this sketch.

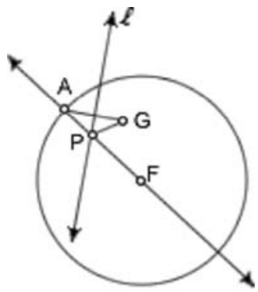
**a.** Construct a circle $F$. Pick a point $A$ on the circle, and another point $G$ inside the circle. Construct $\overline{GA}$.

**b.** Construct the perpendicular bisector of $\overline{GA}$ and label it $\ell$. Construct $\overleftrightarrow{FA}$. Label the intersection of $\ell$ and $\overleftrightarrow{FA}$ as $P$.

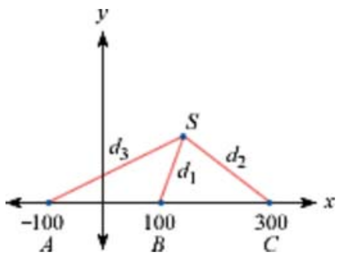
**c.** Explain why the sum of the distances $FP$ and $GP$ is constant.

**d.** Trace point $P$ as you drag point $A$. What shape is traced? Why?

**e.** Now move $G$ to a different location within the circle and repeat 9d. Describe how the shape changes. What happens when you move $G$ outside the circle?




**10.** **APPLICATION** Three LORAN radio transmitters, $A$, $B$, and $C$, are located 200 miles apart along a straight coastline. They simultaneously transmit radio signals at regular intervals. The signals travel at a speed of 980 feet per microsecond. A ship, at $S$, first receives a signal from transmitter $B$. After 264 microseconds, the ship receives the signal from transmitter $C$, and then another 264 microseconds later it receives the signal from transmitter $A$. Use the diagram at right to answer these questions to find the location of the ship.



a. Find $d_2 - d_1$. Express your answer in miles (1 mi = 5280 ft).

b. Find $d_3 - d_1$. Express your answer in miles.

c. Use the fact that $d_2 - d_1$ is constant to write the equation of the hyperbola represented in 10a. Note that transmitters $B$ and $C$ are located at the foci.

d. Write the equation of the hyperbola represented in 10b.

e. Graph the hyperbolas and find the coordinates of the location of the ship. How can you be sure which of the intersection points represents the ship?

**History CONNECTION**

During World War II, LORAN, a long-range navigation system developed at MIT, used radio waves and the definition of a hyperbola to determine the exact location of ships. Today, LORAN-C is operated by the U.S. Coast Guard to monitor U.S. coastal waters. Civil and military air, land, and marine users are provided navigation, location, and timing services by the Coast Guard. Although today global positioning satellites can provide accurate locations, LORAN is a lower-cost alternative because it does not require the launch of satellites.

A technician performs a system check at the LORAN Station at Kodiak, Alaska.

11. Find the equation of the circle that passes through the four intersection points of the ellipses $\frac{x^2}{16}+\frac{y^2}{9}=1$ and $\frac{x^2}{9}+\frac{y^2}{16}=1$.

## Review

12. Find the equations of two parabolas that pass through the points (2, 5), (0, 9), and (– 6, 7). Sketch each parabola.

13. Find the coordinates of the foci of each ellipse.

a. $\left(\frac{x+2}{4}\right)^2+\left(\frac{y-5}{6}\right)^2=1$

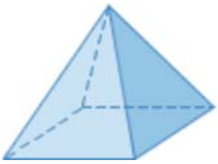
b. $x = \cos t + 1$
$y = 0.5 \sin t - 2$

14. Find the equations of the asymptotes of the hyperbola with vertices (5, 8.5) and (5, 3.5), and foci (5, 12.5) and (5, – 0.5).

15. If the vertices of a triangle are $A(10, 16)$, $B(4, 9)$, and $C(8, 1)$, find $m\angle ABC$.

16. Write the polynomial expression $x^4 - 3x^3 + 4x^2 - 6x + 4$ in factored form.

17. You have seen that a double cone can be intersected with a plane to form a circle, an ellipse, a parabola, and a hyperbola. What shapes can be formed by the intersection of a plane and a square-based pyramid? Draw a sketch of each possibility.



# EXPLORATION

# The Rotation Matrix

Nearly all of the conic sections you have studied have been either horizontally or vertically oriented. You know how to translate and stretch or shrink their graphs. It is also possible to rotate their graphs.

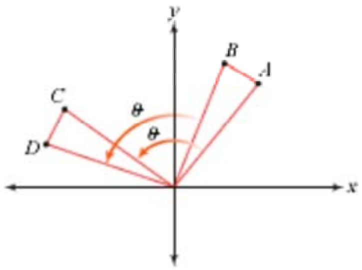
In this exploration you will use rotation matrices to rotate points. This will allow you to graph a rotation of any function or relation.

## Activity

### Around We Go

**You will need**

- graph paper
- a protractor
- a compass or ruler




Step 1 Draw coordinate axes such that the origin is near the center of your graph paper. Mark two points, *A* and *B*, in the first quadrant. Draw $\overline{AB}$, and draw a segment from each point to the origin. Each member of your group must use the same points *A* and *B*.

Step 2 Choose an angle of rotation, $\theta$, between 0° and 180°. Every member of your group must use a different angle. Locate points *C* and *D*, the images of points *A* and *B* after they are rotated $\theta°$ counterclockwise about the origin. Draw $\overline{CD}$. Find the coordinates of points *C* and *D* as accurately as you can.

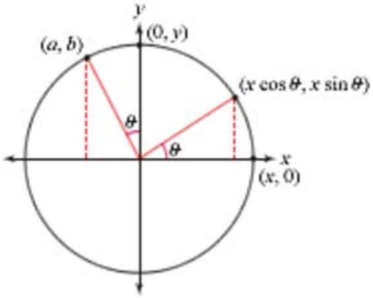
Step 3 You can also rotate segment $\overline{AB}$ by multiplying a rotation matrix by the coordinate matrix of $\overline{AB}$. To find the rotation matrix for your value of $\theta$, substitute the coordinates of the points *A*, *B*, *C*, and *D* into the second and third matrices shown below. ($A_x$ denotes the *x*-coordinate of point *A*.) Use an inverse matrix to find the entries of the rotation matrix, the first matrix shown below. Multiply to check your answer.

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} A_x & B_x \\ A_y & B_y \end{bmatrix} = \begin{bmatrix} C_x & D_x \\ C_y & D_y \end{bmatrix}$$

Step 4 Find the sine and cosine of your rotation angle $\theta$. What is the connection between the entries of your rotation matrix and your sine and cosine values? Compare results with your group. Write a conjecture about a matrix that will rotate a graph $\theta°$ counterclockwise about the origin.

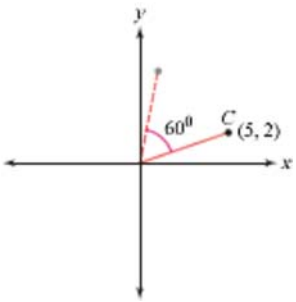
Step 5 To see why the rotation matrix has those entries, look at the effect of rotation on each coordinate. The diagram at right shows the images of points $(x, 0)$ and $(0, y)$ after rotating $\theta°$ counterclockwise about the origin. Identify the lengths, in terms of $x$, of all three sides of the triangle on the right. Do you see why the point $(x, 0)$ rotates to $(x \cos\theta, x \sin\theta)$? Write the coordinates of the point $(a, b)$ in terms of $\cos\theta$ and $\sin\theta$. Then explain why the entries of the rotation matrix make sense.




Step 6 Multiply the rotation matrix you found in Step 4 by the point $\begin{bmatrix} x \\ y \end{bmatrix}$ to find the coordinates of a point $(x, y)$ that has been rotated $\theta°$.

## Questions

**1.** Point $C$ is rotated 60° counterclockwise about the origin. Find the coordinates of the new point.




**2.** The parametric equations of a parabola are $x = 3t^2 + 3$ and $y = 5t$. Rotate the parabola 135° counterclockwise. Write the parametric equations for the new parabola. Verify your results by graphing the original parabola and its rotated image.

**3.** Rotate the hyperbola $x^2 - y^2 = 1$ counterclockwise 45° about the origin. Verify your result by graphing the original hyperbola and its rotated image.

**4.** Consider the parametric equations $x_1 = \tan t$ and $y_1 = \frac{1}{\cos t}$.

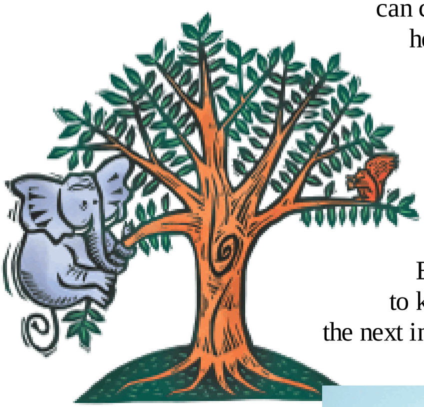
**a.** Predict what the graph will look like.

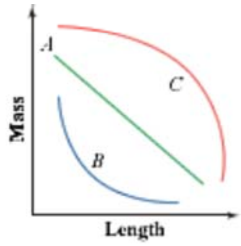
**b.** Predict what the graph of $x_2 = x_1 \cos\theta - y_1 \sin\theta$ and $y_2 = x_1 \sin\theta + y_1 \cos\theta$ will be when $\theta$ equals 50°. [► See Calculator Note 9B to use functions within functions Parametric mode. ◄]

**c.** Draw the graphs to check your predictions.

LESSON 9.6

# Introduction to Rational Functions






You probably know that a lighter tree climber can crawl farther out on a branch than a heavier climber can, before the branch is in danger of breaking. What do you think the graph of (*length*, *mass*) data will look like when mass is added to a length of pole until it breaks? Is the relationship linear, like line $A$, or does it resemble one of the curves, $B$ or $C$?

Engineers study problems like this because they need to know the weight that a beam can safely support. In the next investigation you will collect data and experiment with this relationship.

The Louise M. Davies Symphony Hall, built in 1980, is part of the Civic Center in San Francisco, California. The design of both the balcony and the covered entrance rely upon cantilevers-projecting beams that are supported at only one end.

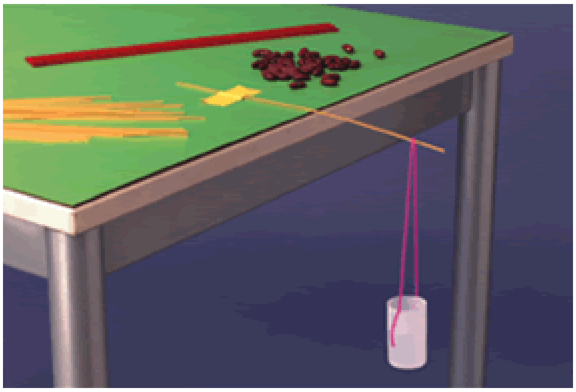
## Investigation
### The Breaking Point

**You will need**

- several pieces of dry spaghetti
- a small film canister
- string
- some weights (pennies, beans, or other small units of mass)
- a ruler
- tape

**Procedure Note**

1. Lay a piece of spaghetti on a table so that its length is perpendicular to one side of the table and the end extends over the edge of the table.
2. Measure the length of the spaghetti that extends beyond the edge of the table. (See the photo on the next page.) Record this information in a table of (*length*, *mass*) data.
3. Tie the string to the film canister so that you can hang it from the end of the spaghetti. (You may need to use tape to hold the string in place.)
4. Place mass units into the container one at a time until the spaghetti breaks. Record the maximum number of weights that the length of spaghetti was able to support.



Step 1 Work with a partner. Follow the procedure note to record at least five data points and then compile your results with those of other group members.

Step 2 Make a graph of your data with length as the independent variable, $x$, and mass as the dependent variable, $y$. Does the relationship appear to be linear? If not, describe the appearance of the graph.

Step 3 Write an equation that is a good fit with the plotted data.

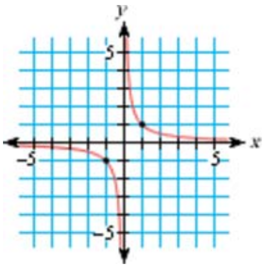
The relationship between length and mass in the investigation is an **inverse variation.** The parent function for an inverse variation curve, $f(x) = \frac{1}{x}$, is the simplest **rational function.**

## Rational Function

A **rational function** is one that can be written as a quotient, $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are both polynomial expressions. The denominator polynomial must be of degree 1 or higher.

This type of function can be transformed just like all the other functions you have previously studied. The function you found in the investigation was probably a transformation of $f(x) = \frac{1}{x}$.

Graph the function $f(x) = \frac{1}{x}$ on your calculator and observe some of its special characteristics. The graph is made up of two branches. One part occurs where $x$ is negative and the other where $x$ is positive. There is no value for this function when $x = 0$. What happens when you try to evaluate $f(0)$? Notice that the graph is a hyperbola that has been rotated 45°, and has vertices (1, 1) and (–1, –1). The $x$- and $y$-axes are the asymptotes. As $x$ gets closer to zero, the $y$-values become increasingly large in absolute value.



Consider these values of the function $f(x) = \frac{1}{x}$.

| $x$ | $-1$ | $-0.1$ | $-0.01$ | $-0.001$ | 0 | 0.001 | 0.01 | 0.1 | 1 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | $-1$ | $-10$ | $-100$ | $-1000$ | undefined | 1000 | 100 | 10 | 1 |

As $x$ approaches zero from the negative side, the $y$-values have an increasingly large absolute value.

So $x = 0$ is a vertical asymptote.

As $x$ approaches zero from the positive side, the $y$-values have an increasingly large absolute value.

The behavior of the $y$-values as $x$ gets closer to zero shows that the $y$-axis is a vertical asymptote for this function.

| $x$ | $-10000$ | $-1000$ | $-100$ | $-10$ | 0 | 10 | 100 | 1000 | 10000 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | $-0.0001$ | $-0.001$ | $-0.01$ | $-0.1$ | undefined | 0.1 | 0.01 | 0.001 | 0.0001 |

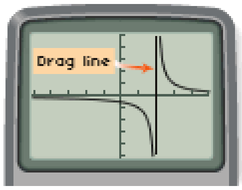
As $x$ takes on larger negative values, the $y$-values approach zero.

So $y = 0$ is a horizontal asymptote.

As $x$ takes on larger positive values, the $y$-values approach zero.

As $x$ approaches the extreme values at the left and right ends of the $x$-axis, the curve approaches the $y$-axis. The horizontal line $y = 0$, then, is a horizontal asymptote. This asymptote is called an end behavior model of the function. In general, the end behavior of a function is its behavior for $x$-values that are large in absolute value.

If you think of $y = \frac{1}{x}$ as a parent function, then $y = \frac{1}{x} + 1$, $y = \frac{1}{x-2}$, and $y = 3\left(\frac{1}{x}\right)$ are examples of transformed rational functions. What happens to a function when $x$ is replaced with $(x - 2)$? The function $y = \frac{1}{x-2}$ is shown at right.



[–5, 5, 1, –5, 5, 1]

Frequently, rational function graphs on the calculator include a nearly vertical drag line. The drag line is not part of the graph! However, it will look much like the graph of the vertical asymptote. [See Calculator Note 9C to learn how to eliminate this line from your graph.]

## EXAMPLE A

Describe the function $f(x) = \frac{2x-5}{x-1}$ as a transformation of the parent function, $f(x) = \frac{1}{x}$. Then sketch a graph.

### Solution

You can change the form of the equation so that the transformations are more obvious. Because the denominator is $(x - 1)$, rather than $x$, try to get the expression $(x - 1)$ in the numerator as well.

$$f(x) = \frac{2x-5}{x-1}$$ Original equation.

$$f(x) = \frac{2(x-1)-3}{x-1}$$ Consider the numerator to be $2x - 2 - 3$, and then factor to get $2(x - 1) - 3$.

Now you want to look for a scale factor and an added term. Separate the rational expression into two fractions:

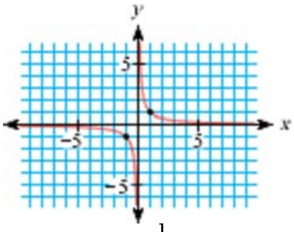
$$f(x) = \frac{2(x-1)}{x-1} - \frac{3}{x-1}$$

Separate the numerator into two numerators over the same denominator.

$$f(x) = 2 - \frac{3}{x-1}$$

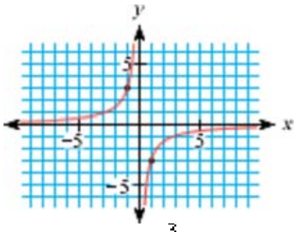
Reduce.

Now you can see that the parent function has been vertically stretched by a factor of – 3, then translated right 1 unit and up 2 units.



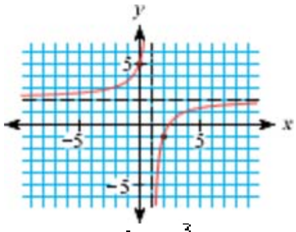

$y = \frac{1}{x}$

The parent rational function, $y = \frac{1}{x}$, has vertices (1, 1) and (–1, –1).




$y = -\frac{3}{x}$

A vertical stretch of -3 moves the vertices to (1, – 3) and (–1, 3). The points (3, 1) and (– 3, –1) are also on the curve. Notice that this graph looks more "spread out" than the graph of $y = \frac{1}{x}$.

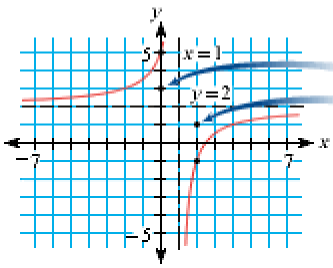



$y = 2 - \frac{3}{x-1}$

A translation right 1 unit and up 2 units moves the vertices to (2, – 1) and (0, 5). The asymptotes are also translated to $x = 1$ and $y = 2$.

Notice that the asymptotes have been translated. How are the equations of the asymptotes related to your final equation above?

To identify an equation that will produce a given graph, do the procedure you used to graph Example A in reverse. You can identify translations by simply looking at the translations of the asymptotes. To identify stretch factors, pick a point, such as a vertex, whose coordinates you would know after the translation of $f(x) = \frac{1}{x}$. Then find a point on the stretched graph that has the same $x$-coordinate. The ratio of the vertical distances from the horizontal asymptote to those two points is the vertical scale factor.




An unstretched (but reflected) inverse variation function with these translations would have vertices (2, 1) and (0, 3), 1 horizontal unit and 1 vertical unit away from the center. Because the distance is now 3 vertical units from the center, include a vertical stretch of 3 to get the equation

$$y = 2 - \frac{3}{x-1}$$

Rational expressions are very useful in chemistry. Scientists use them to model many situations, including the concentration of a solution or mixture as it is diluted.

EXAMPLE B

Suppose you have 100 mL of a solution that is 30% acid and 70% water. How many mL of acid do you need to add to make a solution that is 60% acid? To make it a 90% acid solution? Can it ever be 100% acid?

**Solution**

Of the 100 mL of solution, 30%, or 30 mL, is acid. The percentage, $P$, can be written as $P = \frac{30}{100}$. If $x$ milliliters of acid are added, there will be more acid, but also more solution. The concentration of acid will be

$$P = \frac{30 + x}{100 + x}$$

To find when the solution is 60% acid, substitute 0.6 for $P$ and solve the equation.

$$0.6 = \frac{30 + x}{100 + x} \quad \text{Substitute 0.6 for } P.$$

$$0.6(100 + x) = 30 + x \quad \text{Multiply both sides by } (100 + x).$$

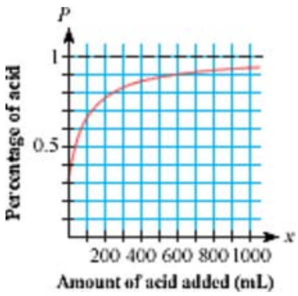
$$60 + 0.6x = 30 + x \quad \text{Distribute.}$$

$$30 = 0.4x \quad \text{Collect like terms.}$$

$$75 = x \quad \text{Divide by 0.4.}$$

Adding 75 mL of acid will make a 60% acid solution.

To find when the solution is 90% acid, solve the equation $0.9 = \frac{30 + x}{100 + x}$. You will find that 600 mL of acid must be added.




The graph of $P = \frac{30 + x}{100 + x}$ shows horizontal asymptote $y = 1$. No matter now many milliliters of acid you add, you will never have a mixture that is 100% acid. This is because the original 70 mL of water will remain, even though it is a smaller and smaller percentage of the entire solution as you continue to add acid.

# EXERCISES

You will need

**Geometry software** for Exercise 13

## Practice Your Skills

**1.** Write an equation and graph each transformation of the parent function $f(x) = \frac{1}{x}$.

**a.** Translate the graph up 2 units.

**b.** Translate the graph right 3 units.

**c.** Translate the graph down 1 unit and left 4 units.

**d.** Vertically stretch the graph by a scale factor of 2.

**e.** Horizontally stretch the graph by a factor of 3, and translate it up 1 unit.

**2.** What are the equations of the asymptotes for each hyperbola?

**a.** $y = \frac{2}{x} + 1$ **b.** $y = \frac{3}{x - 4}$ **c.** $y = \frac{4}{x + 2} - 1$ **d.** $y = \frac{-2}{x + 3} - 4$

3. Solve.

a. $12 = \frac{x-8}{x+3}$  b. $21 = \frac{3x+8}{x+5}$  c. $3 = \frac{2x+5}{4x-7}$  d. $-4 = \frac{-6x+5}{2x+3}$

4. As the rational function $y = \frac{1}{x}$ is translated, its asymptotes are translated also. Write an equation for the translation of $y = \frac{1}{x}$ that has the asymptotes described.

a. horizontal asymptote $y = 2$ and vertical asymptote $x = 0$

b. horizontal asymptote $y = -4$ and vertical asymptote $x = 2$

c. horizontal asymptote $y = 3$ and vertical asymptote $x = -4$

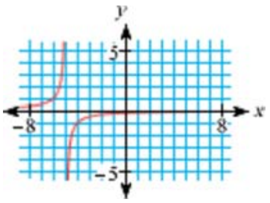
Keymath.com More Practice Your Skills

5. If a basketball team's present record is 42 wins and 36 losses, how many consecutive games must it win so that its winning record reaches 60%?

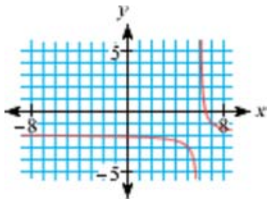
## Reason and Apply

6. Write a rational equation to describe each graph. Some equations will need scale factors.

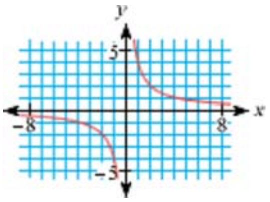
a.



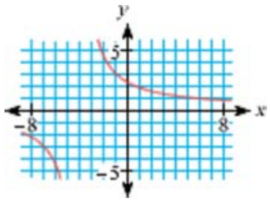

b.



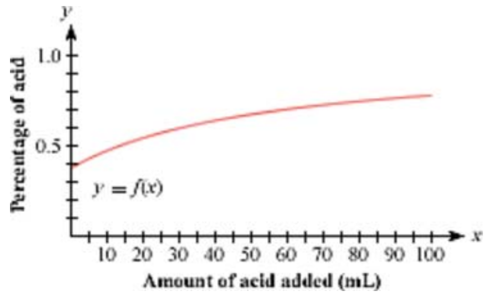

c.




d.




7. **APPLICATION** The graph at right shows the concentration of acid in a solution as pure acid is added. The solution began as 55 mL of a 38% acid solution.




a. How many milliliters of pure acid were in the original solution?

b. Write an equation for $f(x)$.

c. Find the amount of pure acid that must be added to create a solution that is 64% acid.

d. Describe the end behavior of $f(x)$.

8. **APPLICATION** In a container of 2% milk, 2% of the mixture is fat. How much of the liquid in a 1 gal container of 2% milk would need to be emptied and replaced with pure fat so that the container could be labeled as whole (3.25%) milk?

**9.** Consider these functions.

**i.** $y=\frac{2x-13}{x-5}$ **ii.** $y=\frac{3x+11}{x+3}$

**a.** Rewrite each rational function to show how it is a transformation of $y = \frac{1}{x}$.

**b.** Describe the transformations of the graph of $y = \frac{1}{x}$ that will produce graphs of the equations in 9a.

**c.** Graph each equation on your calculator to confirm your answers to 9b.

**10.** Draw the graph of $y = \frac{1}{x}$.

**a.** Label the vertices of the hyperbola.

**b.** The *x*- and *y*-axes are the asymptotes for this hyperbola. Draw the box between the two branches of the hyperbola that has the asymptotes as its diagonals. The vertices should lie on the box.

**c.** The dimensions of the box are $2a$ and $2b$. Find the values of $a$ and $b$.

**d.** The foci of the hyperbola lie on the line passing through the two vertices. In this case, that line is $y = x$. The foci are $c$ units from the center of the hyperbola where $a^2 + b^2 = c^2$. Find the value of $c$ and the coordinates of the two foci.

**11.** Recall that the general quadratic equation is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Let $A = 0$, $B = 4$, $C = 0$, $D = 0$, $E = 0$, and $F = -1$.

**a.** Graph this equation. What type of conic section is formed?

**b.** What is the relationship between the inverse variation function, $y = \frac{1}{x}$, and the conic sections?

**c.** Convert the rational function $y=\frac{1}{x-2}+3$ to general quadratic form. What are the values of *A*, *B*, *C*, *D*, *E*, and *F* in the general quadratic equation?

**12.** **APPLICATION** Ohm's law states that $I=\frac{V}{R}$. This law can be used to determine the amount of current *I*, in amps, flowing in the circuit when a voltage *V*, in volts, is applied to a resistance *R*, in ohms.

**a.** If a hairdryer set on high is using a maximum of 8.33 amps on a 120-volt line, what is the resistance in the heating coils?

**b.** In the United Kingdom, power lines use 240 volts. If a traveler were to plug in a hairdryer, and the resistance in the hairdryer was the same as in 12a, what would be the flow of current?

**c.** The additional current flowing through the hairdryer would cause a meltdown of the coils and the motor wires. In order to reduce the current flow in 12b back to the value in 12a, how much resistance would be needed?

**Consumer CONNECTION**

Many travel appliances, such as hairdryers and shavers, are made with a voltage switch that provides the resistance necessary for the appliance to work properly in different countries. In the United States, a voltage of about 120V is standard, whereas in Europe about 240V is typical. A dimmer switch on a light fixture works in a similar way. When the dimmer switch is set low, there is higher resistance, causing less current to flow, and less illumination is produced. As the dimmer switch is turned, there is less resistance, allowing more current to flow, and more illumination is produced. The volume control on a stereo system works the same way.

13. ***Technology*** Using geometry software, draw the curve $y = \frac{1}{x}$ and plot the foci you found in Exercise 10d. Use measurement tools to verify that $y = \frac{1}{x}$ satisfies the locus definition of a hyperbola.

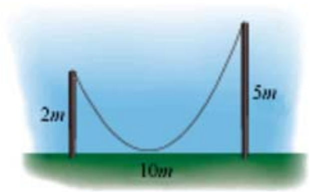
## Review

14. Factor each expression completely.

**a.** $x^2 - 7x + 10$ **b.** $x^3 - 9x$

15. Write the equation of the circle with center (2, – 3) and radius 4.

16. A 2 m rod and a 5 m rod are mounted vertically 10 m apart. One end of a 15 m wire is attached to the top of each rod. Suppose the wire is stretched taut and fastened to the ground between the two rods. How far from the base of the 2 m rod is the wire fastened?

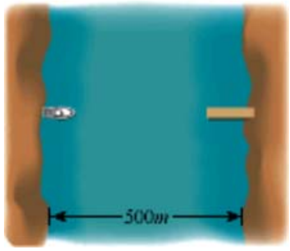



17. Sarah would like to row her boat directly across a river 500 m wide. The current flows 3 km/h and she is able to row 5 km/h.

**a.** At what angle to the riverbank should she point her boat?

**b.** As she starts, how far upstream on the opposite bank should she head?

**c.** Write parametric equations to simulate Sarah's crossing.

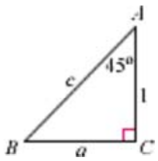



18. Write the general quadratic equations of two concentric circles with center (6, – 4) and radii 5 and 8.

19. Find exact values of missing side lengths for 19a-d.

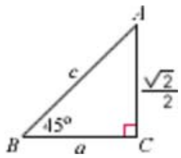
**a.**


**b.**




**c.**


**d.**




**e.** What is the ratio of side lengths in a 45°-45°-90° triangle? A 30°-60°-90° triangle?

LESSON

9.7

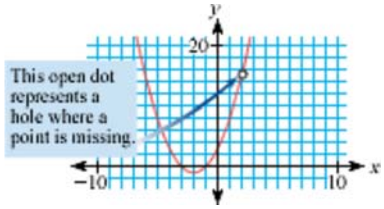
# Graphs of Rational Functions

*Besides learning to see, there is another art to be learned- to see what is not.*

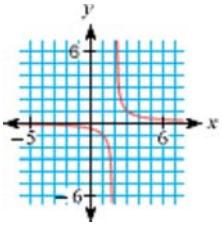
MARIA MITCHELL

Some rational functions create very different kinds of graphs from those you have studied previously. The graphs of these functions are often in two or more parts. This is because the denominator, a polynomial function, may be equal to zero at some point, so the function will be undefined at that point. Sometimes it's difficult to see the different parts of the graph because they may be separated by only one missing point, called a **hole.** At other times you will see two parts that look very similar-one part may look like a reflection or rotation of the other part. Or you might get multiple parts that look totally different from each other. Look for these features in the graphs below.

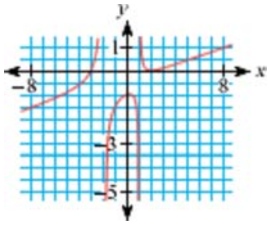
$$y=\frac{x^3+2x^2-5x-6}{x-2}$$



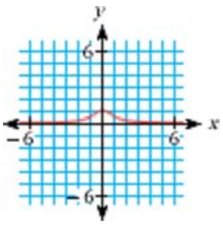

$$y=\frac{1}{x-2}$$



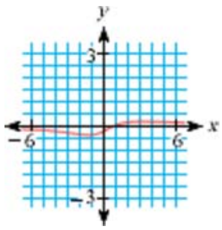

$$y=\frac{x^3-x^2-8x+12}{6x^2+6x-12}$$




$$y=\frac{1}{x^2+1}$$




$$y=\frac{x-1}{x^2+4}$$



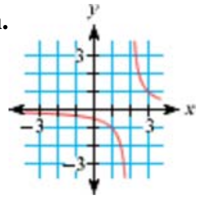
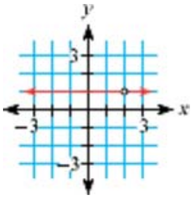
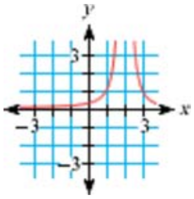
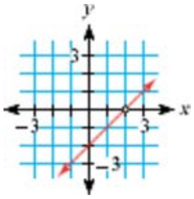

In this lesson you will explore local and end behavior of rational functions, and you will learn how to predict some of the features of a rational function's graph by studying its equation. When examining a rational function, you will often find it helpful to look at the equation in factored form.

## Investigation

# Predicting Asymptotes and Holes

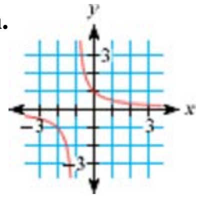
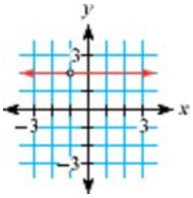
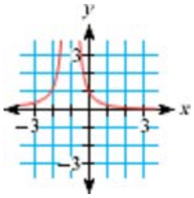
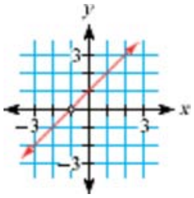
In this investigation you will consider the graphs of four rational functions.

Step 1 Match each rational function with a graph. Use a friendly window as you graph and trace the equations on your calculator. Describe the unusual occurrences at and near $x = 2$, and try to explain what feature in the equation makes the graph look the way it does. (You will not actually see the hole pictured in graph d unless you turn off the coordinate axes on your calculator.)

**a.**  **b.**  **c.**  **d.** 

**A.** $y = \frac{1}{(x-2)^2}$ **B.** $y = \frac{1}{x-2}$ **C.** $y = \frac{(x-2)^2}{x-2}$ **D.** $y = \frac{x-2}{x-2}$

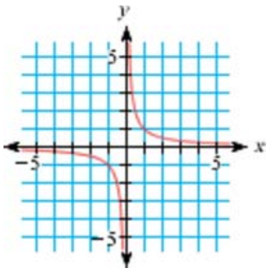
Step 2 Have each group member choose one of the graphs below. Find a rational function equation for your graph, and write a few sentences that explain the appearance of your graph. Share your answers with your group.

**a.**  **b.**  **c.**  **d.** 

Step 3 Write a paragraph explaining how you can use an equation to predict where holes and asymptotes will occur, and how you can use these features in a graph to write an equation.

Step 4 Consider the graph of $y = \frac{x-2}{(x-2)^2}$. What features does it have? What can you generalize about the graph of a function that has a factor that occurs more times in the denominator than in the numerator?

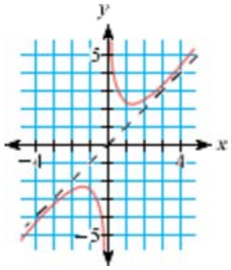
You have seen transformations of the function $y = \frac{1}{x}$ and observed some of the peculiarities involving graphs of more complicated rational functions. You have seen that $y = \frac{1}{x}$ has both horizontal and vertical asymptotes, as shown at right. What do you think the graph would look like if you added $x$ to $\frac{1}{x}$? Reflect on this question for a moment. Then graph $y = x + \frac{1}{x}$ on your calculator.



EXAMPLE A

Describe the graph of $y = x + \frac{1}{x}$.

**▶ Solution**




There is a vertical asymptote or hole at $x = 0$ because the function is undefined when $x = 0$. The graph shows that the feature at $x = 0$ is a vertical asymptote. The values of $\frac{1}{x}$ are added to the values of $x$. This means that as the absolute value of $x$ increases, the absolute value of $y = x + \frac{1}{x}$ also increases. If you graph the curve and the line $y = x$, you see that the function approaches the line $y = x$ instead of the $x$-axis. The line $y = x$ is called a slant asymptote, because it is a diagonal line that the function approaches as $x$-values increase in the positive and negative directions. This slant asymptote describes the end behavior for this function.

Often you can determine the features of a rational function graph before you actually graph it. Values that make the denominator or numerator equal to zero give you important clues about the appearance of the graph.

EXAMPLE B

Describe the features of the graph of $y = \frac{x^2+2x-3}{x^2-2x-8}$.

**▶ Solution**

Features of rational functions are apparent when the numerator and denominator are factored.

$$y = \frac{x^2+2x-3}{x^2-2x-8} = \frac{(x+3)(x-1)}{(x-4)(x+2)}$$

No common factors occur in both the numerator and denominator, so there are no holes.

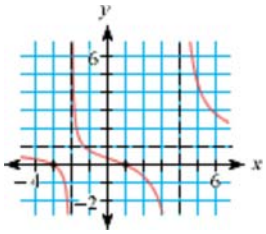
If $x = 4$ or $x = -2$, then the denominator is 0 and the function is undefined. There are vertical asymptotes at these values.

If $x = -3$ or $x = 1$, then the numerator is 0, so the $x$-intercepts are $-3$ and 1.

If $x = 0$, then $y = 0.375$. This is the $y$-intercept.

To find any horizontal asymptotes, consider what happens to $y$-values as $x$-values get increasingly large in absolute value.

| **x** | – 10000 | – 1000 | – 100 | 100 | 1000 | 10000 |
|---|---|---|---|---|---|---|
| **y** | 0.9996001 | 0.9960129 | 0.9612441 | 1.0413603 | 1.0040131 | 1.0004001 |




A table shows that the $y$-values get closer and closer to 1 as $x$ gets farther from 0. So, $y = 1$ is a horizontal asymptote. Note that for large positive values of $x$, $y$-values decrease to 1, whereas for large negative values of $x$, $y$-values increase to 1.

A graph of the function confirms these features.

Rational functions can be written in different forms. The factored form is convenient for locating asymptotes and intercepts. And you saw in the previous lesson how rational functions can be written in a form that shows you clearly how the parent function has been transformed. You can use properties of arithmetic with fractions to convert from one form to another.

**EXAMPLE C** Rewrite $f(x) = \frac{3}{x-2} - 4$ as a rational function.

**Solution** The original form shows that this function is related to the parent function, $f(x) = \frac{1}{x}$. It has been vertically stretched by a factor of 3 and translated right 2 units and down 4 units. To change to rational function form, you must add the two parts to form a single fraction.

| | |
|---|---|
| $f(x) = \frac{3}{x-2} - 4$ | Original equation. |
| $= \frac{3}{x-2} - \frac{4}{1} \cdot \frac{x-2}{x-2}$ | Create a common denominator of $(x - 2)$. |
| $= \frac{3}{x-2} - \frac{4x-8}{x-2}$ | Rewrite second fraction. |
| $= \frac{3-4x+8}{x-2}$ | Combine the two fractions. |
| $f(x) = \frac{11-4x}{x-2}$ | Add like terms. |

There are no common factors in both the numerator and denominator, so there are no holes.

In this form you can see that $x = 2$ is a vertical asymptote, because this value makes the denominator equal to zero. You can also see that the $x$-intercept is $\frac{11}{4}$, because this $x$-value makes the numerator equal to zero. Evaluating the function for large values shows that $y = -4$ is a horizontal asymptote.

| $x$ | – 10000 | – 1000 | – 100 | 100 | 1000 | 10000 |
|---|---|---|---|---|---|---|
| $y$ | – 4.0002999 | – 4.0029940 | – 4.0294118 | – 3.9693878 | – 3.9969940 | – 3.9996999 |

# EXERCISES

## Practice Your Skills

**1.** Rewrite each rational expression in factored form.

**a.** $\frac{x^2 + 7x + 12}{x^2 - 4}$ **b.** $\frac{x^3 - 5x^2 - 14x}{x^2 + 2x + 1}$

**2.** Identify the vertical asymptotes for each equation.

**a.** $y = \frac{x^2 + 7x + 12}{x^2 - 4}$ **b.** $y = \frac{x^3 - 5x^2 - 14x}{x^2 + 2x + 1}$

**3.** Rewrite each expression in rational form (as the quotient of two polynomials).

**a.** $3 + \frac{4x-1}{x-2}$ **b.** $\frac{3x+7}{2x-1} - 5$

**4.** Graph each equation on your calculator, and make a sketch of the graph on your paper. Use a friendly graphing window. Indicate any holes on your sketches.

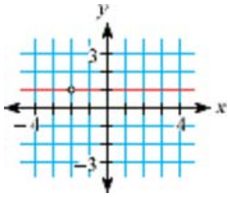
**a.** $y = \dfrac{5 - x}{x - 5}$  **b.** $y = \dfrac{3x + 6}{x + 2}$  **c.** $y = \dfrac{(x + 3)(x - 4)}{x - 4}$

**d.** What causes a hole to appear in the graph?

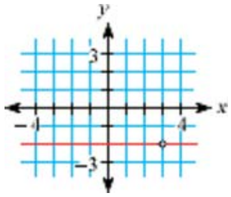
## Reason and Apply

**5.** Write an equation for each graph.

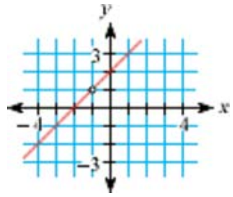
**a.**




**b.**




**c.**




**6.** Graph $y = \dfrac{4}{x - 3}$.

**a.** Describe the end behavior of the graph.

**b.** Describe the behavior of the graph near $x = 3$.

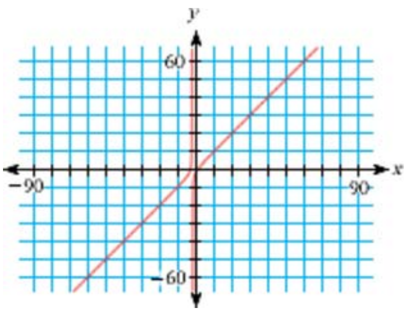
**c.** Rewrite $y = -x + \dfrac{4}{x - 3}$ in rational function form.

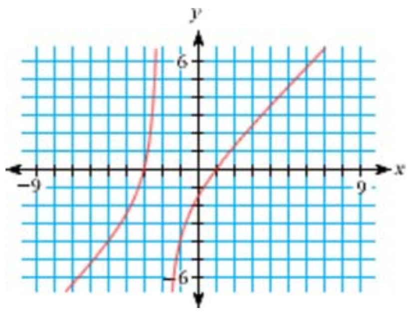
**d.** Factor your answer from 6c. What do the factors tell you about the graph?

**7.** Graph each function on your calculator. List all holes and asymptotes, including slant asymptotes.

**a.** $y = x - 2 + \dfrac{1}{x}$  **b.** $y = -2x + 3 + \dfrac{2}{x - 1}$  **c.** $y = 7 + \dfrac{8 - 4x}{x - 2}$

**8.** The two graphs below show the same function.




**a.** List all the important facts you can about the graph.

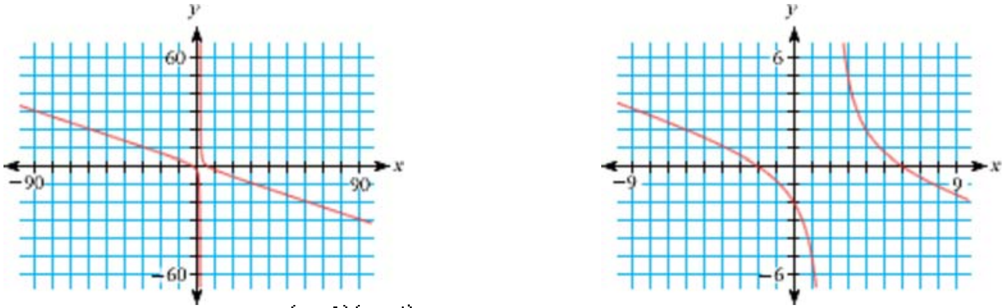
**b.** Find the equation of the slant asymptote in the first graph.

**c.** Give an example of an equation with asymptote $x = -2$.

**d.** Name a polynomial with zeros $x = -3$ and $x = 1$.

**e.** Write an equation for the function shown in the graphs. Graph the function to check your answer.

**9.** The two graphs below show the same function. Write an equation for this function.




**10.** Consider the equation $y = \dfrac{(x-1)(x+4)}{(x-2)(x+3)}$.

**a.** Describe the features of the graph of this function.

**b.** Describe the end behavior of the graph.

**c.** Sketch the graph.

**11.** Solve. Give exact solutions.

**a.** $\dfrac{2}{x-1} + x = 5$

**b.** $\dfrac{2}{x-1} + x = 2$

**12.** **APPLICATION** The functional response curve given by the function $y = \dfrac{60x}{1+0.625x}$ models the number of moose attacked by wolves as the density of moose in an area increases. In this model, $x$ represents the number of moose per 1000 km$^2$, and $y$ represents the number of moose attacked every 100 days.

**a.** How many moose are attacked every 100 days if there is a herd containing 260 moose in a land preserve with area 1000 km$^2$?

**b.** Graph the function.

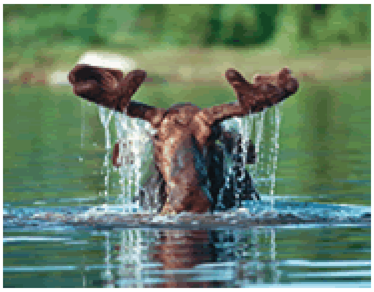
**c.** What are the asymptotes for this function?

**d.** Describe the significance of the asymptotes in this problem.

### Environmental CONNECTION

Ecologists often look for a mathematical model to describe the interrelationship of organisms. C. S. Holling, a Canadian researcher, came up with an equation in the late 1950s for what he called a Type II functional response curve. The equation describes the relationship between the number of prey attacked by a predator and the density of the prey. For example, the wolf population increases through reproduction as moose density increases. Eventually, wolf populations stabilize at about 40 per 1000 km$^2$, which is the optimum size of their range based on defense of their territories.

The functional response curve applies to all species of animals. It could be larvae-eating insects and mosquito larvae, fishers and a particular species of fish, or panda and amount of bamboo in a forest.



A moose rises from the surface of a pond in Baxter State Park, Maine.

**13.** A machine drill removes a core from any cylinder. Suppose you want the amount of material left after the core is removed to remain constant. The table below compares the height and radius needed if the volume of the hollow cylinder is to remain constant.


| **Radius** $x$ | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 | 5.5 | 6.0 | 6.5 |
|---|---|---|---|---|---|---|---|---|---|
| **Height** $h$ | 56.6 | 25.5 | 15.4 | 10.6 | 7.8 | 6.1 | 4.9 | 4.0 | 3.3 |

**a.** Plot the data points, $(x, h)$, and draw a smooth curve through them.

**b.** Explain what happens to the height of the figure as the radius gets smaller. How small can $x$ be?

**c.** Write a formula for the volume of the hollow cylinder, $V$, in terms of $x$ and $h$.

**d.** Solve the formula in 13c for $h$ to get a function that describes the height as a function of the radius.

**e.** What is the constant volume?

## Review

**14.** Find the points of intersection, if any, of the circle with center (2, 1) and radius 5 and the line $x - 7y + 30 = 0$.

**15.** A 500 g jar of mixed nuts contains 30% cashews, 20% almonds, and 50% peanuts.

**a.** How many grams of cashews must you add to the mixture to increase the percentage of cashews to 40%? What is the new percentage of almonds and peanuts?

**b.** How many grams of almonds must you add to the original mixture to make the percentage of almonds the same as the percentage of cashews? Now what is the percentage of each type of nut?

**16.** Solve each quadratic equation.

**a.** $2x^2 - 5x - 3 = 0$ **b.** $x^2 + 4x - 4 = 0$ **c.** $x^2 + 4x + 1 = 0$

## Project

### GOING DOWNHILL FAST

Design an investigation to determine a relationship between the angle of elevation of a long tube and the time it takes a ball to travel the length of the tube.

Your project should include

- A description of your investigation and the data you collect.
- A graph that shows the relationship between the tube's angle and the ball's time.
- The domain and range of the relationship. Include a description of what happens to the time as the angle approaches the extreme values of the domain.
- A description of the features of the graph. Attach real-world meaning to each feature.

LESSON

9.8

# Operations with Rational Expressions

*Images/split the truth/ in fractions.*

DENISE LEVERTOV

In this lesson you will learn to add, subtract, multiply, and divide rational expressions. In the previous lesson you combined a rational expression with a single constant or variable by finding a common denominator. That process was much like adding a fraction to a whole number. Likewise, all of the other arithmetic operations you will do with rational expressions have their counterparts in working with fractions. Keeping the operations with fractions in mind will help you understand the procedures.

Recall that to add $\frac{7}{12}+\frac{3}{10}$, you need a common denominator. The smallest number that has both 12 and 10 as factors is 60. So, use 60 as the common denominator.

| | |
|---|---|
| $\frac{7}{12}+\frac{3}{10}$ | Original expression. |
| $\frac{7}{12}\cdot\frac{5}{5}+\frac{3}{10}\cdot\frac{6}{6}$ | Multiply each fraction by an equivalent of 1 to get a denominator of 60. |
| $\frac{35}{60}+\frac{18}{60}$ | Multiply. |
| $\frac{53}{60}$ | Add. |

You could use other numbers, such as 120, as a common denominator, but using the least common denominator keeps the numbers as small as possible and eliminates some of the reducing afterward. Recall that you can find the least common denominator by factoring the denominators to see which factors they share and are unique to each one. In this example, 12 factors to $2 \cdot 2 \cdot 3$ and 10 factors to $2 \cdot 5$. The least common denominator must include factors that multiply to give each denominator, with no extras. So, in this case you need two 2's, a 3, and a 5. You can use this same process to add two rational expressions.

Rational expressions can be combined, just like fractions, to express the sum of parts. *Circles* (1916-1923) by Johannes Itten (1888-1967) shows many circles divided into fractional parts.

## EXAMPLE A

Add rational expressions to rewrite the right side of this equation as a single rational expression in factored form.

$$y=\frac{x-3}{(x+1)(x-2)}+\frac{2x+1}{(x+2)(x-2)}$$

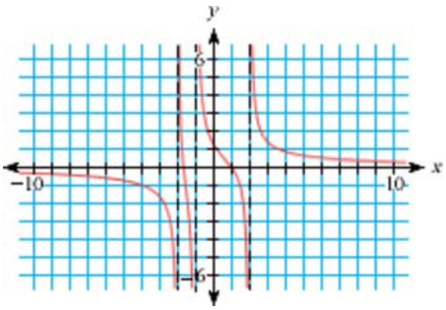
## Solution

First, identify the least common denominator. It must contain all of the factors of each denominator. The factors $(x + 1)$ and $(x - 2)$ are needed to create the first denominator, and an additional $(x + 2)$ is needed for the second denominator. So, use the common denominator $(x + 1)(x - 2)(x + 2)$.

| | |
|---|---|
| $y = \dfrac{x-3}{(x+1)(x-2)} + \dfrac{2x+1}{(x+2)(x-2)}$ | Original equation. |
| $y = \dfrac{x-3}{(x+1)(x-2)} \cdot \dfrac{(x+2)}{(x+2)} + \dfrac{2x+1}{(x+2)(x-2)} \cdot \dfrac{(x+1)}{(x+1)}$ | Multiply each fraction by an equivalent of 1 to get a common denominator. |
| $y = \dfrac{x^2-x-6}{(x+1)(x-2)(x+2)} + \dfrac{2x^2+3x+1}{(x+2)(x-2)(x+1)}$ | Multiply and expand the numerators. |
| $y = \dfrac{3x^2+2x-5}{(x+1)(x-2)(x+2)}$ | Add the two fractions and combine like terms in the numerator. |
| $y = \dfrac{(3x+5)(x-1)}{(x+1)(x-2)(x+2)}$ | Factor the numerator. |

In this case, the numerator factors. Often, however, it will not.

Expressing a rational function as a single rational expression in factored form helps you identify some features of the graph. An $x$-value that makes the expression equal to zero is an $x$-intercept. Here, 1 and $-\frac{5}{3}$ are the $x$-intercepts. An $x$-value that leads to division by zero makes the expression undefined. This results in a vertical asymptote (when the associated factor appears more times in the denominator than in the numerator) or a hole (when the factor appears in both the numerator and denominator, and the factor does not represent a vertical asymptote). The graph of the equation above has vertical asymptotes $x = -1$, $x = 2$, and $x = -2$, as shown.




Subtraction of rational expressions is much like addition.

## EXAMPLE B

Find any $x$-intercepts, vertical asymptotes, or holes in the graph of

$$y = \frac{x+2}{(x-3)(x+4)} - \frac{5}{x+1}$$

## Solution

Begin by finding a common denominator so that you can write the expression on the right side as a single rational expression in factored form. The common denominator is $(x - 3)(x + 4)(x + 1)$.

$$y = \frac{(x+2)}{(x-3)(x+4)} \cdot \frac{(x+1)}{(x+1)} - \frac{5}{(x+1)} \cdot \frac{(x-3)(x+4)}{(x-3)(x+4)}$$

Multiply each fraction by an equivalent of 1 to get a common denominator.

$$y = \frac{x^2+3x+2}{(x-3)(x+4)(x+1)} - \frac{5(x^2+x-12)}{(x+1)(x-3)(x+4)}$$

Expand the numerators.

$$y = \frac{x^2+3x+2-5(x^2+x-12)}{(x-3)(x+4)(x+1)}$$

Write with a single denominator.

$$y = \frac{x^2+3x+2-5x^2-5x+60}{(x-3)(x+4)(x+1)}$$

Use the distributive property.

$$y = \frac{-4x^2-2x+62}{(x-3)(x+4)(x+1)}$$

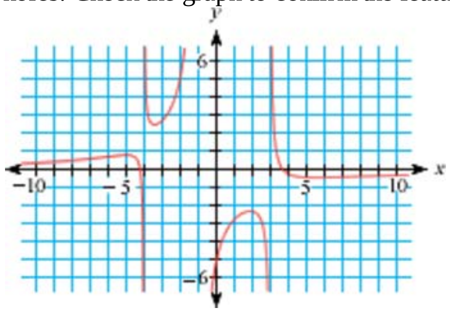
Combine like terms.

Check to see whether the numerator factors. There is a common factor of – 2 in the numerator, so you may rewrite it as $y = \frac{-2(2x^2+x-31)}{(x-3)(x+4)(x+1)}$. The equation is undefined when $x = 3$, $x = -4$, or $x = -1$, so these are the vertical asymptotes. Because the numerator cannot be factored further, the $x$-intercepts are not obvious. But you can use the quadratic formula to find the values that make the expression in the numerator equal to zero.

$$x = \frac{-1 \pm \sqrt{1-4(2)(-31)}}{2(2)}$$

$x \approx 3.695$ or $x \approx -4.195$

Because there are no common factors in the numerator and denominator, there are no holes. Check the graph to confirm the features you've identified.




To multiply and divide rational expressions, you don't need to find a common denominator. Look at the following problems to remind yourself how multiplication and division work with fractions.

$$\frac{5}{12} \cdot \frac{3}{4} = \frac{5 \cdot 3}{12 \cdot 4} = \frac{5}{4 \cdot 4} = \frac{5}{16}$$

Multiply straight across. Reduce common factors if any appear.

$$\frac{\frac{2}{5}}{\frac{6}{7}} = \frac{2}{5} \cdot \frac{7}{6} = \frac{2 \cdot 7}{5 \cdot 6} = \frac{7}{5 \cdot 3} = \frac{7}{15}$$

To divide, multiply by the reciprocal of the denominator.

In multiplication and division problems with rational expressions, it is best to factor all expressions first. This will make it easy to reduce common factors and identify $x$-intercepts, holes, and vertical asymptotes.

## EXAMPLE C

Multiply $\frac{(x+2)}{(x-3)} \cdot \frac{(x+1)(x-3)}{(x-1)(x^2-4)}$.

### Solution

$$\frac{(x+2)}{(x-3)} \cdot \frac{(x+1)(x-3)}{(x-1)(x-2)(x+2)}$$

Factor any expressions that you can. $(x^2-4)$ factors to $(x-2)(x+2)$.

$$\frac{(x+2)(x+1)(x-3)}{(x-3)(x-1)(x-2)(x+2)}$$

Combine the two expressions.

$$\frac{\cancel{(x+2)}(x+1)\cancel{(x-3)}}{\cancel{(x-3)}(x-1)(x-2)\cancel{(x+2)}}$$

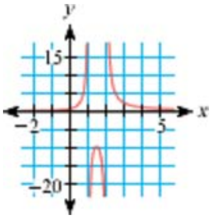
Reduce common factors.

$$\frac{(x+1)}{(x-1)(x-2)}$$

Rewrite.

The expression $\frac{(x+2)}{(x-3)} \cdot \frac{(x+1)(x-3)}{(x-1)(x^2-4)}$ reduces to $\frac{(x+1)}{(x-1)(x-2)}$, but their graphs are slightly different. The original multiplication expression had two factors, $(x-3)$ and $(x+2)$, that were eventually reduced. When a factor can be reduced, and the factor no longer appears in the denominator after being reduced, it represents a hole. So the graph of $y = \frac{(x+2)}{(x-3)} \cdot \frac{(x+1)(x-3)}{(x-1)(x^2-4)}$ has holes at $x = 3$ and $x = -2$, whereas the graph of $y = \frac{(x+1)}{(x-1)(x-2)}$ does not. Both graphs have vertical asymptotes $x = 1$ and $x = 2$ and $x$-intercept $-1$.

$$y = \frac{(x+1)}{(x-1)(x-2)}$$




Multiplication and division with rational expressions will give you a good chance to practice your factoring skills.

## EXAMPLE D

Divide $\dfrac{\frac{x^2-1}{x^2+5x+6}}{\frac{x^2-3x+2}{x+3}}$.

### Solution

$$\frac{x^2-1}{x^2+5x+6} \cdot \frac{x+3}{x^2-3x+2}$$

Invert the fraction in the denominator and multiply.

$$\frac{(x+1)(x-1)}{(x+2)(x+3)} \cdot \frac{(x+3)}{(x-2)(x-1)}$$

Factor all expressions.

$$\frac{(x+1)(x-1)(x+3)}{(x+2)(x+3)(x-2)(x-1)}$$

Multiply.

$$\frac{(x+1)}{(x-1)(x-2)}$$

Reduce all common factors.

Rational expressions like those in Example D can look intimidating. However, the rules are the same as for regular fraction arithmetic. Work carefully and stay organized. Check the graph of the original problem and any step along the way to see whether you have made an error. The graphs should be identical except for holes after you have reduced common factors.

# EXERCISES

## Practice Your Skills

**1.** Factor each expression completely and reduce common factors.

**a.** $\dfrac{x^2+2x}{x^2-4}$ **b.** $\dfrac{x^2-5x+4}{x^2-1}$ **c.** $\dfrac{3x^2-6x}{x^2-6x+8}$ **d.** $\dfrac{x^2+3x-10}{x^2-25}$

**2.** What is the least common denominator for each pair of rational expressions?

**a.** $\dfrac{x}{(x+3)(x-2)}, \dfrac{x-1}{(x-3)(x-2)}$

**b.** $\dfrac{x^2}{(2x+1)(x-4)}, \dfrac{x}{(x+1)(x-2)}$

**c.** $\dfrac{2}{x^2-4}, \dfrac{x}{(x+3)(x-2)}$

**d.** $\dfrac{x+1}{(x-3)(x+2)}, \dfrac{x-2}{x^2+5x+6}$

**3.** Add or subtract as indicated.

**a.** $\dfrac{x}{(x+3)(x-2)} + \dfrac{x-1}{(x-3)(x-2)}$

**b.** $\dfrac{2}{x^2-4} - \dfrac{x}{(x+3)(x-2)}$

**c.** $\dfrac{x+1}{(x-3)(x+2)} + \dfrac{x-2}{x^2+5x+6}$

**d.** $\dfrac{2x}{(x+1)(x-2)} - \dfrac{3}{x^2-1}$

**4.** Multiply or divide as indicated. Reduce any common factors to simplify.

**a.** $\dfrac{x+1}{(x+2)(x-3)} \cdot \dfrac{x^2-4}{x^2-x-2}$

**b.** $\dfrac{x^2-16}{x+5} \div \dfrac{x^2+8x+16}{x^2+3x-10}$

**c.** $\dfrac{x^2+7x+6}{x^2+5x-6} \cdot \dfrac{2x^2-2x}{x+1}$

**d.** $\dfrac{\dfrac{x+3}{x^2-8x+15}}{\dfrac{x^2-9}{x^2-4x-5}}$

## Reason and Apply

**5.** Rewrite as a single rational expression.

**a.** $\dfrac{1-\dfrac{x}{x+2}}{\dfrac{x+1}{x^2-4}}$

**b.** $\dfrac{\dfrac{1}{x-1}+\dfrac{1}{x+1}}{\dfrac{x}{x-1}-\dfrac{x}{x+1}}$

**6.** Graph $y = \dfrac{x+1}{x^2-7x-8} - \dfrac{x}{2(x-8)}$ on your calculator.

**a.** List all asymptotes, holes, and intercepts based on your calculator's graph.

**b.** Rewrite the right side of the equation as a single rational expression.

**c.** Use your answer from 6b to verify your observations in 6a. Explain.

7. Consider the equation $y = \frac{x-3}{x^2-4}$.

a. Without graphing, identify the zeros and asymptotes of the graph of the equation. Explain your methods.

b. Verify your answers by graphing the function.

8. Consider the equation $y = x + 1 + \frac{1}{x-1}$.

a. Without graphing, name the asymptotes of the function.

b. Rewrite the equation as a single rational function.

c. Sketch a graph of the function without using your calculator.

d. Confirm your work by graphing the function with your calculator.

9. ***Mini-Investigation*** You saw in Lesson 9.7 that a transformation of the parent function $f(x) = \frac{1}{x}$ is a rotated hyperbola. Are there other kinds of rational functions that are also rotated hyperbolas? Recall that any conic section can be written in the general quadratic form, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

a. Look back at your graph of $y = x + 1 + \frac{1}{x-1}$ from Exercise 8. Does the graph appear to be a rotated hyperbola?

b. Rewrite the equation $y = x + 1 + \frac{1}{x-1}$ in general quadratic form, if possible. Is the graph actually a rotated hyperbola?

c. Look back at your graph of $y = \frac{x-3}{x^2-4}$ from Exercise 7. Does the graph appear to be a rotated hyperbola?

d. Rewrite the equation $y = \frac{x-3}{x^2-4}$ in general quadratic form, if possible. Is the graph actually a rotated hyperbola?

e. Write a conjecture that describes how you can tell, based on an equation, what rational functions are rotated hyperbolas.

10. **APPLICATION** How long should a traffic light stay yellow before turning red? One study suggests that for a car approaching a 40 ft wide intersection under normal driving conditions, the length of time, $y$, that a light should stay yellow, in seconds, is given by the equation $y = 1 + \frac{v}{25} + \frac{50}{v}$, where $v$ is the velocity of an approaching car in feet per second.

a. Rewrite the equation in rational function form.

b. Enter the original equation as Y1 and your simplified equation as Y2 into your calculator. Check using the table feature that the values of both functions are the same. What does this tell you?

c. If the speed limit at a particular intersection is 45 mi/h, how long should the light stay yellow?

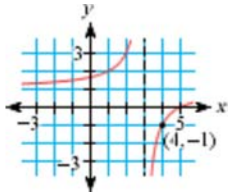
d. If cars typically travel at speeds ranging from 25 mi/h to 55 mi/h at that intersection, what is the possible range of times that a light should stay yellow?

**Career CONNECTION**

Traffic engineers time traffic signals to minimize the "dilemma zone." The dilemma zone occurs when a driver has to decide to brake hard to stop or to accelerate to get through the intersection. Either decision can be risky. It is estimated that 22% of traffic accidents occur when a driver runs a red light.

## Review

**11.** The graph at right is the image of $y = \frac{1}{x}$ after a transformation.




**a.** Write an equation for each asymptote.

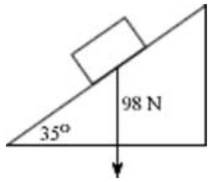
**b.** What translations are involved in transforming $y = \frac{1}{x}$ to its image?

**c.** The point (4, – 1) is on the image. What is the vertical scale factor in the transformation?

**d.** Write an equation of the image.

**e.** Name the intercepts.

**12.** A block with mass 10 kg is sliding down a 35° incline, acted on by a gravity force vector of 98 N (newtons).




**a.** Sketch this gravity vector in two components, one parallel to the incline, $v_i$ , and one perpendicular to the incline, $v_n$.

**b.** Find the magnitude of each component.

**13.** If you invest $1000 at 6.5% interest for 5 years, how much interest do you earn in each of these scenarios?

**a.** The interest is compounded annually.

**b.** The interest is compounded monthly.

**c.** The interest is compounded weekly.

**d.** The interest is compounded daily.

## Project

### CYCLIC HYPERBOLAS

Consider the rational function $f(x) = \frac{x-3}{x+1}$ , whose graph is a hyperbola. You can use this function as a recursive formula. Choose any starting value for $x$ and find the first six terms of the sequence. The values should repeat. Choose another value for $x$. Does the same thing happen? Graph the function using web graphs and explore what happens for various values of $x$. [ See **Calculator Note 5D** to learn how to make a web graph. ] Then use function composition to prove that such repetition always happens for this function. Hyperbolas with this property are called *cyclic hyperbolas*.

Your project is to explore what can happen when you use a rational function whose graph is a hyperbola as a recursive rule and to look for other cyclic hyperbolas. You might also explore what happens when you use other functions you've studied as a recursive rule. Are any of them cyclic?

Your project should include

- Your calculations and web graphs for $f(x) = \frac{x-3}{x+1}$, and other functions you explore.
- Any other cyclic hyperbolas you find, including a proof that they are cyclic.
- Any research you do about recursive sequences on hyperbolas or other functions.

# CHAPTER 9 REVIEW

Keymath.com Links to Resources

In this chapter you saw some special relations. Each relation was described as a set of points, or **locus,** that satisfied some criteria. These relations are called **conic sections** because the shapes can be formed by slicing a double cone at various angles. The simplest of the conic sections is the **circle,** the set of points a fixed distance from a fixed point called the **center.** Closely related to the circle is the **ellipse.** The ellipse can be defined as the set of points such that the sum of their distances from two fixed points, the **foci,** is a constant. The **parabola** is another conic section, which you studied in earlier chapters. It can be defined as the set of points that are equidistant from a fixed point called the focus and a fixed line called the **directrix.** The last of the conic sections is the **hyperbola.** The definition of a hyperbola is similar to the definition of an ellipse, except that the difference between the distances from the foci remains constant. The equations for these conic sections can be written parametrically or in standard form, or as a general quadratic equation. And each of these conic sections can be either vertically oriented or horizontally oriented. You learned how to convert between the general quadratic equation and standard form, and how to solve systems of quadratic equations to find the intersections of two conic sections.

You were also introduced to **rational functions** in the form $y = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial expressions and the degree of $q(x)$ is at least 1. The graphs of rational functions contain **vertical asymptotes** or **holes** when the denominator is undefined. You also learned how to do arithmetic with rational expressions.

## EXERCISES

**1.** Sketch the graph of each equation. Label foci when appropriate.

**a.** $\frac{x-3}{-2} = (y-2)^2$

**b.** $x^2 + (y-2)^2 = 16$

**c.** $\left(\frac{y+2.5}{3}\right)^2 - \left(\frac{x-1}{4}\right)^2 = 1$

**d.** $\left(\frac{x}{5}\right)^2 + 3y^2 = 1$

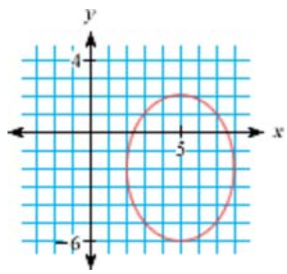
**2.** Consider this ellipse.

**a.** Write the equation for the graph shown in standard form.

**b.** Write the parametric equations for this graph.

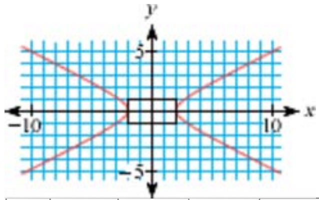
**c.** Name the coordinates of the center and foci.

**d.** Write the general quadratic form of the equation for this graph.



**3.** Consider the hyperbola graphed at right.




**a.** Write the equations of the asymptotes for this hyperbola.

**b.** Write the general quadratic equation for this hyperbola.

**c.** Write an equation that will give the vertical distance, $d$, between the asymptote with positive slope and a point on the upper portion of the right branch of the hyperbola as a function of the point's $x$-coordinate, $x$.

**d.** Use the function from 3c to fill in this table. What does this tell you about the relationship between the function and its asymptote?

| ***x*** | 2 | 10 | 20 | 100 |
|---|---|---|---|---|
| ***d*** | | | | |

**4.** Write the general quadratic equation $x^2 + y^2 + 8x - 2y - 8 = 0$ in standard form. Identify the shape described by the equation and describe its features.

**5.** Write the general quadratic equation $y^2 - 8y - 4x + 28 = 0$ in standard form. Determine the vertex, focus, and directrix of the parabola defined by this equation. Sketch a graph.

**6.** **APPLICATION** Pure gold is too soft to be used for jewelry, so gold is always mixed with other metals. 18-karat gold is 75% gold and 25% other metals. How much pure gold must be mixed with 5 oz of 18-karat gold to make a 22–karat (91.7%) gold mixture?

**7.** Write an equation of each rational function described as a translation of the graph of $y = \frac{1}{x}$.

**a.** The rational function has asymptotes $x = -2$ and $y = 1$.

**b.** The rational function has asymptotes $x = 0$ and $y = -4$.

**8.** Graph $y = \frac{2x-14}{x-5}$. Write equations for the horizontal and vertical asymptotes.

**9.** How can you modify the equation $y = \frac{2x-14}{x-5}$ so that the graph of the new equation is the same as the original graph except for a hole at $x = -3$? Verify your new equation by graphing it on your calculator.

**10.** On her way to school, Ellen drives at a steady speed for the first 2 mi. After glancing at her watch, she drives 20 mi/h faster during the remaining 3.5 mi. How fast does she drive during the two portions of this trip if the total time of her trip is 10 min?

**11.** Rewrite each expression as a single rational expression in factored form.

**a.** $\dfrac{2x}{(x-2)(x+1)} + \dfrac{x+3}{x^2-4}$ **b.** $\dfrac{x^2}{x+1} \cdot \dfrac{3x-6}{x^2-2x}$ **c.** $\dfrac{x^2-5x-6}{x} \div \dfrac{x^2-8x+12}{x^2-1}$

**12.** Solve this system of equations algebraically, then confirm your answer graphically.

$$\begin{cases} x^2 + y^2 = 4 \\ (x+1)^2 - \dfrac{y^2}{3} = 1 \end{cases}$$

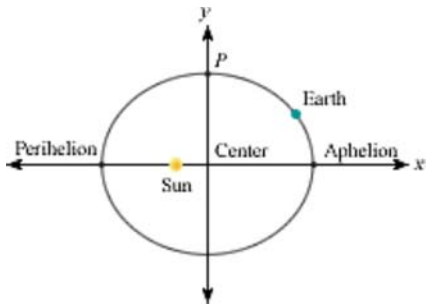
## MIXED REVIEW

**13.** Write an equation of the image of the absolute-value function, $y = |x|$, after performing each of the following transformations in order. Sketch a graph of your final equation.

**a.** Stretch vertically by a factor of 2.
**b.** Then translate right 4 units.
**c.** Then translate down 3 units.

**14.** Earth's orbit is an ellipse with the Sun at one of the foci. Perihelion is the point at which Earth is closest to the Sun, and aphelion is the point at which it is farthest from the Sun. The distances from perihelion to the Sun and from the Sun to aphelion are in an approximate ratio of 59:61. If the total distance from aphelion to perihelion along the major axis is about 186 million miles, approximate

**a.** The distance from perihelion to the Sun.
**b.** The distance from aphelion to the Sun.
**c.** The distance from aphelion to the center.
**d.** The distance from the center to the Sun.
**e.** In this ellipse it can be shown that the distance from the Sun to point $P$ equals the distance from aphelion to the center. Using this information, find the distance from the center to $P$.
**f.** Write an equation that models the orbit of Earth around the Sun.




These four circular snow sculptures by British environmental sculptor Andy Goldsworthy (b 1956) were made from bricks of packed snow at the Arctic Circle.

**15.** Perform the following matrix computations. If a computation is not possible, explain why.

$$[A] = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix} \qquad [B] = \begin{bmatrix} 0 & -2 \\ 5 & 0 \\ 3 & -1 \end{bmatrix} \qquad [C] = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

**a.** $[A][B]$
**b.** $[A]+[B]$
**c.** $[A]-[C]$
**d.** $[B][A]$
**e.** $3[C]-[A]$

**16.** Identify each sequence as arithmetic, geometric, or neither. State the next three terms, and then write a recursive formula to generate each sequence.

**a.** 9,12,15,18, . . . **b.** 1, 1, 2, 3, 5, 8, 13, . . . **c.** – 3, 6, – 12, 24, . . .

**17.** Evan is standing at one corner of the football field when he sees his dog, Spot, start to run diagonally across the field as shown. Evan knows that Spot can run to the opposite corner in 15 s. The dimensions of the

football field are 100 yd by 52 yd.

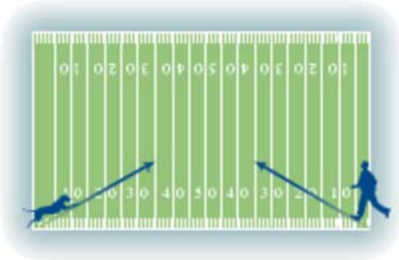
**a.** What is Spot's rate in yards per second?

**b.** What is Spot's angle with the horizontal axis?

**c.** Write equations that model the motion of Spot running from corner to corner.

**d.** Write equations that show Evan running at the same rate as Spot from one corner to an opposite corner as shown in the diagram above.

**e.** If Spot and Evan both start at the same time, when and where do they meet?



**18.** D'Andre surveyed a randomly chosen group of 15 teachers at his school and asked them how many students were enrolled in their third-period classes. Here is the data set he collected.

{27, 29, 18, 34, 42, 38, 34, 33, 25, 28, 45, 35, 32, 19, 36}

**a.** List the mean, median, and mode.

**b.** Make a box plot of the data.

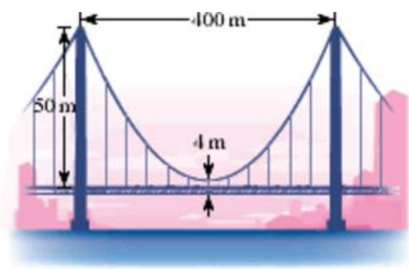
**c.** Calculate the standard deviation. What does this tell you about the data? If the standard deviation were smaller, what would it tell you about the data?

**19.** Rewrite each equation in standard form, and identify the type of curve.

**a.** $25x^2 - 4y^2 + 100 = 0$ **b.** $4y^2 - 10x + 16y + 36 = 0$

**c.** $4x^2 + 4y^2 + 24x - 8y + 39 = 0$ **d.** $3x^2 + 5y^2 - 12x + 20y + 8 = 0$

**20.** The towers of a parabolic suspension bridge are 400 m apart and reach 50 m above the suspended roadway. The cable is 4 m above the roadway at the halfway point. Write an equation that models the shape of the cable. Assume the origin, (0, 0), is located at the halfway point of the roadway.




**21.** Solve algebraically. Round answers to the nearest hundredth.

**a.** $4 + 5^x = 18$ **b.** $12(0.5)^{2x} = 30$ **c.** $\log_3 15 = \dfrac{\log x}{\log 3}$

**d.** $\log_6 100 = x$ **e.** $2 \log x = 2.5$ **f.** $\log_5 5^3 = x$

**g.** $4 \log x = \log 16$ **h.** $\log(5 + x) - \log 5 = 2$ **i.** $x \log 5^x = 12$

**22.** The chart below shows average fuel efficiency of new U.S. passenger cars.

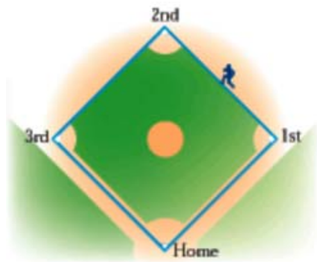
**Average Fuel Efficiency**

| Year | 1970 | 1975 | 1980 | 1985 | 1990 | 1995 | 1998 | 1999 |
|---|---|---|---|---|---|---|---|---|
| Fuel efficiency (mi/gal) | 14.1 | 15.1 | 22.6 | 26.3 | 26.9 | 27.7 | 28.1 | 28.2 |

(*The New York Times Almanac 2002*)

**a.** Find the median-median line for the data.
**b.** What is the root mean square error for the median-median line model?
**c.** What is the real-world meaning of the root mean square error in 22b?
**d.** Is the median-median line a good model to use to predict fuel efficiency in the future?

**23.** The bases on a baseball diamond form a square that is 90 ft on each side. Deanna has a 12 ft lead and can run the remaining 78 ft from first to second base at 28 ft/s. The catcher releases the ball from home plate toward second base 1.5 s after Deanna starts to steal the base, and the ball travels 125 ft/s. Write parametric equations to simulate this situation and determine whether Deanna is successful. Explain your solution.




**24.** Use $P = 1 + 3i$, $Q = -2 + i$, and $R = 3 - 5i$ to evaluate each expression. Give answers in the form $a + bi$.

**a.** $P + Q - R$ **b.** $PQ$ **c.** $Q^2$ **d.** $P \div Q$

## TAKE ANOTHER LOOK

**1.** The equation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ is the standard form of an ellipse. What shape will the equations $\left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 = 1$, $\left(\frac{x}{a}\right)^4 + \left(\frac{y}{b}\right)^4 = 1$, or generally $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 1$ create? These equations are called Lamé curves, or superellipses, when $n > 2$. Investigate several Lamé curve graphs in a friendly graphing window. Assume that $a > 0$ and $b > 0$. You already know what effect the values of $a$ and $b$ have on an ellipse. Do they have the same effect on a Lamé curve? For fixed values of $a$ and $b$, try different $n$-values, including positive, negative, whole-number, and rational values. Explore the graph shapes and properties for different values of $n$. Summarize your discoveries.

**2.** How can you find the horizontal asymptote of a rational function without graphing or using a table? Use a calculator to explore these functions, and look for patterns. Some equations may not have horizontal asymptotes. Make a conjecture about how to determine the equation of a horizontal asymptote just by looking at the equation.

$y = \dfrac{3x^2 + 4x - 5}{2x^4 + 2}$ $\quad y = \dfrac{4x^4 - 2x^3 - 2}{x^5 - 5x}$ $\quad y = \dfrac{3x^2 + 4x - 5}{2x^2}$

$y = \dfrac{4x^5 - 2x^3 - 2}{x^5 - 5x}$ $\quad y = \dfrac{-x^3}{x^2 - 2x + 5}$ $\quad y = \dfrac{3x^4 + 2x}{5x^2 - 1}$

**3.** You have seen that for an ellipse, the sum of the distances from any point to the two foci is constant. For a hyperbola, the difference of the distances from any point to the two foci is constant. What shape is created if the product of the distances is constant? What if the ratio of the distances is constant? You may want to use geometry software to explore these patterns.

**4.** In Lesson 9.5, you explored the number of points of intersection of two conic sections. You saw that, for example, a hyperbola and an ellipse can intersect 0, 1, 2, 3, or 4 times. However, points of intersection on a coordinate plane only include the real solutions to a system of equations. If you include nonreal answers, how many solutions can a system of conic sections have? Consider each pair of the four conic sections. Explain how the number of solutions is related to the exponents in the original equations.

## Assessing What You've Learned

GIVE A PRESENTATION By yourself or with a group, demonstrate how to factor the equation of a rational function. Then describe how to find asymptotes, holes, and intercepts, and graph the equation. Or present your solution to a project or Take Another Look activity from this chapter.

PERFORMANCE ASSESSMENT While a classmate, teacher, or family member observes, demonstrate how to convert the equation of a conic section in general quadratic form to standard form. Describe the features of the conic section, then draw a graph of the equation.

ORGANIZE YOUR NOTEBOOK Update your notebook to include the equations for the four conic sections. Include methods of graphing and how to find foci, vertices, centers, and asymptotes.