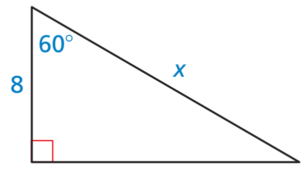
9.1–9.4 Quiz

1. In a right triangle, θ is an acute angle and $\sin \theta = \frac{2}{7}$. Evaluate the other five trigonometric functions of θ. *(Section 9.1)*

Find the value of x for the right triangle. *(Section 9.1)*

2.



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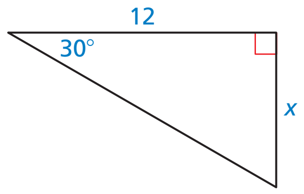

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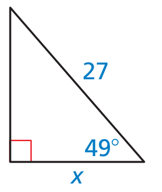


Draw an angle with the given measure in standard position. Then find one positive angle and one negative angle that are coterminal with the given angle. *(Section 9.2)*

5. 40°

6. $\frac{5\pi}{6}$

7. −960°

Convert the degree measure to radians or the radian measure to degrees. *(Section 9.2)*

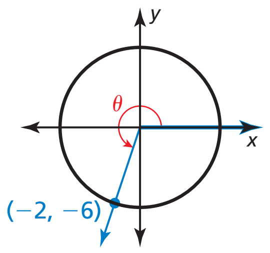
8. $\frac{3\pi}{10}$

9. −60°

10. 72°

Evaluate the six trigonometric functions of θ. *(Section 9.3)*

11.



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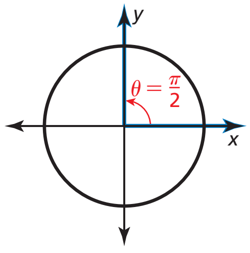

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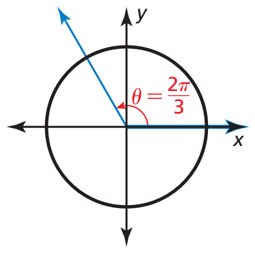


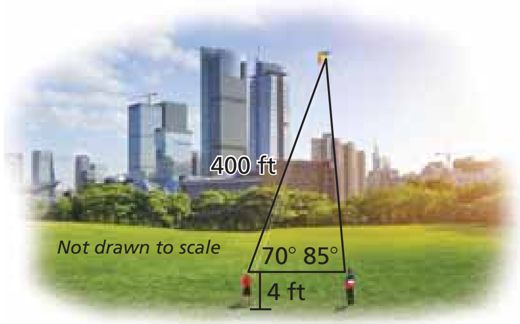
14. Identify the amplitude and period of $g(x) = 3 \sin x$. Then graph the function and describe the graph of g as a transformation of the graph of $f(x) = \sin x$. *(Section 9.4)*

15. Identify the amplitude and period of $g(x) = \cos 5\pi x + 3$. Then graph the function and describe the graph of g as a transformation of the graph of $f(x) = \cos x$. *(Section 9.4)*

16. You are flying a kite at an angle of 70°. You have let out a total of 400 feet of string and are holding the reel steady 4 feet above the ground. *(Section 9.1)*

a. How high above the ground is the kite?

b. A friend watching the kite estimates that the angle of elevation to the kite is 85°. How far from your friend are you standing?



17. The top of the Space Needle in Seattle, Washington, is a revolving, circular restaurant. The restaurant has a radius of 47.25 feet and makes one complete revolution in about an hour. You have dinner at a window table from 7:00 P.M. to 8:55 P.M. Compare the distance you revolve with the distance of a person seated 5 feet away from the windows. *(Section 9.2)*

9.5 Graphing Other Trigonometric Functions

Essential Question What are the characteristics of the graph of the tangent function?

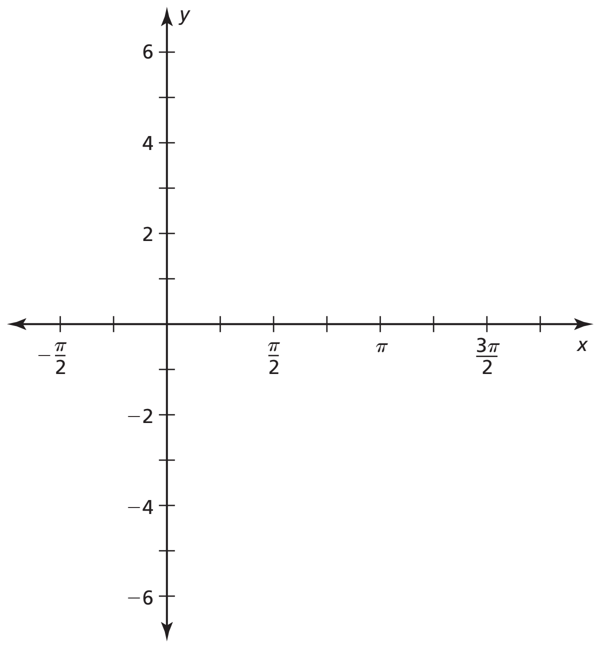
EXPLORATION 1 Graphing the Tangent Function

Work with a partner.

a. Complete the table for $y = \tan x$, where x is an angle measure in radians.

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$y = \tan x$									
x	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$
$y = \tan x$									

b. The graph of $y = \tan x$ has vertical asymptotes at x-values where $\tan x$ is undefined. Plot the points (x, y) from part (a). Then use the asymptotes to sketch the graph of $y = \tan x$.



c. For the graph of $y = \tan x$, identify the asymptotes, the x-intercepts, and the intervals for which the function is increasing or decreasing over $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$. Is the tangent function *even*, *odd*, or *neither*?

MAKING SENSE OF PROBLEMS

To be proficient in math, you need to consider analogous problems and try special cases of the original problem in order to gain insight into its solution.

Communicate Your Answer

2. What are the characteristics of the graph of the tangent function?

3. Describe the asymptotes of the graph of $y = \cot x$ on the interval $-\frac{\pi}{2} < x < \frac{3\pi}{2}$.

9.5 Lesson

Core Vocabulary

Previous
asymptote
period
amplitude
x-intercept
transformations

What You Will Learn

- Explore characteristics of tangent and cotangent functions.
- Graph tangent and cotangent functions.
- Graph secant and cosecant functions.

Exploring Tangent and Cotangent Functions

The graphs of tangent and cotangent functions are related to the graphs of the parent functions $y = \tan x$ and $y = \cot x$, which are graphed below.

← x approaches $-\frac{\pi}{2}$ | x approaches $\frac{\pi}{2}$ →

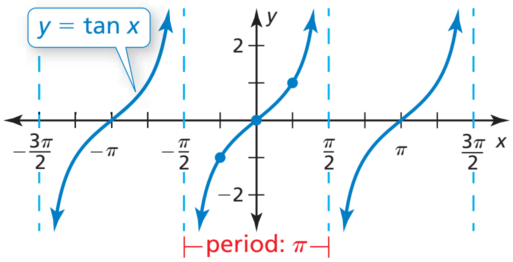
x	$-\frac{\pi}{2}$	-1.57	-1.5	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	1.5	1.57	$\frac{\pi}{2}$
$y = \tan x$	Undef.	-1256	-14.10	-1	0	1	14.10	1256	Undef.

← $\tan x$ approaches $-\infty$ | $\tan x$ approaches ∞ →

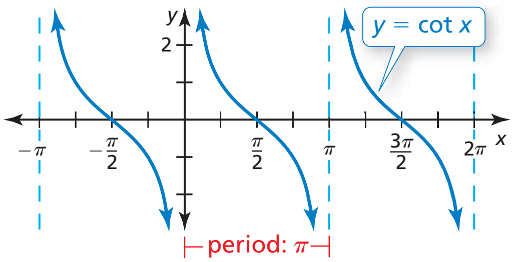
Because $\tan x = \frac{\sin x}{\cos x}$, $\tan x$ is undefined for x-values at which $\cos x = 0$, such as

$x = \pm\frac{\pi}{2} \approx \pm 1.571.$

The table indicates that the graph has asymptotes at these values. The table represents one cycle of the graph, so the period of the graph is π.



You can use a similar approach to graph $y = \cot x$. Because $\cot x = \frac{\cos x}{\sin x}$, $\cot x$ is undefined for x-values at which $\sin x = 0$, which are multiples of π. The graph has asymptotes at these values. The period of the graph is also π.



Core Concept

Characteristics of $y = \tan x$ and $y = \cot x$

The functions $y = \tan x$ and $y = \cot x$ have the following characteristics.

- The domain of $y = \tan x$ is all real numbers except odd multiples of $\frac{\pi}{2}$. At these x-values, the graph has vertical asymptotes.
- The domain of $y = \cot x$ is all real numbers except multiples of π. At these x-values, the graph has vertical asymptotes.
- The range of each function is all real numbers. So, the functions do not have maximum or minimum values, and the graphs do not have an amplitude.
- The period of each graph is π.
- The x-intercepts for $y = \tan x$ occur when $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.
- The x-intercepts for $y = \cot x$ occur when $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$.

STUDY TIP

Odd multiples of $\frac{\pi}{2}$ are values such as these:

$\pm 1 \cdot \frac{\pi}{2} = \pm\frac{\pi}{2}$

$\pm 3 \cdot \frac{\pi}{2} = \pm\frac{3\pi}{2}$

$\pm 5 \cdot \frac{\pi}{2} = \pm\frac{5\pi}{2}$

Graphing Tangent and Cotangent Functions

The graphs of $y = a \tan bx$ and $y = a \cot bx$ represent transformations of their parent functions. The value of a indicates a vertical stretch ($a > 1$) or a vertical shrink ($0 < a < 1$). The value of b indicates a horizontal stretch ($0 < b < 1$) or a horizontal shrink ($b > 1$) and changes the period of the graph.

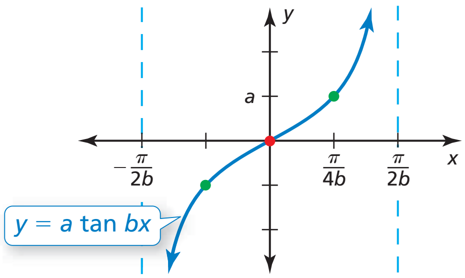
Core Concept

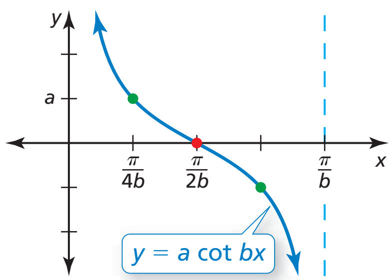
Period and Vertical Asymptotes of $y = a \tan bx$ and $y = a \cot bx$

The period and vertical asymptotes of the graphs of $y = a \tan bx$ and $y = a \cot bx$, where a and b are nonzero real numbers, are as follows.

- The period of the graph of each function is $\frac{\pi}{|b|}$.
- The vertical asymptotes for $y = a \tan bx$ are at odd multiples of $\frac{\pi}{2|b|}$.
- The vertical asymptotes for $y = a \cot bx$ are at multiples of $\frac{\pi}{|b|}$.

Each graph below shows five key x-values that you can use to sketch the graphs of $y = a \tan bx$ and $y = a \cot bx$ for $a > 0$ and $b > 0$. These are the x-intercept, the x-values where the asymptotes occur, and the x-values halfway between the x-intercept and the asymptotes. At each halfway point, the value of the function is either a or $-a$.



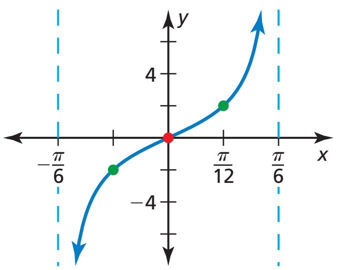


EXAMPLE 1 **Graphing a Tangent Function**

Graph one period of $g(x) = 2 \tan 3x$. Describe the graph of g as a transformation of the graph of $f(x) = \tan x$.

SOLUTION

The function is of the form $g(x) = a \tan bx$ where $a = 2$ and $b = 3$. So, the period is $\frac{\pi}{|b|} = \frac{\pi}{3}$.



Intercept: $(0, 0)$

Asymptotes: $x = \frac{\pi}{2|b|} = \frac{\pi}{2(3)}$, or $x = \frac{\pi}{6}$; $\quad x = -\frac{\pi}{2|b|} = -\frac{\pi}{2(3)}$, or $x = -\frac{\pi}{6}$

Halfway points: $\left(\frac{\pi}{4b}, a\right) = \left(\frac{\pi}{4(3)}, 2\right) = \left(\frac{\pi}{12}, 2\right)$;

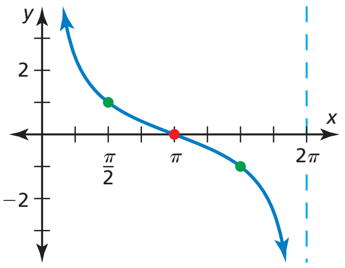
$$\left(-\frac{\pi}{4b}, -a\right) = \left(-\frac{\pi}{4(3)}, -2\right) = \left(-\frac{\pi}{12}, -2\right)$$

▶ The graph of g is a vertical stretch by a factor of 2 and a horizontal shrink by a factor of $\frac{1}{3}$ of the graph of f.

EXAMPLE 2 Graphing a Cotangent Function

Graph one period of $g(x) = \cot \frac{1}{2}x$. Describe the graph of g as a transformation of the graph of $f(x) = \cot x$.

SOLUTION



The function is of the form $g(x) = a \cot bx$ where $a = 1$ and $b = \frac{1}{2}$. So, the period is

$$\frac{\pi}{|b|} = \frac{\pi}{\frac{1}{2}} = 2\pi.$$

Intercept: $\left(\frac{\pi}{2b}, 0\right) = \left(\frac{\pi}{2\left(\frac{1}{2}\right)}, 0\right) = (\pi, 0)$

Asymptotes: $x = 0$; $x = \frac{\pi}{|b|} = \frac{\pi}{\frac{1}{2}}$, or $x = 2\pi$

Halfway points: $\left(\frac{\pi}{4b}, a\right) = \left(\frac{\pi}{4\left(\frac{1}{2}\right)}, 1\right) = \left(\frac{\pi}{2}, 1\right)$; $\left(\frac{3\pi}{4b}, -a\right) = \left(\frac{3\pi}{4\left(\frac{1}{2}\right)}, -1\right) = \left(\frac{3\pi}{2}, -1\right)$

▶ The graph of g is a horizontal stretch by a factor of 2 of the graph of f.

Monitoring Progress

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Graph one period of the function. Describe the graph of g as a transformation of the graph of its parent function.

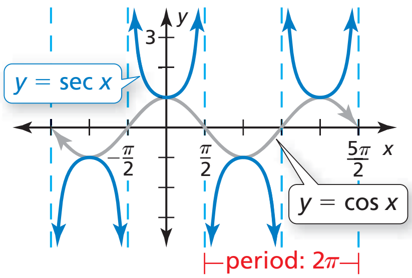
1. $g(x) = \tan 2x$ **2.** $g(x) = \frac{1}{3} \cot x$ **3.** $g(x) = 2 \cot 4x$ **4.** $g(x) = 5 \tan \pi x$

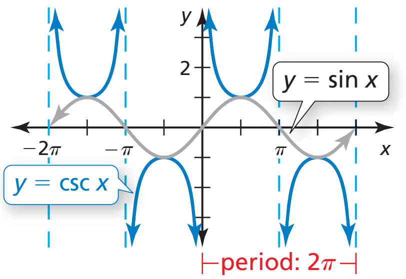
STUDY TIP

Because $\sec x = \frac{1}{\cos x}$, $\sec x$ is undefined for x-values at which $\cos x = 0$. The graph of $y = \sec x$ has vertical asymptotes at these x-values. You can use similar reasoning to understand the vertical asymptotes of the graph of $y = \csc x$.

Graphing Secant and Cosecant Functions

The graphs of secant and cosecant functions are related to the graphs of the parent functions $y = \sec x$ and $y = \csc x$, which are shown below.





Core Concept

Characteristics of $y = \sec x$ and $y = \csc x$

The functions $y = \sec x$ and $y = \csc x$ have the following characteristics.

- The domain of $y = \sec x$ is all real numbers except odd multiples of $\frac{\pi}{2}$. At these x-values, the graph has vertical asymptotes.
- The domain of $y = \csc x$ is all real numbers except multiples of π. At these x-values, the graph has vertical asymptotes.
- The range of each function is $y \leq -1$ and $y \geq 1$. So, the graphs do not have an amplitude.
- The period of each graph is 2π.

To graph $y = a \sec bx$ or $y = a \csc bx$, first graph the function $y = a \cos bx$ or $y = a \sin bx$, respectively. Then use the asymptotes and several points to sketch a graph of the function. Notice that the value of b represents a horizontal stretch or shrink by a factor of $\frac{1}{b}$, so the period of $y = a \sec bx$ and $y = a \csc bx$ is $\frac{2\pi}{|b|}$.

EXAMPLE 3 Graphing a Secant Function

Graph one period of $g(x) = 2 \sec x$. Describe the graph of g as a transformation of the graph of $f(x) = \sec x$.

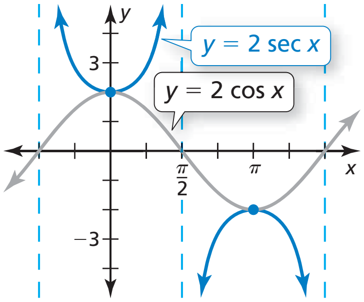
SOLUTION

Step 1 Graph the function $y = 2 \cos x$.

The period is $\frac{2\pi}{1} = 2\pi$.

Step 2 Graph asymptotes of g. Because the asymptotes of g occur when $2 \cos x = 0$, graph $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$.

Step 3 Plot points on g, such as $(0, 2)$ and $(\pi, -2)$. Then use the asymptotes to sketch the curve.



▶ The graph of g is a vertical stretch by a factor of 2 of the graph of f.

EXAMPLE 4 Graphing a Cosecant Function

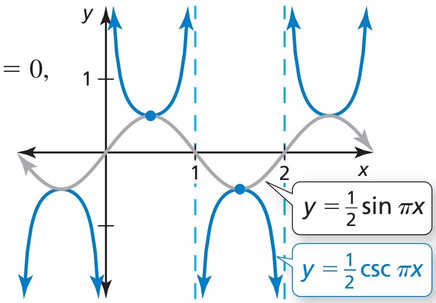
Graph one period of $g(x) = \frac{1}{2} \csc \pi x$. Describe the graph of g as a transformation of the graph of $f(x) = \csc x$.

SOLUTION

Step 1 Graph the function $y = \frac{1}{2} \sin \pi x$. The period is $\frac{2\pi}{\pi} = 2$.

Step 2 Graph asymptotes of g. Because the asymptotes of g occur when $\frac{1}{2} \sin \pi x = 0$, graph $x = 0$, $x = 1$, and $x = 2$.

Step 3 Plot points on g, such as $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{3}{2}, -\frac{1}{2}\right)$. Then use the asymptotes to sketch the curve.



▶ The graph of g is a vertical shrink by a factor of $\frac{1}{2}$ and a horizontal shrink by a factor of $\frac{1}{\pi}$ of the graph of f.

LOOKING FOR A PATTERN

In Examples 3 and 4, notice that the plotted points are on both graphs. Also, these points represent a local maximum on one graph and a local minimum on the other graph.

Monitoring Progress

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Graph one period of the function. Describe the graph of g as a transformation of the graph of its parent function.

5. $g(x) = \csc 3x$ **6.** $g(x) = \frac{1}{2} \sec x$ **7.** $g(x) = 2 \csc 2x$ **8.** $g(x) = 2 \sec \pi x$

9.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Explain why the graphs of the tangent, cotangent, secant, and cosecant functions do not have an amplitude.

2. **COMPLETE THE SENTENCE** The ________ and ________ functions are undefined for x-values at which $\sin x = 0$.

3. **COMPLETE THE SENTENCE** The period of the function $y = \sec x$ is ______, and the period of $y = \cot x$ is ______.

4. **WRITING** Explain how to graph a function of the form $y = a \sec bx$.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, graph one period of the function. Describe the graph of g as a transformation of the graph of its parent function. *(See Examples 1 and 2.)*

5. $g(x) = 2 \tan x$
6. $g(x) = 3 \tan x$
7. $g(x) = \cot 3x$
8. $g(x) = \cot 2x$
9. $g(x) = 3 \cot \frac{1}{4}x$
10. $g(x) = 4 \cot \frac{1}{2}x$
11. $g(x) = \frac{1}{2} \tan \pi x$
12. $g(x) = \frac{1}{3} \tan 2\pi x$

13. **ERROR ANALYSIS** Describe and correct the error in finding the period of the function $y = \cot 3x$.

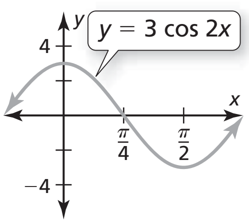
✗ Period: $\frac{2\pi}{|b|} = \frac{2\pi}{3}$

14. **ERROR ANALYSIS** Describe and correct the error in describing the transformation of $f(x) = \tan x$ represented by $g(x) = 2 \tan 5x$.

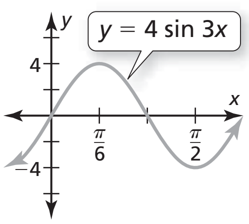
✗ A vertical stretch by a factor of 5 and a horizontal shrink by a factor of $\frac{1}{2}$.

15. **ANALYZING RELATIONSHIPS** Use the given graph to graph each function.

a. $f(x) = 3 \sec 2x$



b. $f(x) = 4 \csc 3x$



16. **USING EQUATIONS** Which of the following are asymptotes of the graph of $y = 3 \tan 4x$?

Ⓐ $x = \frac{\pi}{8}$　　Ⓑ $x = \frac{\pi}{4}$

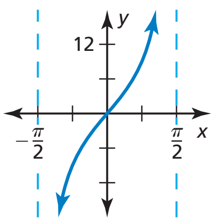
Ⓒ $x = 0$　　Ⓓ $x = -\frac{5\pi}{8}$

In Exercises 17–24, graph one period of the function. Describe the graph of g as a transformation of the graph of its parent function. *(See Examples 3 and 4.)*

17. $g(x) = 3 \csc x$
18. $g(x) = 2 \csc x$
19. $g(x) = \sec 4x$
20. $g(x) = \sec 3x$
21. $g(x) = \frac{1}{2} \sec \pi x$
22. $g(x) = \frac{1}{4} \sec 2\pi x$
23. $g(x) = \csc \frac{\pi}{2}x$
24. $g(x) = \csc \frac{\pi}{4}x$

ATTENDING TO PRECISION In Exercises 25–28, use the graph to write a function of the form $y = a \tan bx$.

25.



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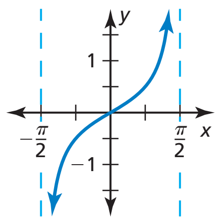

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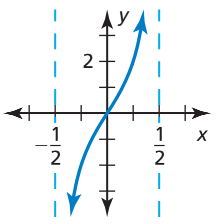

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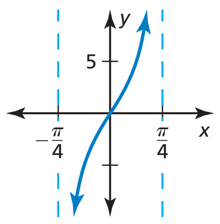
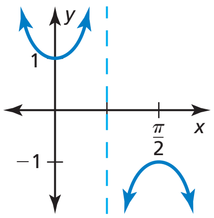


USING STRUCTURE In Exercises 29–34, match the equation with the correct graph. Explain your reasoning.

29. $g(x) = 4 \tan x$ **30.** $g(x) = 4 \cot x$

31. $g(x) = 4 \csc \pi x$ **32.** $g(x) = 4 \sec \pi x$

33. $g(x) = \sec 2x$ **34.** $g(x) = \csc 2x$

A.


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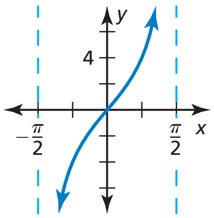

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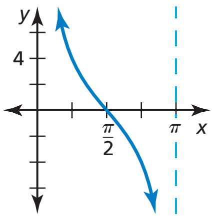

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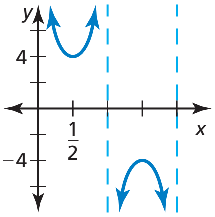

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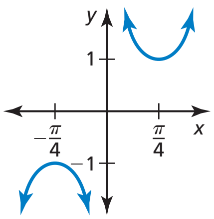
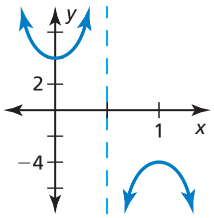


F.


35. WRITING Explain why there is more than one tangent function whose graph passes through the origin and has asymptotes at $x = -\pi$ and $x = \pi$.

36. USING EQUATIONS Graph one period of each function. Describe the transformation of the graph of its parent function.

a. $g(x) = \sec x + 3$ **b.** $g(x) = \csc x - 2$

c. $g(x) = \cot(x - \pi)$ **d.** $g(x) = -\tan x$

WRITING EQUATIONS In Exercises 37–40, write a rule for *g* that represents the indicated transformation of the graph of *f*.

37. $f(x) = \cot 2x$; translation 3 units up and $\frac{\pi}{2}$ units left

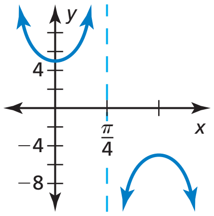
38. $f(x) = 2 \tan x$; translation π units right, followed by a horizontal shrink by a factor of $\frac{1}{3}$

39. $f(x) = 5 \sec (x - \pi)$; translation 2 units down, followed by a reflection in the x-axis

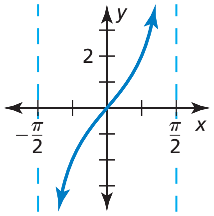
40. $f(x) = 4 \csc x$; vertical stretch by a factor of 2 and a reflection in the x-axis

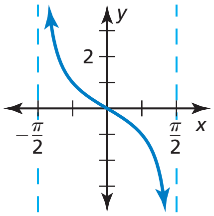
41. MULTIPLE REPRESENTATIONS Which function has a greater local maximum value? Which has a greater local minimum value? Explain.

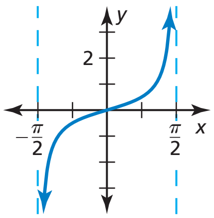
A. $f(x) = \frac{1}{4} \csc \pi x$

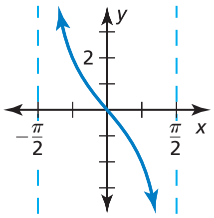
B.


42. ANALYZING RELATIONSHIPS Order the functions from the least average rate of change to the greatest average rate of change over the interval $-\frac{\pi}{4} < x < \frac{\pi}{4}$.

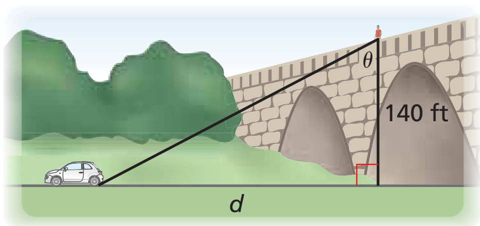
A.


B.


C.


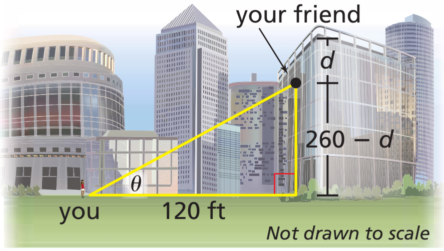
D.


43. REASONING You are standing on a bridge 140 feet above the ground. You look down at a car traveling away from the underpass. The distance d (in feet) the car is from the base of the bridge can be modeled by $d = 140 \tan \theta$. Graph the function. Describe what happens to θ as d increases.



44. USING TOOLS You use a video camera to pan up the Statue of Liberty. The height h (in feet) of the part of the Statue of Liberty that can be seen through your video camera after time t (in seconds) can be modeled by $h = 100 \tan \frac{\pi}{36} t$. Graph the function using a graphing calculator. What viewing window did you use? Explain.

45. **MODELING WITH MATHEMATICS** You are standing 120 feet from the base of a 260-foot building. You watch your friend go down the side of the building in a glass elevator.



a. Write an equation that gives the distance d (in feet) your friend is from the top of the building as a function of the angle of elevation θ.

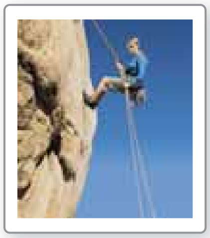
b. Graph the function found in part (a). Explain how the graph relates to this situation.

46. **MODELING WITH MATHEMATICS** You are standing 300 feet from the base of a 200-foot cliff. Your friend is rappelling down the cliff.

a. Write an equation that gives the distance d (in feet) your friend is from the top of the cliff as a function of the angle of elevation θ.

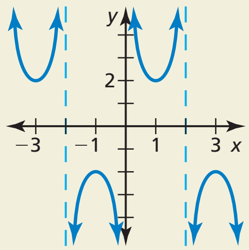
b. Graph the function found in part (a).

c. Use a graphing calculator to determine the angle of elevation when your friend has rappelled halfway down the cliff.



47. **MAKING AN ARGUMENT** Your friend states that it is not possible to write a cosecant function that has the same graph as $y = \sec x$. Is your friend correct? Explain your reasoning.

48. **HOW DO YOU SEE IT?** Use the graph to answer each question.



a. What is the period of the graph?

b. What is the range of the function?

c. Is the function of the form $f(x) = a \csc bx$ or $f(x) = a \sec bx$? Explain.

49. **ABSTRACT REASONING** Rewrite $a \sec bx$ in terms of $\cos bx$. Use your results to explain the relationship between the local maximums and minimums of the cosine and secant functions.

50. **THOUGHT PROVOKING** A trigonometric equation that is true for all values of the variable for which both sides of the equation are defined is called a *trigonometric identity*. Use a graphing calculator to graph the function

$$y = \frac{1}{2}\left(\tan \frac{x}{2} + \cot \frac{x}{2}\right).$$

Use your graph to write a trigonometric identity involving this function. Explain your reasoning.

51. **CRITICAL THINKING** Find a tangent function whose graph intersects the graph of $y = 2 + 2 \sin x$ only at minimum points of the sine function.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write a cubic function whose graph passes through the given points. *(Section 4.9)*

52. $(-1, 0), (1, 0), (3, 0), (0, 3)$

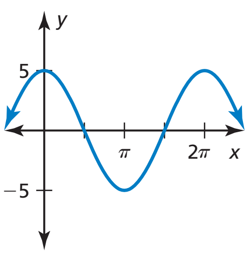
53. $(-2, 0), (1, 0), (3, 0), (0, -6)$

54. $(-1, 0), (2, 0), (3, 0), (1, -2)$

55. $(-3, 0), (-1, 0), (3, 0), (-2, 1)$

Find the amplitude and period of the graph of the function. *(Section 9.4)*

56.



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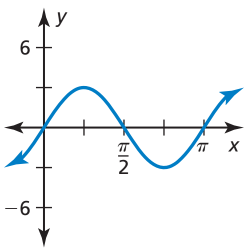

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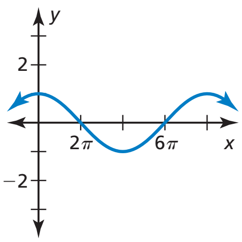


9.6 Modeling with Trigonometric Functions

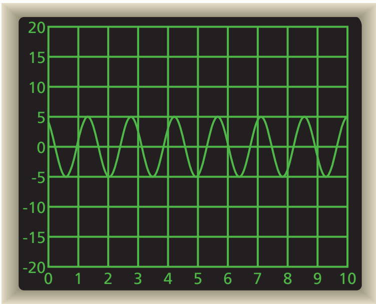
Essential Question

What are the characteristics of the real-life problems that can be modeled by trigonometric functions?

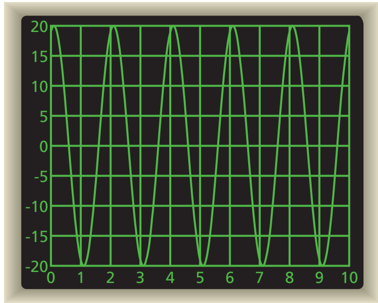
EXPLORATION 1 Modeling Electric Currents

Work with a partner. Find a sine function that models the electric current shown in each oscilloscope screen. State the amplitude and period of the graph.

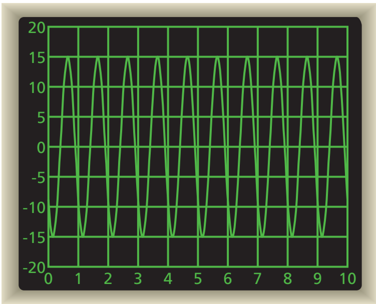
a.



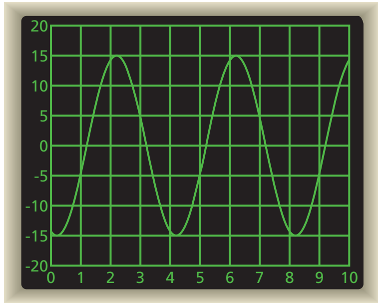
b.



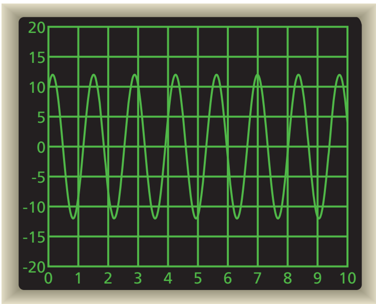
c.



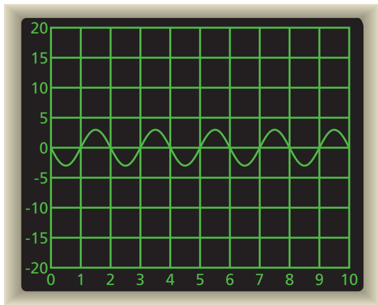
d.



e.



f.



MODELING WITH MATHEMATICS

To be proficient in math, you need to apply the mathematics you know to solve problems arising in everyday life.

Communicate Your Answer

2. What are the characteristics of the real-life problems that can be modeled by trigonometric functions?

3. Use the Internet or some other reference to find examples of real-life situations that can be modeled by trigonometric functions.

9.6 Lesson

Core Vocabulary

frequency, *p. 506*
sinusoid, *p. 507*

Previous
amplitude
period
midline

What You Will Learn

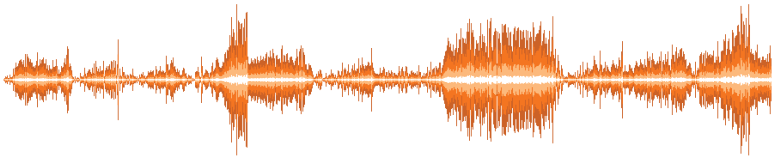
- Interpret and use frequency.
- Write trigonometric functions.
- Use technology to find trigonometric models.

Frequency

The periodic nature of trigonometric functions makes them useful for modeling *oscillating* motions or repeating patterns that occur in real life. Some examples are sound waves, the motion of a pendulum, and seasons of the year. In such applications, the reciprocal of the period is called the **frequency**, which gives the number of cycles per unit of time.

EXAMPLE 1 Using Frequency

A sound consisting of a single frequency is called a *pure tone*. An audiometer produces pure tones to test a person's auditory functions. An audiometer produces a pure tone with a frequency f of 2000 hertz (cycles per second). The maximum pressure P produced from the pure tone is 2 millipascals. Write and graph a sine model that gives the pressure P as a function of the time t (in seconds).



SOLUTION

Step 1 Find the values of a and b in the model $P = a \sin bt$. The maximum pressure is 2, so $a = 2$. Use the frequency f to find b.

$$\text{frequency} = \frac{1}{\text{period}}$$ Write relationship involving frequency and period.

$$2000 = \frac{b}{2\pi}$$ Substitute.

$$4000\pi = b$$ Multiply each side by 2π.

The pressure P as a function of time t is given by $P = 2 \sin 4000\pi t$.

Step 2 Graph the model. The amplitude is $a = 2$ and the period is

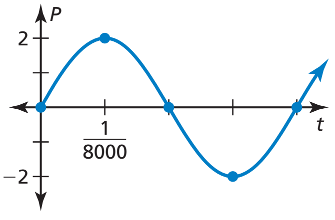
$$\frac{1}{f} = \frac{1}{2000}.$$

The key points are:

$$\text{Intercepts: } (0, 0);\ \left(\frac{1}{2} \cdot \frac{1}{2000}, 0\right) = \left(\frac{1}{4000}, 0\right);\ \left(\frac{1}{2000}, 0\right)$$

$$\text{Maximum: } \left(\frac{1}{4} \cdot \frac{1}{2000}, 2\right) = \left(\frac{1}{8000}, 2\right)$$

$$\text{Minimum: } \left(\frac{3}{4} \cdot \frac{1}{2000}, -2\right) = \left(\frac{3}{8000}, -2\right)$$



The graph of $P = 2 \sin 4000\pi t$ is shown at the left.

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1. **WHAT IF?** In Example 1, how would the function change when the audiometer produced a pure tone with a frequency of 1000 hertz?

Writing Trigonometric Functions

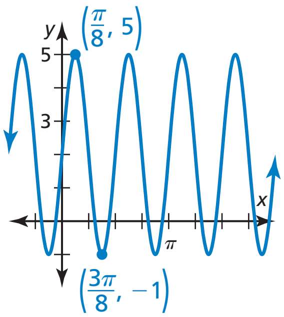
Graphs of sine and cosine functions are called **sinusoids**. One method to write a sine or cosine function that models a sinusoid is to find the values of a, b, h, and k for

$$y = a \sin b(x - h) + k \qquad \text{or} \qquad y = a \cos b(x - h) + k$$

where $|a|$ is the amplitude, $\frac{2\pi}{b}$ is the period ($b > 0$), h is the horizontal shift, and k is the vertical shift.

EXAMPLE 2 **Writing a Trigonometric Function**

Write a function for the sinusoid shown.



SOLUTION

Step 1 Find the maximum and minimum values. From the graph, the maximum value is 5 and the minimum value is -1.

Step 2 Identify the vertical shift, k. The value of k is the mean of the maximum and minimum values.

$$k = \frac{(\text{maximum value}) + (\text{minimum value})}{2} = \frac{5 + (-1)}{2} = \frac{4}{2} = 2$$

Step 3 Decide whether the graph should be modeled by a sine or cosine function. Because the graph crosses the midline $y = 2$ on the y-axis, the graph is a sine curve with no horizontal shift. So, $h = 0$.

STUDY TIP

Because the graph repeats every $\frac{\pi}{2}$ units, the period is $\frac{\pi}{2}$.

Step 4 Find the amplitude and period. The period is

$$\frac{\pi}{2} = \frac{2\pi}{b} \quad \Rightarrow \quad b = 4.$$

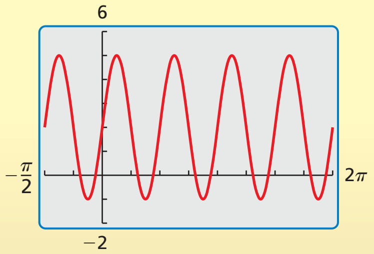
The amplitude is

$$|a| = \frac{(\text{maximum value}) - (\text{minimum value})}{2} = \frac{5 - (-1)}{2} = \frac{6}{2} = 3.$$

The graph is not a reflection, so $a > 0$. Therefore, $a = 3$.

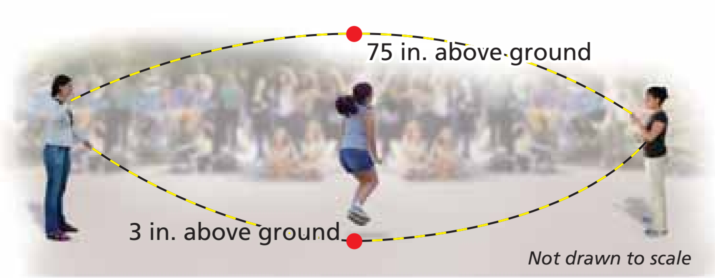
▶ The function is $y = 3 \sin 4x + 2$. Check this by graphing the function on a graphing calculator.

Check



EXAMPLE 3 Modeling Circular Motion

Two people swing jump ropes, as shown in the diagram. The highest point of the middle of each rope is 75 inches above the ground, and the lowest point is 3 inches. The rope makes 2 revolutions per second. Write a model for the height h (in inches) of a rope as a function of the time t (in seconds) given that the rope is at its lowest point when $t = 0$.



SOLUTION

A rope oscillates between 3 inches and 75 inches above the ground. So, a sine or cosine function may be an appropriate model for the height over time.

Step 1 Identify the maximum and minimum values. The maximum height of a rope is 75 inches. The minimum height is 3 inches.

Step 2 Identify the vertical shift, k.

$$k = \frac{(\text{maximum value}) + (\text{minimum value})}{2} = \frac{75 + 3}{2} = 39$$

Step 3 Decide whether the height should be modeled by a sine or cosine function. When $t = 0$, the height is at its minimum. So, use a cosine function whose graph is a reflection in the x-axis with no horizontal shift ($h = 0$).

Step 4 Find the amplitude and period.

The amplitude is $|a| = \frac{(\text{maximum value}) - (\text{minimum value})}{2} = \frac{75 - 3}{2} = 36.$

Because the graph is a reflection in the x-axis, $a < 0$. So, $a = -36$. Because a rope is rotating at a rate of 2 revolutions per second, one revolution is completed in 0.5 second. So, the period is $\frac{2\pi}{b} = 0.5$, and $b = 4\pi$.

▶ A model for the height of a rope is $h(t) = -36 \cos 4\pi t + 39$.

Check

Use the *table* feature of a graphing calculator to check your model.

X	Y1
0	3
.25	75
.5	3
.75	75
1	3
1.25	75
1.5	3

X=0

2 revolutions

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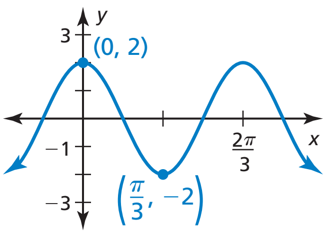
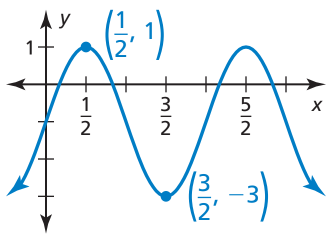
Write a function for the sinusoid.

2.



3.

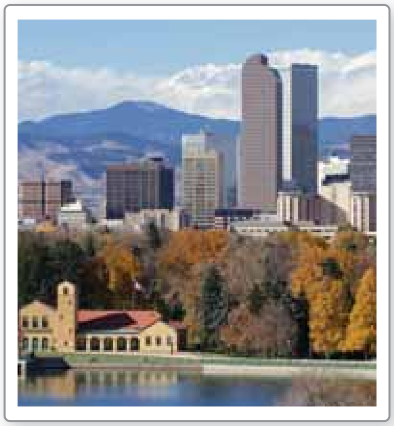


4. **WHAT IF?** Describe how the model in Example 3 changes when the lowest point of a rope is 5 inches above the ground and the highest point is 70 inches above the ground.

Using Technology to Find Trigonometric Models

Another way to model sinusoids is to use a graphing calculator that has a sinusoidal regression feature.

EXAMPLE 4 Using Sinusoidal Regression



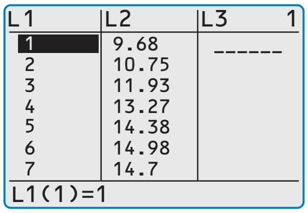
The table shows the numbers N of hours of daylight in Denver, Colorado, on the 15th day of each month, where $t = 1$ represents January. Write a model that gives N as a function of t and interpret the period of its graph.

t	1	2	3	4	5	6
N	9.68	10.75	11.93	13.27	14.38	14.98

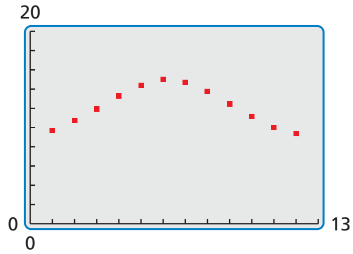
t	7	8	9	10	11	12
N	14.70	13.73	12.45	11.17	9.98	9.38

SOLUTION

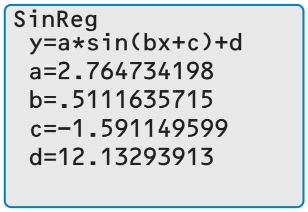
Step 1 Enter the data in a graphing calculator.



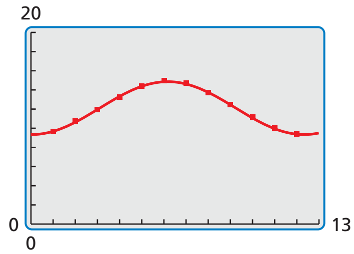
Step 2 Make a scatter plot.



Step 3 The scatter plot appears sinusoidal. So, perform a sinusoidal regression.



Step 4 Graph the data and the model in the same viewing window.



STUDY TIP

Notice that the *sinusoidal regression* feature finds a model of the form $y = a \sin(bx + c) + d$. This function has a period of $\frac{2\pi}{b}$ because it can be written as $y = a \sin b\left(x + \frac{c}{b}\right) + d$.

▶ The model appears to be a good fit. So, a model for the data is $N = 2.76 \sin(0.511t - 1.59) + 12.1$. The period, $\frac{2\pi}{0.511} \approx 12$, makes sense because there are 12 months in a year and you would expect this pattern to continue in following years.

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5. The table shows the average daily temperature T (in degrees Fahrenheit) for a city each month, where $m = 1$ represents January. Write a model that gives T as a function of m and interpret the period of its graph.

m	1	2	3	4	5	6	7	8	9	10	11	12
T	29	32	39	48	59	68	74	72	65	54	45	35

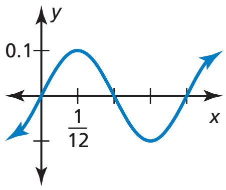
9.6 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** Graphs of sine and cosine functions are called __________.

2. **WRITING** Describe how to find the frequency of the function whose graph is shown.



Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, find the frequency of the function.

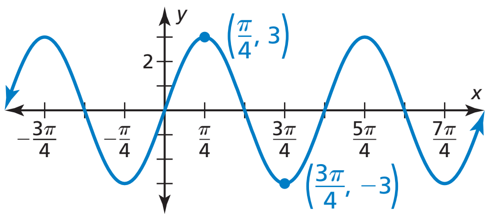
3. $y = \sin x$
4. $y = \sin 3x$
5. $y = \cos 4x + 2$
6. $y = -\cos 2x$
7. $y = \sin 3\pi x$
8. $y = \cos \frac{\pi x}{4}$
9. $y = \frac{1}{2} \cos 0.75x - 8$
10. $y = 3 \sin 0.2x + 6$

11. **MODELING WITH MATHEMATICS** The lowest frequency of sounds that can be heard by humans is 20 hertz. The maximum pressure P produced from a sound with a frequency of 20 hertz is 0.02 millipascal. Write and graph a sine model that gives the pressure P as a function of the time t (in seconds). *(See Example 1.)*

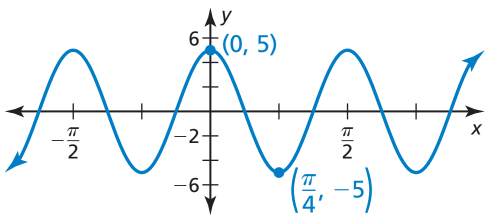
12. **MODELING WITH MATHEMATICS** A middle-A tuning fork vibrates with a frequency f of 440 hertz (cycles per second). You strike a middle-A tuning fork with a force that produces a maximum pressure of 5 pascals. Write and graph a sine model that gives the pressure P as a function of the time t (in seconds).

In Exercises 13–16, write a function for the sinusoid. *(See Example 2.)*

13.



14.



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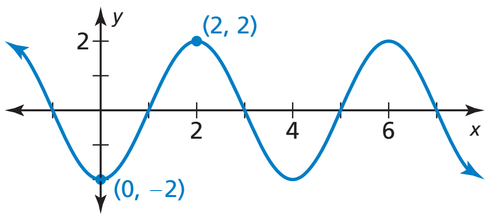

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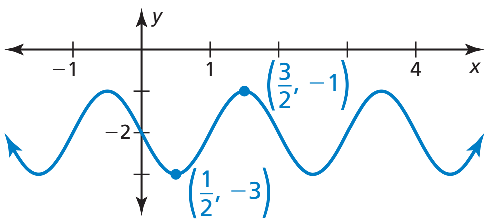


17. **ERROR ANALYSIS** Describe and correct the error in finding the amplitude of a sinusoid with a maximum point at (2, 10) and a minimum point at (4, −6).

✗
$$|a| = \frac{(\text{maximum value}) + (\text{minimum value})}{2} = \frac{10 - 6}{2} = 2$$

18. **ERROR ANALYSIS** Describe and correct the error in finding the vertical shift of a sinusoid with a maximum point at (3, −2) and a minimum point at (7, −8).

✗
$$k = \frac{(\text{maximum value}) + (\text{minimum value})}{2} = \frac{7 + 3}{2} = 5$$

19. **MODELING WITH MATHEMATICS** One of the largest sewing machines in the world has a *flywheel* (which turns as the machine sews) that is 5 feet in diameter. The highest point of the handle at the edge of the flywheel is 9 feet above the ground, and the lowest point is 4 feet. The wheel makes a complete turn every 2 seconds. Write a model for the height h (in feet) of the handle as a function of the time t (in seconds) given that the handle is at its lowest point when $t = 0$. *(See Example 3.)*

20. **MODELING WITH MATHEMATICS** The Great Laxey Wheel, located on the Isle of Man, is the largest working water wheel in the world. The highest point of a bucket on the wheel is 70.5 feet above the viewing platform, and the lowest point is 2 feet below the viewing platform. The wheel makes a complete turn every 24 seconds. Write a model for the height h (in feet) of the bucket as a function of time t (in seconds) given that the bucket is at its lowest point when $t = 0$.

USING TOOLS **In Exercises 21 and 22, the time t is measured in months, where $t = 1$ represents January. Write a model that gives the average monthly high temperature D as a function of t and interpret the period of the graph.** *(See Example 4.)*

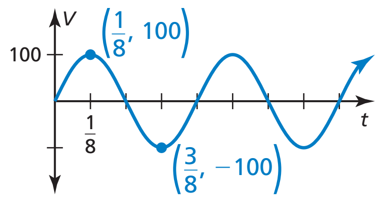
21.

Air Temperatures in Apple Valley, CA						
t	1	2	3	4	5	6
D	60	63	69	75	85	94
t	7	8	9	10	11	12
D	99	99	93	81	69	60

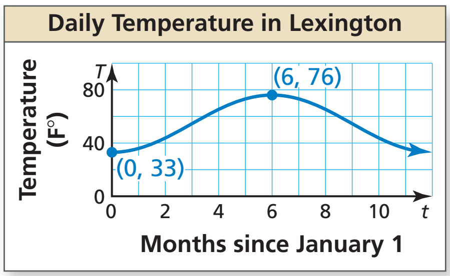
22.

Water Temperatures at Miami Beach, FL						
t	1	2	3	4	5	6
D	71	73	75	78	81	85
t	7	8	9	10	11	12
D	86	85	84	81	76	73

23. **MODELING WITH MATHEMATICS** A circuit has an alternating voltage of 100 volts that peaks every 0.5 second. Write a sinusoidal model for the voltage V as a function of the time t (in seconds).



24. **MULTIPLE REPRESENTATIONS** The graph shows the average daily temperature of Lexington, Kentucky. The average daily temperature of Louisville, Kentucky, is modeled by $y = -22 \cos \frac{\pi}{6}t + 57$, where y is the temperature (in degrees Fahrenheit) and t is the number of months since January 1. Which city has the greater average daily temperature? Explain.



25. USING TOOLS The table shows the numbers of employees N (in thousands) at a sporting goods company each year for 11 years. The time t is measured in years, with $t = 1$ representing the first year.

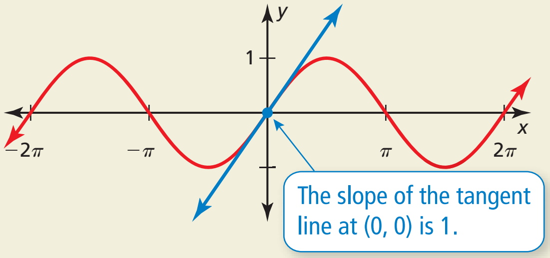
t	1	2	3	4	5	6
N	20.8	22.7	24.6	23.2	20	17.5

t	7	8	9	10	11	12
N	16.7	17.8	21	22	24.1	

a. Use sinusoidal regression to find a model that gives N as a function of t.

b. Predict the number of employees at the company in the 12th year.

26. THOUGHT PROVOKING The figure shows a tangent line drawn to the graph of the function $y = \sin x$. At several points on the graph, draw a tangent line to the graph and estimate its slope. Then plot the points (x, m), where m is the slope of the tangent line. What can you conclude?



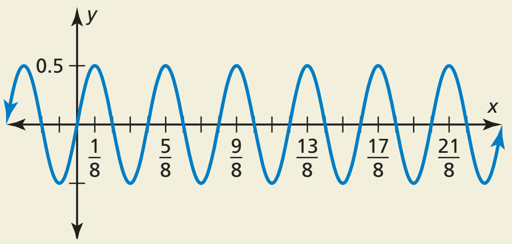
27. REASONING Determine whether you would use a sine or cosine function to model each sinusoid with the y-intercept described. Explain your reasoning.

a. The y-intercept occurs at the maximum value of the function.

b. The y-intercept occurs at the minimum value of the function.

c. The y-intercept occurs halfway between the maximum and minimum values of the function.

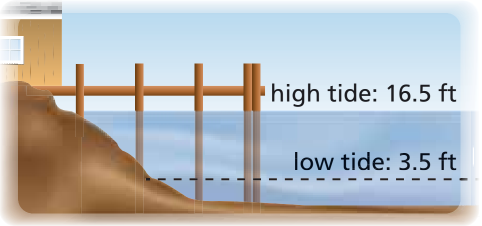
28. HOW DO YOU SEE IT? What is the frequency of the function whose graph is shown? Explain.



29. USING STRUCTURE During one cycle, a sinusoid has a minimum at $\left(\frac{\pi}{2}, 3\right)$ and a maximum at $\left(\frac{\pi}{4}, 8\right)$. Write a sine function *and* a cosine function for the sinusoid. Use a graphing calculator to verify that your answers are correct.

30. MAKING AN ARGUMENT Your friend claims that a function with a frequency of 2 has a greater period than a function with a frequency of $\frac{1}{2}$. Is your friend correct? Explain your reasoning.

31. PROBLEM SOLVING The low tide at a port is 3.5 feet and occurs at midnight. After 6 hours, the port is at high tide, which is 16.5 feet.



a. Write a sinusoidal model that gives the tide depth d (in feet) as a function of the time t (in hours). Let $t = 0$ represent midnight.

b. Find all the times when low and high tides occur in a 24-hour period.

c. Explain how the graph of the function you wrote in part (a) is related to a graph that shows the tide depth d at the port t hours after 3:00 A.M.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the expression. *(Section 5.2)*

32. $\frac{17}{\sqrt{2}}$ **33.** $\frac{3}{\sqrt{6} - 2}$ **34.** $\frac{8}{\sqrt{10} + 3}$ **35.** $\frac{13}{\sqrt{3} + \sqrt{11}}$

Expand the logarithmic expression. *(Section 6.5)*

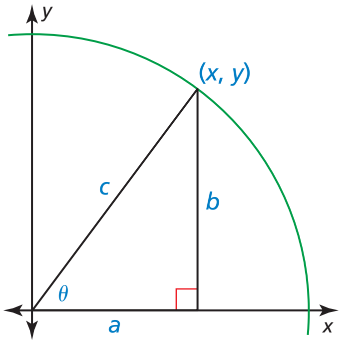
36. $\log_8 \frac{x}{7}$ **37.** $\ln 2x$ **38.** $\log_3 5x^3$ **39.** $\ln \frac{4x^6}{y}$

9.7 Using Trigonometric Identities

Essential Question How can you verify a trigonometric identity?

EXPLORATION 1 Writing a Trigonometric Identity

Work with a partner. In the figure, the point (x, y) is on a circle of radius c with center at the origin.



a. Write an equation that relates a, b, and c.

b. Write expressions for the sine and cosine ratios of angle θ.

c. Use the results from parts (a) and (b) to find the sum of $\sin^2\theta$ and $\cos^2\theta$. What do you observe?

d. Complete the table to verify that the identity you wrote in part (c) is valid for angles (of your choice) in each of the four quadrants.

	θ	$\sin^2 \theta$	$\cos^2 \theta$	$\sin^2 \theta + \cos^2 \theta$
QI				
QII				
QIII				
QIV				

EXPLORATION 2 Writing Other Trigonometric Identities

Work with a partner. The trigonometric identity you derived in Exploration 1 is called a Pythagorean identity. There are two other Pythagorean identities. To derive them, recall the four relationships:

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \qquad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta} \qquad \csc \theta = \frac{1}{\sin \theta}$$

a. Divide each side of the Pythagorean identity you derived in Exploration 1 by $\cos^2\theta$ and simplify. What do you observe?

b. Divide each side of the Pythagorean identity you derived in Exploration 1 by $\sin^2\theta$ and simplify. What do you observe?

REASONING ABSTRACTLY

To be proficient in math, you need to know and flexibly use different properties of operations and objects.

Communicate Your Answer

3. How can you verify a trigonometric identity?

4. Is $\sin \theta = \cos \theta$ a trigonometric identity? Explain your reasoning.

5. Give some examples of trigonometric identities that are different than those in Explorations 1 and 2.

9.7 Lesson

Core Vocabulary

trigonometric identity, *p. 514*

Previous

unit circle

STUDY TIP

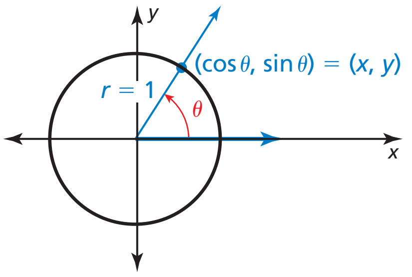
Note that $\sin^2 \theta$ represents $(\sin \theta)^2$ and $\cos^2 \theta$ represents $(\cos \theta)^2$.

What You Will Learn

- Use trigonometric identities to evaluate trigonometric functions and simplify trigonometric expressions.
- Verify trigonometric identities.

Using Trigonometric Identities

Recall that when an angle θ is in standard position with its terminal side intersecting the unit circle at (x, y), then $x = \cos \theta$ and $y = \sin \theta$. Because (x, y) is on a circle centered at the origin with radius 1, it follows that



$$x^2 + y^2 = 1$$

and

$$\cos^2 \theta + \sin^2 \theta = 1.$$

The equation $\cos^2 \theta + \sin^2 \theta = 1$ is true for any value of θ. A trigonometric equation that is true for all values of the variable for which both sides of the equation are defined is called a **trigonometric identity**. In Section 9.1, you used reciprocal identities to find the values of the cosecant, secant, and cotangent functions. These and other fundamental trigonometric identities are listed below.

Core Concept

Fundamental Trigonometric Identities

Reciprocal Identities

$$\csc \theta = \frac{1}{\sin \theta} \qquad \sec \theta = \frac{1}{\cos \theta} \qquad \cot \theta = \frac{1}{\tan \theta}$$

Tangent and Cotangent Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \qquad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

Pythagorean Identities

$$\sin^2 \theta + \cos^2 \theta = 1 \qquad 1 + \tan^2 \theta = \sec^2 \theta \qquad 1 + \cot^2 \theta = \csc^2 \theta$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta \qquad \tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$$

Negative Angle Identities

$$\sin(-\theta) = -\sin \theta \qquad \cos(-\theta) = \cos \theta \qquad \tan(-\theta) = -\tan \theta$$

In this section, you will use trigonometric identities to do the following.

- Evaluate trigonometric functions.
- Simplify trigonometric expressions.
- Verify other trigonometric identities.

EXAMPLE 1 Finding Trigonometric Values

Given that $\sin\theta = \frac{4}{5}$ and $\frac{\pi}{2} < \theta < \pi$, find the values of the other five trigonometric functions of θ.

SOLUTION

Step 1 Find $\cos\theta$.

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{Write Pythagorean identity.}$$

$$\left(\frac{4}{5}\right)^2 + \cos^2\theta = 1 \quad \text{Substitute } \frac{4}{5} \text{ for } \sin\theta.$$

$$\cos^2\theta = 1 - \left(\frac{4}{5}\right)^2 \quad \text{Subtract } \left(\frac{4}{5}\right)^2 \text{ from each side.}$$

$$\cos^2\theta = \frac{9}{25} \quad \text{Simplify.}$$

$$\cos\theta = \pm\frac{3}{5} \quad \text{Take square root of each side.}$$

$$\cos\theta = -\frac{3}{5} \quad \text{Because } \theta \text{ is in Quadrant II, } \cos\theta \text{ is negative.}$$

Step 2 Find the values of the other four trigonometric functions of θ using the values of $\sin\theta$ and $\cos\theta$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{-\frac{3}{5}}{\frac{4}{5}} = -\frac{3}{4}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{4}{5}} = \frac{5}{4} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{1}{-\frac{3}{5}} = -\frac{5}{3}$$

EXAMPLE 2 Simplifying Trigonometric Expressions

Simplify (a) $\tan\left(\frac{\pi}{2} - \theta\right)\sin\theta$ and (b) $\sec\theta\tan^2\theta + \sec\theta$.

SOLUTION

a. $\tan\left(\frac{\pi}{2} - \theta\right)\sin\theta = \cot\theta\sin\theta$ — Cofunction identity

$= \left(\frac{\cos\theta}{\sin\theta}\right)(\sin\theta)$ — Cotangent identity

$= \cos\theta$ — Simplify.

b. $\sec\theta\tan^2\theta + \sec\theta = \sec\theta(\sec^2\theta - 1) + \sec\theta$ — Pythagorean identity

$= \sec^3\theta - \sec\theta + \sec\theta$ — Distributive Property

$= \sec^3\theta$ — Simplify.

Monitoring Progress

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1. Given that $\cos\theta = \frac{1}{6}$ and $0 < \theta < \frac{\pi}{2}$, find the values of the other five trigonometric functions of θ.

Simplify the expression.

2. $\sin x \cot x \sec x$ **3.** $\cos\theta - \cos\theta\sin^2\theta$ **4.** $\frac{\tan x \csc x}{\sec x}$

Verifying Trigonometric Identities

You can use the fundamental identities from this chapter to verify new trigonometric identities. When verifying an identity, begin with the expression on one side. Use algebra and trigonometric properties to manipulate the expression until it is identical to the other side.

EXAMPLE 3 Verifying a Trigonometric Identity

Verify the identity $\dfrac{\sec^2 \theta - 1}{\sec^2 \theta} = \sin^2 \theta$.

SOLUTION

$$\frac{\sec^2 \theta - 1}{\sec^2 \theta} = \frac{\sec^2 \theta}{\sec^2 \theta} - \frac{1}{\sec^2 \theta} \qquad \text{Write as separate fractions.}$$

$$= 1 - \left(\frac{1}{\sec \theta}\right)^2 \qquad \text{Simplify.}$$

$$= 1 - \cos^2 \theta \qquad \text{Reciprocal identity}$$

$$= \sin^2 \theta \qquad \text{Pythagorean identity}$$

Notice that verifying an identity is not the same as solving an equation. When verifying an identity, you cannot assume that the two sides of the equation are equal because you are trying to verify that they are equal. So, you cannot use any properties of equality, such as adding the same quantity to each side of the equation.

EXAMPLE 4 Verifying a Trigonometric Identity

Verify the identity $\sec x + \tan x = \dfrac{\cos x}{1 - \sin x}$.

LOOKING FOR STRUCTURE

To verify the identity, you must introduce $1 - \sin x$ into the denominator. Multiply the numerator and the denominator by $1 - \sin x$ so you get an equivalent expression.

SOLUTION

$$\sec x + \tan x = \frac{1}{\cos x} + \tan x \qquad \text{Reciprocal identity}$$

$$= \frac{1}{\cos x} + \frac{\sin x}{\cos x} \qquad \text{Tangent identity}$$

$$= \frac{1 + \sin x}{\cos x} \qquad \text{Add fractions.}$$

$$= \frac{1 + \sin x}{\cos x} \cdot \frac{1 - \sin x}{1 - \sin x} \qquad \text{Multiply by } \frac{1 - \sin x}{1 - \sin x}.$$

$$= \frac{1 - \sin^2 x}{\cos x(1 - \sin x)} \qquad \text{Simplify numerator.}$$

$$= \frac{\cos^2 x}{\cos x(1 - \sin x)} \qquad \text{Pythagorean identity}$$

$$= \frac{\cos x}{1 - \sin x} \qquad \text{Simplify.}$$

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Verify the identity.

5. $\cot(-\theta) = -\cot \theta$

6. $\csc^2 x(1 - \sin^2 x) = \cot^2 x$

7. $\cos x \csc x \tan x = 1$

8. $(\tan^2 x + 1)(\cos^2 x - 1) = -\tan^2 x$

9.7 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Describe the difference between a trigonometric identity and a trigonometric equation.

2. **WRITING** Explain how to use trigonometric identities to determine whether $\sec(-\theta) = \sec\theta$ or $\sec(-\theta) = -\sec\theta$.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, find the values of the other five trigonometric functions of θ. *(See Example 1.)*

3. $\sin\theta = \frac{1}{3}, 0 < \theta < \frac{\pi}{2}$

4. $\sin\theta = -\frac{7}{10}, \pi < \theta < \frac{3\pi}{2}$

5. $\tan\theta = -\frac{3}{7}, \frac{\pi}{2} < \theta < \pi$

6. $\cot\theta = -\frac{2}{5}, \frac{\pi}{2} < \theta < \pi$

7. $\cos\theta = -\frac{5}{6}, \pi < \theta < \frac{3\pi}{2}$

8. $\sec\theta = \frac{9}{4}, \frac{3\pi}{2} < \theta < 2\pi$

9. $\cot\theta = -3, \frac{3\pi}{2} < \theta < 2\pi$

10. $\csc\theta = -\frac{5}{3}, \pi < \theta < \frac{3\pi}{2}$

In Exercises 11–20, simplify the expression. *(See Example 2.)*

11. $\sin x \cot x$

12. $\cos\theta(1 + \tan^2\theta)$

13. $\frac{\sin(-\theta)}{\cos(-\theta)}$

14. $\frac{\cos^2 x}{\cot^2 x}$

15. $\frac{\cos\left(\frac{\pi}{2} - x\right)}{\csc x}$

16. $\sin\left(\frac{\pi}{2} - \theta\right)\sec\theta$

17. $\frac{\csc^2 x - \cot^2 x}{\sin(-x)\cot x}$

18. $\frac{\cos^2 x \tan^2(-x) - 1}{\cos^2 x}$

19. $\frac{\cos\left(\frac{\pi}{2} - \theta\right)}{\csc\theta} + \cos^2\theta$

20. $\frac{\sec x \sin x + \cos\left(\frac{\pi}{2} - x\right)}{1 + \sec x}$

ERROR ANALYSIS In Exercises 21 and 22, describe and correct the error in simplifying the expression.

21. ✗
$$\begin{aligned} 1 - \sin^2\theta &= 1 - (1 + \cos^2\theta) \\ &= 1 - 1 - \cos^2\theta \\ &= -\cos^2\theta \end{aligned}$$

22. ✗
$$\begin{aligned} \tan x \csc x &= \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} \\ &= \frac{\cos x}{\sin^2 x} \end{aligned}$$

In Exercises 23–30, verify the identity. *(See Examples 3 and 4.)*

23. $\sin x \csc x = 1$

24. $\tan\theta \csc\theta \cos\theta = 1$

25. $\cos\left(\frac{\pi}{2} - x\right)\cot x = \cos x$

26. $\sin\left(\frac{\pi}{2} - x\right)\tan x = \sin x$

27. $\frac{\cos\left(\frac{\pi}{2} - \theta\right) + 1}{1 - \sin(-\theta)} = 1$

28. $\frac{\sin^2(-x)}{\tan^2 x} = \cos^2 x$

29. $\frac{1 + \cos x}{\sin x} + \frac{\sin x}{1 + \cos x} = 2\csc x$

30. $\frac{\sin x}{1 - \cos(-x)} = \csc x + \cot x$

31. **USING STRUCTURE** A function f is *odd* when $f(-x) = -f(x)$. A function f is *even* when $f(-x) = f(x)$. Which of the six trigonometric functions are odd? Which are even? Justify your answers using identities and graphs.

32. **ANALYZING RELATIONSHIPS** As the value of $\cos\theta$ increases, what happens to the value of $\sec\theta$? Explain your reasoning.

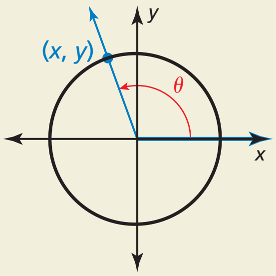
33. MAKING AN ARGUMENT Your friend simplifies an expression and obtains $\sec x \tan x - \sin x$. You simplify the same expression and obtain $\sin x \tan^2 x$. Are your answers equivalent? Justify your answer.

34. HOW DO YOU SEE IT? The figure shows the unit circle and the angle θ.

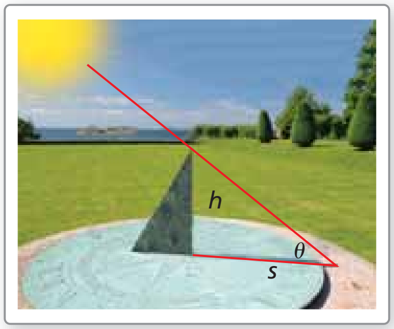
a. Is $\sin \theta$ positive or negative? $\cos \theta$? $\tan \theta$?

b. In what quadrant does the terminal side of $-\theta$ lie?

c. Is $\sin(-\theta)$ positive or negative? $\cos(-\theta)$? $\tan(-\theta)$?



35. MODELING WITH MATHEMATICS A vertical *gnomon* (the part of a sundial that projects a shadow) has height h. The length s of the shadow cast by the gnomon when the angle of the Sun above the horizon is θ can be modeled by the equation below. Show that the equation below is equivalent to $s = h \cot \theta$.



$$s = \frac{h \sin(90° - \theta)}{\sin \theta}$$

36. THOUGHT PROVOKING Explain how you can use a trigonometric identity to find all the values of x for which $\sin x = \cos x$.

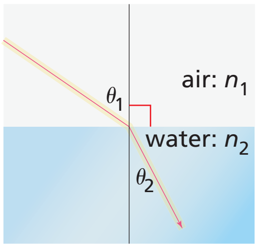
37. DRAWING CONCLUSIONS *Static friction* is the amount of force necessary to keep a stationary object on a flat surface from moving. Suppose a book weighing W pounds is lying on a ramp inclined at an angle θ. The coefficient of static friction u for the book can be found using the equation $uW \cos \theta = W \sin \theta$.

a. Solve the equation for u and simplify the result.

b. Use the equation from part (a) to determine what happens to the value of u as the angle θ increases from 0° to 90°.

38. PROBLEM SOLVING When light traveling in a medium (such as air) strikes the surface of a second medium (such as water) at an angle θ_1, the light begins to travel at a different angle θ_2. This change of direction is defined by Snell's law, $n_1 \sin \theta_1 = n_2 \sin \theta_2$, where n_1 and n_2 are the *indices of refraction* for the two mediums. Snell's law can be derived from the equation

$$\frac{n_1}{\sqrt{\cot^2 \theta_1 + 1}} = \frac{n_2}{\sqrt{\cot^2 \theta_2 + 1}}.$$



a. Simplify the equation to derive Snell's law.

b. What is the value of n_1 when $\theta_1 = 55°$, $\theta_2 = 35°$, and $n_2 = 2$?

c. If $\theta_1 = \theta_2$, then what must be true about the values of n_1 and n_2? Explain when this situation would occur.

39. WRITING Explain how transformations of the graph of the parent function $f(x) = \sin x$ support the cofunction identity $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$.

40. USING STRUCTURE Verify each identity.

a. $\ln|\sec \theta| = -\ln|\cos \theta|$

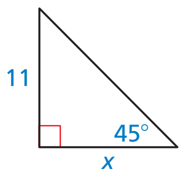
b. $\ln|\tan \theta| = \ln|\sin \theta| - \ln|\cos \theta|$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find the value of x for the right triangle. *(Section 9.1)*

41.



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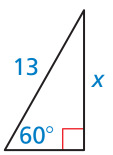

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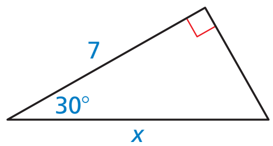


9.8 Using Sum and Difference Formulas

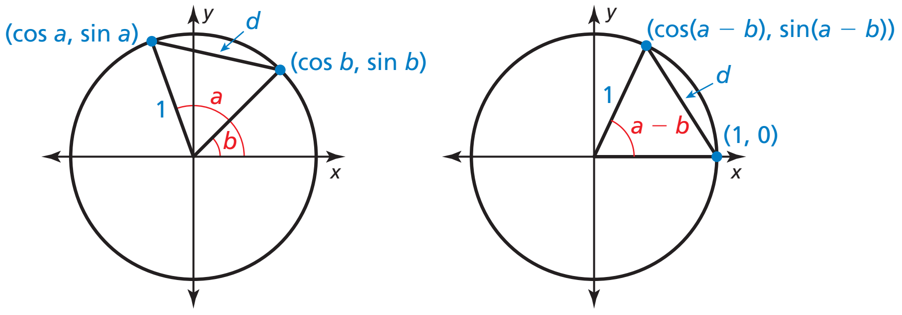
Essential Question

How can you evaluate trigonometric functions of the sum or difference of two angles?

EXPLORATION 1 Deriving a Difference Formula

Work with a partner.

a. Explain why the two triangles shown are congruent.



b. Use the Distance Formula to write an expression for d in the first unit circle.

c. Use the Distance Formula to write an expression for d in the second unit circle.

d. Write an equation that relates the expressions in parts (b) and (c). Then simplify this equation to obtain a formula for $\cos(a - b)$.

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to understand and use stated assumptions, definitions, and previously established results.

EXPLORATION 2 Deriving a Sum Formula

Work with a partner. Use the difference formula you derived in Exploration 1 to write a formula for $\cos(a + b)$ in terms of sine and cosine of a and b. *Hint*: Use the fact that

$$\cos(a + b) = \cos[a - (-b)].$$

EXPLORATION 3 Deriving Difference and Sum Formulas

Work with a partner. Use the formulas you derived in Explorations 1 and 2 to write formulas for $\sin(a - b)$ and $\sin(a + b)$ in terms of sine and cosine of a and b. *Hint*: Use the cofunction identities

$$\sin\left(\frac{\pi}{2} - a\right) = \cos a \text{ and } \cos\left(\frac{\pi}{2} - a\right) = \sin a$$

and the fact that

$$\cos\left[\left(\frac{\pi}{2} - a\right) + b\right] = \sin(a - b) \text{ and } \sin(a + b) = \sin[a - (-b)].$$

Communicate Your Answer

4. How can you evaluate trigonometric functions of the sum or difference of two angles?

5. a. Find the exact values of $\sin 75°$ and $\cos 75°$ using sum formulas. Explain your reasoning.

b. Find the exact values of $\sin 75°$ and $\cos 75°$ using difference formulas. Compare your answers to those in part (a).

9.8 Lesson

Core Vocabulary

Previous
ratio

What You Will Learn

- Use sum and difference formulas to evaluate and simplify trigonometric expressions.
- Use sum and difference formulas to solve trigonometric equations and rewrite real-life formulas.

Using Sum and Difference Formulas

In this lesson, you will study formulas that allow you to evaluate trigonometric functions of the sum or difference of two angles.

Core Concept

Sum and Difference Formulas

Sum Formulas	Difference Formulas
$\sin(a + b) = \sin a \cos b + \cos a \sin b$	$\sin(a - b) = \sin a \cos b - \cos a \sin b$
$\cos(a + b) = \cos a \cos b - \sin a \sin b$	$\cos(a - b) = \cos a \cos b + \sin a \sin b$
$\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$	$\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$

In general, $\sin(a + b) \neq \sin a + \sin b$. Similar statements can be made for the other trigonometric functions of sums and differences.

EXAMPLE 1 Evaluating Trigonometric Expressions

Find the exact value of (a) $\sin 15°$ and (b) $\tan \frac{7\pi}{12}$.

SOLUTION

a. $\sin 15° = \sin(60° - 45°)$ — Substitute 60° − 45° for 15°.

$= \sin 60° \cos 45° - \cos 60° \sin 45°$ — Difference formula for sine

$= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)$ — Evaluate.

$= \frac{\sqrt{6} - \sqrt{2}}{4}$ — Simplify.

▶ The exact value of $\sin 15°$ is $\frac{\sqrt{6} - \sqrt{2}}{4}$. Check this with a calculator.

Check

```
sin(15°)
      .2588190451
(√(6)-√(2))/4
      .2588190451
```

b. $\tan \frac{7\pi}{12} = \tan\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$ — Substitute $\frac{\pi}{3} + \frac{\pi}{4}$ for $\frac{7\pi}{12}$.

$= \dfrac{\tan \frac{\pi}{3} + \tan \frac{\pi}{4}}{1 - \tan \frac{\pi}{3} \tan \frac{\pi}{4}}$ — Sum formula for tangent

$= \dfrac{\sqrt{3} + 1}{1 - \sqrt{3} \cdot 1}$ — Evaluate.

$= -2 - \sqrt{3}$ — Simplify.

▶ The exact value of $\tan \frac{7\pi}{12}$ is $-2 - \sqrt{3}$. Check this with a calculator.

Check

```
tan(7π/12)
      -3.732050808
-2-√(3)
      -3.732050808
```

EXAMPLE 2 Using a Difference Formula

ANOTHER WAY

You can also use a Pythagorean identity and quadrant signs to find $\sin a$ and $\cos b$.

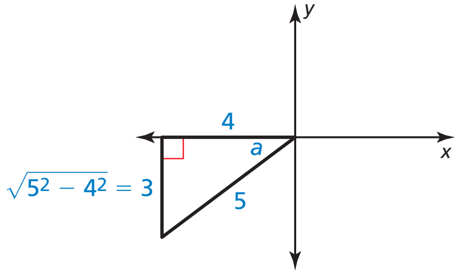
Find $\cos(a - b)$ given that $\cos a = -\frac{4}{5}$ with $\pi < a < \frac{3\pi}{2}$ and $\sin b = \frac{5}{13}$ with $0 < b < \frac{\pi}{2}$.

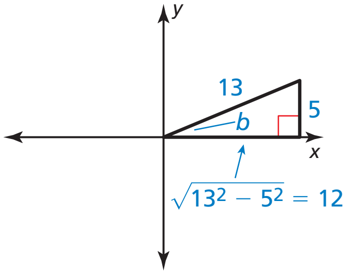
SOLUTION

Step 1 Find $\sin a$ and $\cos b$.

Because $\cos a = -\frac{4}{5}$ and a is in Quadrant III, $\sin a = -\frac{3}{5}$, as shown in the figure.

Because $\sin b = \frac{5}{13}$ and b is in Quadrant I, $\cos b = \frac{12}{13}$, as shown in the figure.





Step 2 Use the difference formula for cosine to find $\cos(a - b)$.

$$\cos(a - b) = \cos a \cos b + \sin a \sin b \qquad \text{Difference formula for cosine}$$

$$= -\frac{4}{5}\left(\frac{12}{13}\right) + \left(-\frac{3}{5}\right)\left(\frac{5}{13}\right) \qquad \text{Evaluate.}$$

$$= -\frac{63}{65} \qquad \text{Simplify.}$$

▶ The value of $\cos(a - b)$ is $-\frac{63}{65}$.

EXAMPLE 3 Simplifying an Expression

Simplify the expression $\cos(x + \pi)$.

SOLUTION

$$\cos(x + \pi) = \cos x \cos \pi - \sin x \sin \pi \qquad \text{Sum formula for cosine}$$

$$= (\cos x)(-1) - (\sin x)(0) \qquad \text{Evaluate.}$$

$$= -\cos x \qquad \text{Simplify.}$$

Monitoring Progress

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Find the exact value of the expression.

1. $\sin 105°$ **2.** $\cos 15°$ **3.** $\tan \frac{5\pi}{12}$ **4.** $\cos \frac{\pi}{12}$

5. Find $\sin(a - b)$ given that $\sin a = \frac{8}{17}$ with $0 < a < \frac{\pi}{2}$ and $\cos b = -\frac{24}{25}$ with $\pi < b < \frac{3\pi}{2}$.

Simplify the expression.

6. $\sin(x + \pi)$ **7.** $\cos(x - 2\pi)$ **8.** $\tan(x - \pi)$

Solving Equations and Rewriting Formulas

EXAMPLE 4 Solving a Trigonometric Equation

Solve $\sin\left(x + \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{3}\right) = 1$ for $0 \le x < 2\pi$.

SOLUTION

$$\sin\left(x + \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{3}\right) = 1 \quad \text{Write equation.}$$

$$\sin x \cos \frac{\pi}{3} + \cos x \sin \frac{\pi}{3} + \sin x \cos \frac{\pi}{3} - \cos x \sin \frac{\pi}{3} = 1 \quad \text{Use formulas.}$$

$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x = 1 \quad \text{Evaluate.}$$

$$\sin x = 1 \quad \text{Simplify.}$$

▶ In the interval $0 \le x < 2\pi$, the solution is $x = \frac{\pi}{2}$.

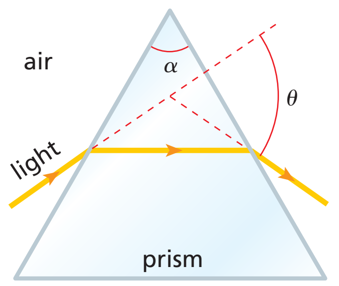
ANOTHER WAY

You can also solve the equation by using a graphing calculator. First, graph each side of the original equation. Then use the *intersect* feature to find the *x*-value(s) where the expressions are equal.

EXAMPLE 5 Rewriting a Real-Life Formula

The *index of refraction* of a transparent material is the ratio of the speed of light in a vacuum to the speed of light in the material. A triangular prism, like the one shown, can be used to measure the index of refraction using the formula

$$n = \frac{\sin\left(\frac{\theta}{2} + \frac{\alpha}{2}\right)}{\sin \frac{\theta}{2}}.$$



For $\alpha = 60°$, show that the formula can be rewritten as $n = \frac{\sqrt{3}}{2} + \frac{1}{2}\cot \frac{\theta}{2}$.

SOLUTION

$$n = \frac{\sin\left(\frac{\theta}{2} + 30°\right)}{\sin \frac{\theta}{2}} \quad \text{Write formula with } \frac{\alpha}{2} = \frac{60°}{2} = 30°.$$

$$= \frac{\sin \frac{\theta}{2} \cos 30° + \cos \frac{\theta}{2} \sin 30°}{\sin \frac{\theta}{2}} \quad \text{Sum formula for sine}$$

$$= \frac{\left(\sin \frac{\theta}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\cos \frac{\theta}{2}\right)\left(\frac{1}{2}\right)}{\sin \frac{\theta}{2}} \quad \text{Evaluate.}$$

$$= \frac{\frac{\sqrt{3}}{2}\sin \frac{\theta}{2}}{\sin \frac{\theta}{2}} + \frac{\frac{1}{2}\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} \quad \text{Write as separate fractions.}$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\cot \frac{\theta}{2} \quad \text{Simplify.}$$

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9. Solve $\sin\left(\frac{\pi}{4} - x\right) - \sin\left(x + \frac{\pi}{4}\right) = 1$ for $0 \le x < 2\pi$.

9.8 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** Write the expression $\cos 130° \cos 40° - \sin 130° \sin 40°$ as the cosine of an angle.

2. **WRITING** Explain how to evaluate $\tan 75°$ using either the sum or difference formula for tangent.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, find the exact value of the expression. *(See Example 1.)*

3. $\tan(-15°)$
4. $\tan 195°$
5. $\sin \frac{23\pi}{12}$
6. $\sin(-165°)$
7. $\cos 105°$
8. $\cos \frac{11\pi}{12}$
9. $\tan \frac{17\pi}{12}$
10. $\sin\left(-\frac{7\pi}{12}\right)$

In Exercises 11–16, evaluate the expression given that $\cos a = \frac{4}{5}$ with $0 < a < \frac{\pi}{2}$ and $\sin b = -\frac{15}{17}$ with $\frac{3\pi}{2} < b < 2\pi$. *(See Example 2.)*

11. $\sin(a + b)$
12. $\sin(a - b)$
13. $\cos(a - b)$
14. $\cos(a + b)$
15. $\tan(a + b)$
16. $\tan(a - b)$

In Exercises 17–22, simplify the expression. *(See Example 3.)*

17. $\tan(x + \pi)$
18. $\cos\left(x - \frac{\pi}{2}\right)$
19. $\cos(x + 2\pi)$
20. $\tan(x - 2\pi)$
21. $\sin\left(x - \frac{3\pi}{2}\right)$
22. $\tan\left(x + \frac{\pi}{2}\right)$

ERROR ANALYSIS **In Exercises 23 and 24, describe and correct the error in simplifying the expression.**

23. ✗

$$\tan\left(x + \frac{\pi}{4}\right) = \frac{\tan x + \tan \frac{\pi}{4}}{1 + \tan x \tan \frac{\pi}{4}} = \frac{\tan x + 1}{1 + \tan x} = 1$$

24. ✗

$$\sin\left(x - \frac{\pi}{4}\right) = \sin \frac{\pi}{4} \cos x - \cos \frac{\pi}{4} \sin x = \frac{\sqrt{2}}{2} \cos x - \frac{\sqrt{2}}{2} \sin x = \frac{\sqrt{2}}{2}(\cos x - \sin x)$$

25. What are the solutions of the equation $2 \sin x - 1 = 0$ for $0 \le x < 2\pi$?

Ⓐ $\frac{\pi}{3}$ Ⓑ $\frac{\pi}{6}$ Ⓒ $\frac{2\pi}{3}$ Ⓓ $\frac{5\pi}{6}$

26. What are the solutions of the equation $\tan x + 1 = 0$ for $0 \le x < 2\pi$?

Ⓐ $\frac{\pi}{4}$ Ⓑ $\frac{3\pi}{4}$ Ⓒ $\frac{5\pi}{4}$ Ⓓ $\frac{7\pi}{4}$

In Exercises 27–32, solve the equation for $0 \le x < 2\pi$. *(See Example 4.)*

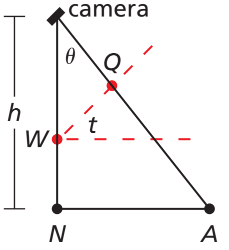
27. $\sin\left(x + \frac{\pi}{2}\right) = \frac{1}{2}$
28. $\tan\left(x - \frac{\pi}{4}\right) = 0$
29. $\cos\left(x + \frac{\pi}{6}\right) - \cos\left(x - \frac{\pi}{6}\right) = 1$
30. $\sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right) = 0$
31. $\tan(x + \pi) - \tan(\pi - x) = 0$
32. $\sin(x + \pi) + \cos(x + \pi) = 0$

33. **USING EQUATIONS** Derive the cofunction identity $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$ using the difference formula for sine.

34. MAKING AN ARGUMENT Your friend claims it is possible to use the difference formula for tangent to derive the cofunction identity $\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$. Is your friend correct? Explain your reasoning.

35. MODELING WITH MATHEMATICS A photographer is at a height h taking aerial photographs with a 35-millimeter camera. The ratio of the image length WQ to the length NA of the actual object is given by the formula

$$\frac{WQ}{NA} = \frac{35\tan(\theta - t) + 35\tan t}{h \tan \theta}$$



where θ is the angle between the vertical line perpendicular to the ground and the line from the camera to point A and t is the tilt angle of the film. When $t = 45°$, show that the formula can be rewritten as $\frac{WQ}{NA} = \frac{70}{h(1 + \tan\theta)}$. *(See Example 5.)*

36. MODELING WITH MATHEMATICS When a wave travels through a taut string, the displacement y of each point on the string depends on the time t and the point's position x. The equation of a *standing wave* can be obtained by adding the displacements of two waves traveling in opposite directions. Suppose a standing wave can be modeled by the formula

$y = A\cos\left(\frac{2\pi t}{3} - \frac{2\pi x}{5}\right) + A\cos\left(\frac{2\pi t}{3} + \frac{2\pi x}{5}\right)$.

When $t = 1$, show that the formula can be rewritten as

$y = -A\cos\frac{2\pi x}{5}$.

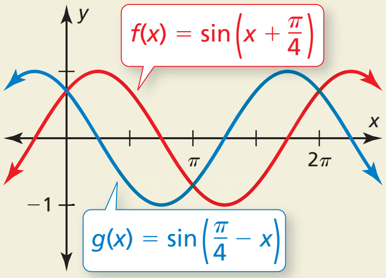
37. MODELING WITH MATHEMATICS The busy signal on a touch-tone phone is a combination of two tones with frequencies of 480 hertz and 620 hertz. The individual tones can be modeled by the equations:

480 hertz: $y_1 = \cos 960\pi t$

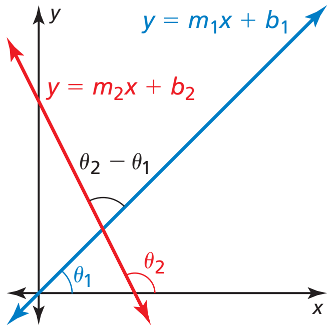
620 hertz: $y_2 = \cos 1240\pi t$

The sound of the busy signal can be modeled by $y_1 + y_2$. Show that $y_1 + y_2 = 2\cos 1100\pi t \cos 140\pi t$.

38. HOW DO YOU SEE IT? Explain how to use the figure to solve the equation $\sin\left(x + \frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4} - x\right) = 0$ for $0 \le x < 2\pi$.



39. MATHEMATICAL CONNECTIONS The figure shows the acute angle of intersection, $\theta_2 - \theta_1$, of two lines with slopes m_1 and m_2.



a. Use the difference formula for tangent to write an equation for $\tan(\theta_2 - \theta_1)$ in terms of m_1 and m_2.

b. Use the equation from part (a) to find the acute angle of intersection of the lines $y = x - 1$ and $y = \left(\frac{1}{\sqrt{3} - 2}\right)x + \frac{4 - \sqrt{3}}{2 - \sqrt{3}}$.

40. THOUGHT PROVOKING Rewrite each function. Justify your answers.

a. Write $\sin 3x$ as a function of $\sin x$.

b. Write $\cos 3x$ as a function of $\cos x$.

c. Write $\tan 3x$ as a function of $\tan x$.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. Check your solution(s). *(Section 7.5)*

41. $1 - \frac{9}{x - 2} = -\frac{7}{2}$

42. $\frac{12}{x} + \frac{3}{4} = \frac{8}{x}$

43. $\frac{2x - 3}{x + 1} = \frac{10}{x^2 - 1} + 5$

9.5–9.8 What Did You Learn?

Core Vocabulary

frequency, *p. 506*
sinusoid, *p. 507*
trigonometric identity, *p. 514*

Core Concepts

Section 9.5

Characteristics of $y = \tan x$ and $y = \cot x$, *p. 498*
Period and Vertical Asymptotes of $y = a \tan bx$ and $y = a \cot bx$, *p. 499*
Characteristics of $y = \sec x$ and $y = \csc x$, *p. 500*

Section 9.6

Frequency, *p. 506*
Writing Trigonometric Functions, *p. 507*
Using Technology to Find Trigonometric Models, *p. 509*

Section 9.7

Fundamental Trigonometric Identities, *p. 514*

Section 9.8

Sum and Difference Formulas, *p. 520*
Trigonometric Equations and Real-Life Formulas, *p. 522*

Mathematical Practices

1. Explain why the relationship between θ and d makes sense in the context of the situation in Exercise 43 on page 503.
2. How can you use definitions to relate the slope of a line with the tangent of an angle in Exercise 39 on page 524?

Performance Task

Lightening the Load

You need to move a heavy table across the room. What is the easiest way to move it? Should you push it? Should you tie a rope around one leg of the table and pull it? How can trigonometry help you make the right decision?

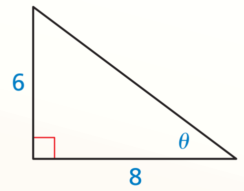
To explore the answers to these questions and more, go to ***BigIdeasMath.com***.

9 Chapter Review

Dynamic Solutions available at *BigIdeasMath.com*

9.1 Right Triangle Trigonometry *(pp. 461–468)*

Evaluate the six trigonometric functions of the angle θ.



From the Pythagorean Theorem, the length of the hypotenuse is

$$\begin{aligned}\text{hyp.} &= \sqrt{6^2 + 8^2}\\ &= \sqrt{100}\\ &= 10.\end{aligned}$$

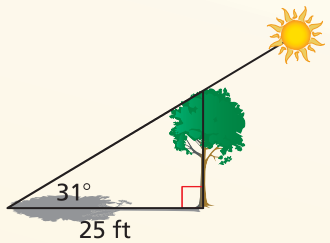
Using adj. = 8, opp. = 6, and hyp. = 10, the values of the six trigonometric functions of θ are:

$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} = \frac{6}{10} = \frac{3}{5} \qquad \cos\theta = \frac{\text{adj.}}{\text{hyp.}} = \frac{8}{10} = \frac{4}{5} \qquad \tan\theta = \frac{\text{opp.}}{\text{adj.}} = \frac{6}{8} = \frac{3}{4}$$

$$\csc\theta = \frac{\text{hyp.}}{\text{opp.}} = \frac{10}{6} = \frac{5}{3} \qquad \sec\theta = \frac{\text{hyp.}}{\text{adj.}} = \frac{10}{8} = \frac{5}{4} \qquad \cot\theta = \frac{\text{adj.}}{\text{opp.}} = \frac{8}{6} = \frac{4}{3}$$

1. In a right triangle, θ is an acute angle and $\cos\theta = \frac{6}{11}$. Evaluate the other five trigonometric functions of θ.

2. The shadow of a tree measures 25 feet from its base. The angle of elevation to the Sun is 31°. How tall is the tree?



9.2 Angles and Radian Measure *(pp. 469–476)*

Convert the degree measure to radians or the radian measure to degrees.

a. 110°

$$110^\circ = 110\ \text{degrees}\left(\frac{\pi\ \text{radians}}{180\ \text{degrees}}\right) = \frac{11\pi}{18}$$

b. $\frac{7\pi}{12}$

$$\frac{7\pi}{12} = \frac{7\pi}{12}\ \text{radians}\left(\frac{180^\circ}{\pi\ \text{radians}}\right) = 105^\circ$$

3. Find one positive angle and one negative angle that are coterminal with 382°.

Convert the degree measure to radians or the radian measure to degrees.

4. 30° **5.** 225° **6.** $\frac{3\pi}{4}$ **7.** $\frac{5\pi}{3}$

8. A sprinkler system on a farm rotates 140° and sprays water up to 35 meters. Draw a diagram that shows the region that can be irrigated with the sprinkler. Then find the area of the region.

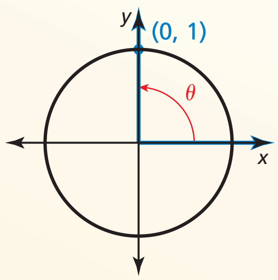
9.3 Trigonometric Functions of Any Angle (pp. 477–484)

Evaluate csc 210°.

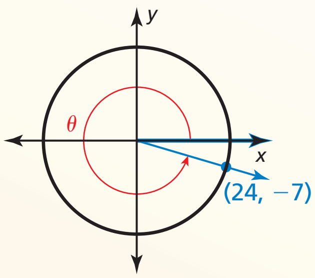
The reference angle is $\theta' = 210° - 180° = 30°$. The cosecant function is negative in Quadrant III, so $\csc 210° = -\csc 30° = -2$.

Evaluate the six trigonometric functions of θ.

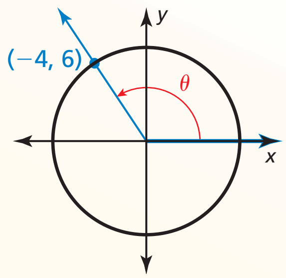
9.



10.



11.



Evaluate the function without using a calculator.

12. $\tan 330°$ **13.** $\sec(-405°)$ **14.** $\sin \frac{13\pi}{6}$ **15.** $\sec \frac{11\pi}{3}$

9.4 Graphing Sine and Cosine Functions (pp. 485–494)

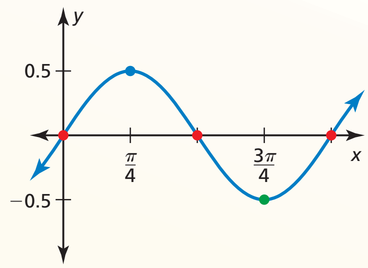
Identify the amplitude and period of $g(x) = \frac{1}{2} \sin 2x$. Then graph the function and describe the graph of g as a transformation of the graph of $f(x) = \sin x$.

The function is of the form $g(x) = a \sin bx$, where $a = \frac{1}{2}$ and $b = 2$. So, the amplitude is $a = \frac{1}{2}$ and the period is $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$.

Intercepts: $(0, 0)$; $\left(\frac{1}{2} \cdot \pi, 0\right) = \left(\frac{\pi}{2}, 0\right)$; $(\pi, 0)$

Maximum: $\left(\frac{1}{4} \cdot \pi, \frac{1}{2}\right) = \left(\frac{\pi}{4}, \frac{1}{2}\right)$

Minimum: $\left(\frac{3}{4} \cdot \pi, -\frac{1}{2}\right) = \left(\frac{3\pi}{4}, -\frac{1}{2}\right)$



▶ The graph of g is a vertical shrink by a factor of $\frac{1}{2}$ and a horizontal shrink by a factor of $\frac{1}{2}$ of the graph of f.

Identify the amplitude and period of the function. Then graph the function and describe the graph of g as a transformation of the graph of the parent function.

16. $g(x) = 8 \cos x$ **17.** $g(x) = 6 \sin \pi x$ **18.** $g(x) = \frac{1}{4} \cos 4x$

Graph the function.

19. $g(x) = \cos(x + \pi) + 2$ **20.** $g(x) = -\sin x - 4$ **21.** $g(x) = 2 \sin\left(x + \frac{\pi}{2}\right)$

9.5 Graphing Other Trigonometric Functions (pp. 497–504)

a. Graph one period of $g(x) = 7 \cot \pi x$. Describe the graph of g as a transformation of the graph of $f(x) = \cot x$.

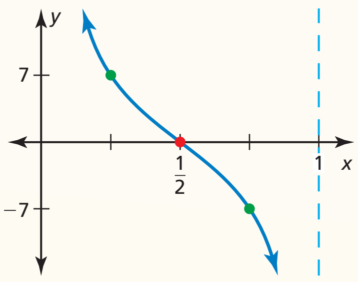
The function is of the form $g(x) = a \cot bx$, where $a = 7$ and $b = \pi$. So, the period is $\frac{\pi}{|b|} = \frac{\pi}{\pi} = 1$.

Intercepts: $\left(\frac{\pi}{2b}, 0\right) = \left(\frac{\pi}{2\pi}, 0\right) = \left(\frac{1}{2}, 0\right)$

Asymptotes: $x = 0$; $x = \frac{\pi}{|b|} = \frac{\pi}{\pi}$, or $x = 1$

Halfway points: $\left(\frac{\pi}{4b}, a\right) = \left(\frac{\pi}{4\pi}, 7\right) = \left(\frac{1}{4}, 7\right)$;

$\left(\frac{3\pi}{4b}, -a\right) = \left(\frac{3\pi}{4\pi}, -7\right) = \left(\frac{3}{4}, -7\right)$



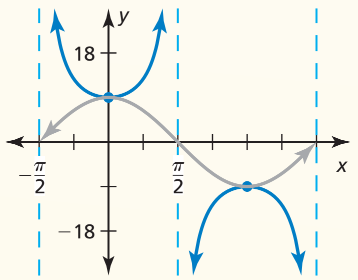
▶ The graph of g is a vertical stretch by a factor of 7 and a horizontal shrink by a factor of $\frac{1}{\pi}$ of the graph of f.

b. Graph one period of $g(x) = 9 \sec x$. Describe the graph of g as a transformation of the graph of $f(x) = \sec x$.

Step 1 Graph the function $y = 9 \cos x$. The period is $\frac{2\pi}{1} = 2\pi$.

Step 2 Graph asymptotes of g. Because the asymptotes of g occur when $9 \cos x = 0$, graph $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$.

Step 3 Plot the points on g, such as $(0, 9)$ and $(\pi, -9)$. Then use the asymptotes to sketch the curve.



▶ The graph of g is a vertical stretch by a factor of 9 of the graph of f.

Graph one period of the function. Describe the graph of g as a transformation of the graph of its parent function.

22. $g(x) = \tan \frac{1}{2}x$

23. $g(x) = 2 \cot x$

24. $g(x) = 4 \tan 3\pi x$

Graph the function.

25. $g(x) = 5 \csc x$

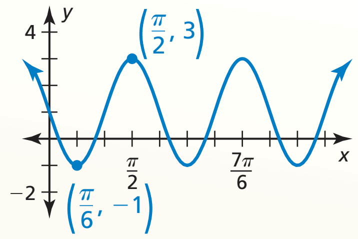
26. $g(x) = \sec \frac{1}{2}x$

27. $g(x) = 5 \sec \pi x$

28. $g(x) = \frac{1}{2} \csc \frac{\pi}{4}x$

9.6 Modeling with Trigonometric Functions (pp. 505–512)

Write a function for the sinusoid shown.



Step 1 Find the maximum and minimum values. From the graph, the maximum value is 3 and the minimum value is −1.

Step 2 Identify the vertical shift, k. The value of k is the mean of the maximum and minimum values.

$$k = \frac{(\text{maximum value}) + (\text{minimum value})}{2} = \frac{3 + (-1)}{2} = \frac{2}{2} = 1$$

Step 3 Decide whether the graph should be modeled by a sine or cosine function. Because the graph crosses the midline $y = 1$ on the y-axis and then decreases to its minimum value, the graph is a sine curve with a reflection in the x-axis and no horizontal shift. So, $h = 0$.

Step 4 Find the amplitude and period.

The period is $\frac{2\pi}{3} = \frac{2\pi}{b}$. So, $b = 3$.

The amplitude is

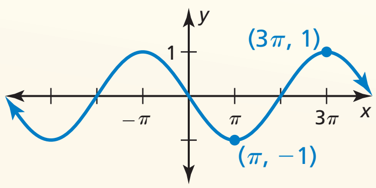
$$|a| = \frac{(\text{maximum value}) - (\text{minimum value})}{2} = \frac{3 - (-1)}{2} = \frac{4}{2} = 2.$$

Because the graph is a reflection in the x-axis, $a < 0$. So, $a = -2$.

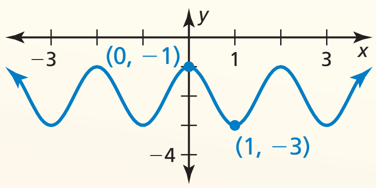
▶ The function is $y = -2 \sin 3x + 1$.

Write a function for the sinusoid.

29.



30.



31. You put a reflector on a spoke of your bicycle wheel. The highest point of the reflector is 25 inches above the ground, and the lowest point is 2 inches. The reflector makes 1 revolution per second. Write a model for the height h (in inches) of a reflector as a function of time t (in seconds) given that the reflector is at its lowest point when $t = 0$.

32. The table shows the monthly precipitation P (in inches) for Bismarck, North Dakota, where $t = 1$ represents January. Write a model that gives P as a function of t and interpret the period of its graph.

t	1	2	3	4	5	6	7	8	9	10	11	12
P	0.5	0.5	0.9	1.5	2.2	2.6	2.6	2.2	1.6	1.3	0.7	0.4

9.7 Using Trigonometric Identities (pp. 513–518)

Verify the identity $\dfrac{\cot^2\theta}{\csc\theta} = \csc\theta - \sin\theta$.

$$\frac{\cot^2\theta}{\csc\theta} = \frac{\csc^2\theta - 1}{\csc\theta} \qquad \text{Pythagorean identity}$$

$$= \frac{\csc^2\theta}{\csc\theta} - \frac{1}{\csc\theta} \qquad \text{Write as separate fractions.}$$

$$= \csc\theta - \frac{1}{\csc\theta} \qquad \text{Simplify.}$$

$$= \csc\theta - \sin\theta \qquad \text{Reciprocal identity}$$

Simplify the expression.

33. $\cot^2 x - \cot^2 x \cos^2 x$

34. $\dfrac{(\sec x + 1)(\sec x - 1)}{\tan x}$

35. $\sin\left(\dfrac{\pi}{2} - x\right)\tan x$

Verify the identity.

36. $\dfrac{\cos x \sec x}{1 + \tan^2 x} = \cos^2 x$

37. $\tan\left(\dfrac{\pi}{2} - x\right)\cot x = \csc^2 x - 1$

9.8 Using Sum and Difference Formulas (pp. 519–524)

Find the exact value of sin 105°.

$$\sin 105° = \sin(45° + 60°) \qquad \text{Substitute } 45° + 60° \text{ for } 105°.$$

$$= \sin 45° \cos 60° + \cos 45° \sin 60° \qquad \text{Sum formula for sine}$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} \qquad \text{Evaluate.}$$

$$= \frac{\sqrt{2} + \sqrt{6}}{4} \qquad \text{Simplify.}$$

▶ The exact value of sin 105° is $\dfrac{\sqrt{2} + \sqrt{6}}{4}$.

Find the exact value of the expression.

38. $\sin 75°$

39. $\tan(-15°)$

40. $\cos\dfrac{\pi}{12}$

41. Find $\tan(a + b)$, given that $\tan a = \dfrac{1}{4}$ with $\pi < a < \dfrac{3\pi}{2}$ and $\tan b = \dfrac{3}{7}$ with $0 < b < \dfrac{\pi}{2}$.

Solve the equation for $0 \le x < 2\pi$.

42. $\cos\left(x + \dfrac{3\pi}{4}\right) + \cos\left(x - \dfrac{3\pi}{4}\right) = 1$

43. $\tan(x + \pi) + \cos\left(x + \dfrac{\pi}{2}\right) = 0$

9 Chapter Test

Verify the identity.

1. $\dfrac{\cos^2 x + \sin^2 x}{1 + \tan^2 x} = \cos^2 x$

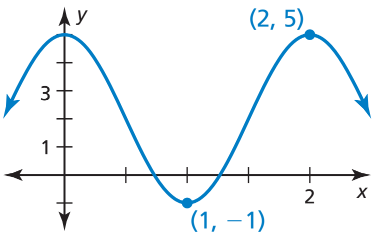
2. $\dfrac{1 + \sin x}{\cos x} + \dfrac{\cos x}{1 + \sin x} = 2 \sec x$

3. $\cos\left(x + \dfrac{3\pi}{2}\right) = \sin x$

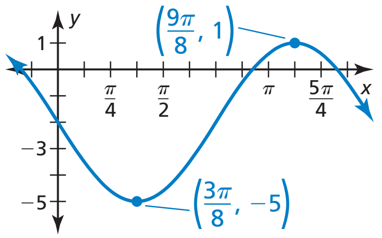
4. Evaluate $\sec(-300°)$ without using a calculator.

Write a function for the sinusoid.

5.



6.



Graph the function. Then describe the graph of g as a transformation of the graph of its parent function.

7. $g(x) = -4 \tan 2x$

8. $g(x) = -2 \cos \frac{1}{3}x + 3$

9. $g(x) = 3 \csc \pi x$

Convert the degree measure to radians or the radian measure to degrees. Then find one positive angle and one negative angle that are coterminal with the given angle.

10. $-50°$

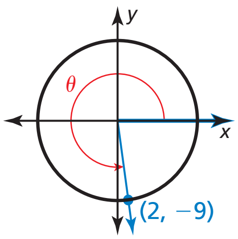
11. $\dfrac{4\pi}{5}$

12. $\dfrac{8\pi}{3}$

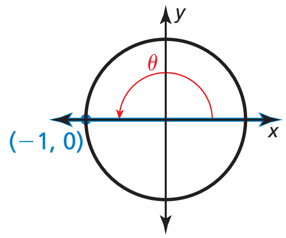
13. Find the arc length and area of a sector with radius $r = 13$ inches and central angle $\theta = 40°$.

Evaluate the six trigonometric functions of the angle θ.

14.

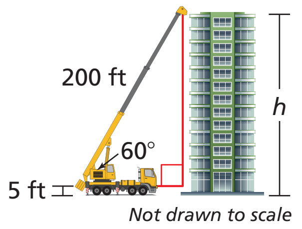


15.



16. In which quadrant does the terminal side of θ lie when $\cos \theta < 0$ and $\tan \theta > 0$? Explain.

17. How tall is the building? Justify your answer.



18. The table shows the average daily high temperatures T (in degrees Fahrenheit) in Baltimore, Maryland, where $m = 1$ represents January. Write a model that gives T as a function of m and interpret the period of its graph.

m	1	2	3	4	5	6	7	8	9	10	11	12
T	41	45	54	65	74	83	87	85	78	67	56	45

9 Cumulative Assessment

1. Which expressions are equivalent to 1?

$\tan x \sec x \cos x$	$\sin^2 x + \cos^2 x$	$\dfrac{\cos^2(-x)\tan^2 x}{\sin^2(-x)}$	$\cos\left(\dfrac{\pi}{2} - x\right)\csc x$

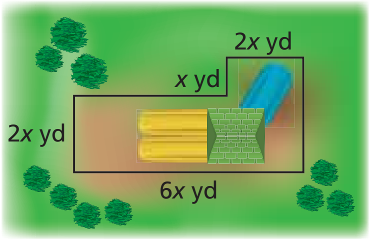
2. Which rational expression represents the ratio of the perimeter to the area of the playground shown in the diagram?

Ⓐ $\dfrac{9}{7x}$

Ⓑ $\dfrac{11}{14x}$

Ⓒ $\dfrac{1}{x}$

Ⓓ $\dfrac{1}{2x}$



3. The chart shows the average monthly temperatures (in degrees Fahrenheit) and the gas usages (in cubic feet) of a household for 12 months.

a. Use a graphing calculator to find trigonometric models for the average temperature y_1 as a function of time and the gas usage y_2 (in thousands of cubic feet) as a function of time. Let $t = 1$ represent January.

b. Graph the two regression equations in the same coordinate plane on your graphing calculator. Describe the relationship between the graphs.

January	February	March	April
32°F	21°F	15°F	22°F
20,000 ft^3	27,000 ft^3	23,000 ft^3	22,000 ft^3
May	**June**	**July**	**August**
35°F	49°F	62°F	78°F
21,000 ft^3	14,000 ft^3	8,000 ft^3	9,000 ft^3
September	**October**	**November**	**December**
71°F	63°F	55°F	40°F
13,000 ft^3	15,000 ft^3	19,000 ft^3	23,000 ft^3

4. Evaluate each logarithm using $\log_2 5 \approx 2.322$ and $\log_2 3 \approx 1.585$, if necessary. Then order the logarithms by value from least to greatest.

a. $\log 1000$

b. $\log_2 15$

c. $\ln e$

d. $\log_2 9$

e. $\log_2 \frac{5}{3}$

f. $\log_2 1$

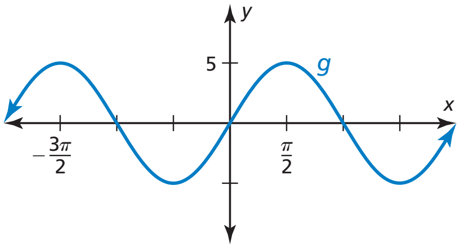
5. Which function is *not* represented by the graph?

(A) $y = 5 \sin x$

(B) $y = 5 \cos\left(\frac{\pi}{2} - x\right)$

(C) $y = 5 \cos\left(x + \frac{\pi}{2}\right)$

(D) $y = -5 \sin(x + \pi)$

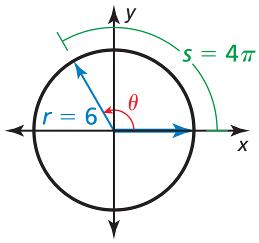


6. Complete each statement with < or > so that each statement is true.

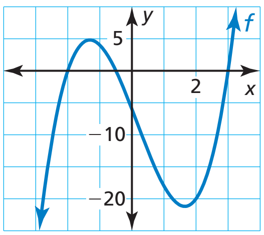
a. θ ☐ 3 radians

b. $\tan \theta$ ☐ 0

c. θ' ☐ 45°



7. Use the Rational Root Theorem and the graph to find all the real zeros of the function $f(x) = 2x^3 - x^2 - 13x - 6$.



8. Your friend claims $-210°$ is coterminal with the angle $\frac{5\pi}{6}$. Is your friend correct? Explain your reasoning.

9. Company A and Company B offer the same starting annual salary of \$20,000. Company A gives a \$1000 raise each year. Company B gives a 4% raise each year.

a. Write rules giving the salaries a_n and b_n for your nth year of employment at Company A and Company B, respectively. Tell whether the sequence represented by each rule is *arithmetic*, *geometric*, or *neither*.

b. Graph each sequence in the same coordinate plane.

c. Under what conditions would you choose to work for Company B?

d. After 20 years of employment, compare your total earnings.