9 Trigonometric Ratios and Functions

9.1 Right Triangle Trigonometry
9.2 Angles and Radian Measure
9.3 Trigonometric Functions of Any Angle
9.4 Graphing Sine and Cosine Functions
9.5 Graphing Other Trigonometric Functions
9.6 Modeling with Trigonometric Functions
9.7 Using Trigonometric Identities
9.8 Using Sum and Difference Formulas

Sundial *(p. 518)*

Tuning Fork *(p. 510)*

Ferris Wheel *(p. 494)*

Terminator *(p. 476)*

Parasailing *(p. 465)*

Maintaining Mathematical Proficiency

Absolute Value

Example 1 **Order the expressions by value from least to greatest: $|6|$, $|-3|$, $\frac{2}{|-4|}$, $|10-6|$**

$$|6| = 6 \qquad |-3| = 3$$

The absolute value of a negative number is positive.

$$\frac{2}{|-4|} = \frac{2}{4} = \frac{1}{2} \qquad |10-6| = |4| = 4$$

▶ So, the order is $\frac{2}{|-4|}$, $|-3|$, $|10-6|$, and $|6|$.

Order the expressions by value from least to greatest.

1. $|4|$, $|2-9|$, $|6+4|$, $-|7|$

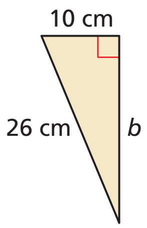
2. $|9-3|$, $|0|$, $|-4|$, $\frac{|-5|}{|2|}$

3. $|-8^3|$, $|-2 \cdot 8|$, $|9-1|$, $|9| + |-2| - |1|$

4. $|-4+20|$, $-|4^2|$, $|5| - |3 \cdot 2|$, $|-15|$

Pythagorean Theorem

Example 2 **Find the missing side length of the triangle.**

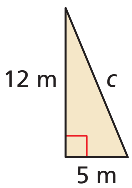


$a^2 + b^2 = c^2$	Write the Pythagorean Theorem.
$10^2 + b^2 = 26^2$	Substitute 10 for a and 26 for c.
$100 + b^2 = 676$	Evaluate powers.
$b^2 = 576$	Subtract 100 from each side.
$b = 24$	Take positive square root of each side.

▶ So, the length is 24 centimeters.

Find the missing side length of the triangle.

5.



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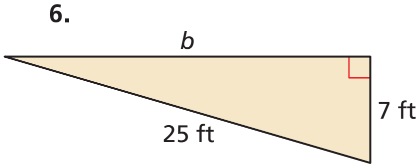

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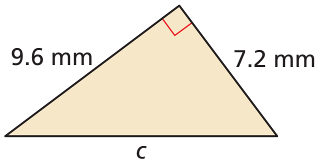

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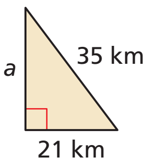

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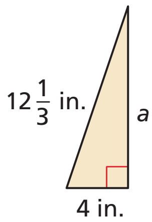

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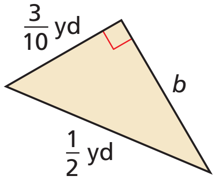


11. ABSTRACT REASONING The line segments connecting the points (x_1, y_1), (x_2, y_1), and (x_2, y_2) form a triangle. Is the triangle a right triangle? Justify your answer.

Mathematical Practices

Mathematically proficient students reason quantitatively by creating valid representations of problems.

Reasoning Abstractly and Quantitatively

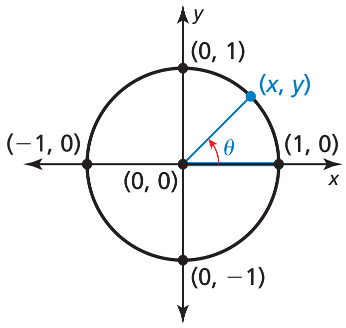
Core Concept

The Unit Circle

The *unit circle* is a circle in the coordinate plane. Its center is at the origin, and it has a radius of 1 unit. The equation of the unit circle is

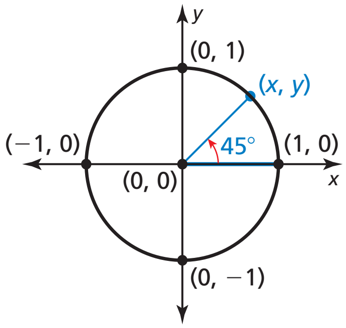
$x^2 + y^2 = 1.$ Equation of unit circle

As the point (x, y) starts at $(1, 0)$ and moves counterclockwise around the unit circle, the angle θ (the Greek letter *theta*) moves from 0° through 360°.



EXAMPLE 1 **Finding Coordinates of a Point on the Unit Circle**

Find the exact coordinates of the point (x, y) on the unit circle.



SOLUTION

Because $\theta = 45°$, (x, y) lies on the line $y = x$.

$x^2 + y^2 = 1$ Write equation of unit circle.

$x^2 + x^2 = 1$ Substitute x for y.

$2x^2 = 1$ Add like terms.

$x^2 = \frac{1}{2}$ Divide each side by 2.

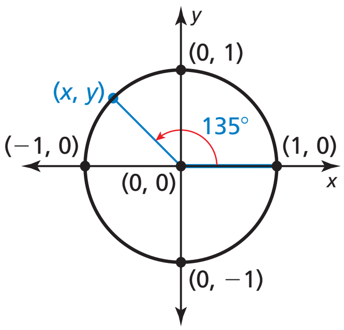
$x = \frac{1}{\sqrt{2}}$ Take positive square root of each side.

▶ The coordinates of (x, y) are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, or $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Monitoring Progress

Find the exact coordinates of the point (x, y) on the unit circle.

1.



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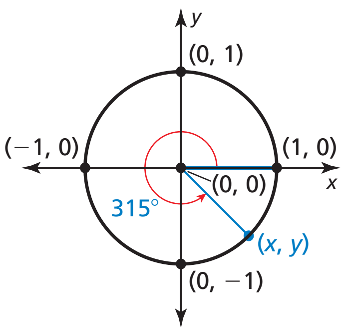

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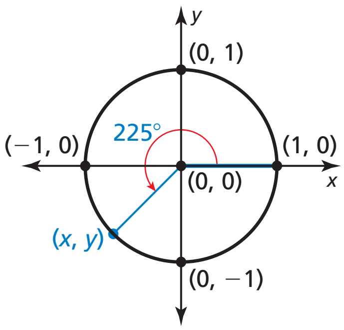
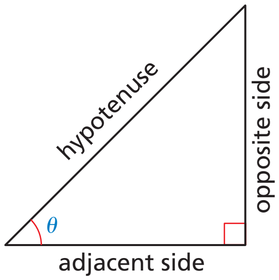


9.1 Right Triangle Trigonometry

Essential Question

How can you find a trigonometric function of an acute angle θ?

Consider one of the acute angles θ of a right triangle. Ratios of a right triangle's side lengths are used to define the six *trigonometric functions*, as shown.



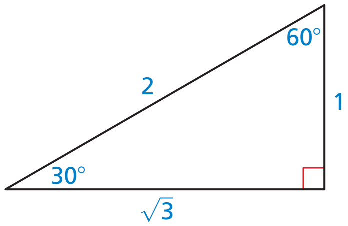
Sine $\sin\theta = \dfrac{\text{opp.}}{\text{hyp.}}$ **Cosine** $\cos\theta = \dfrac{\text{adj.}}{\text{hyp.}}$

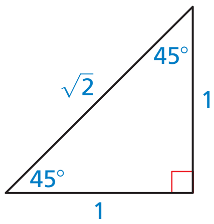
Tangent $\tan\theta = \dfrac{\text{opp.}}{\text{adj.}}$ **Cotangent** $\cot\theta = \dfrac{\text{adj.}}{\text{opp.}}$

Secant $\sec\theta = \dfrac{\text{hyp.}}{\text{adj.}}$ **Cosecant** $\csc\theta = \dfrac{\text{hyp.}}{\text{opp.}}$

EXPLORATION 1 Trigonometric Functions of Special Angles

Work with a partner. Find the exact values of the sine, cosine, and tangent functions for the angles 30°, 45°, and 60° in the right triangles shown.





CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to understand and use stated assumptions, definitions, and previously established results in constructing arguments.

EXPLORATION 2 Exploring Trigonometric Identities

Work with a partner.

Use the definitions of the trigonometric functions to explain why each *trigonometric identity* is true.

a. $\sin\theta = \cos(90° - \theta)$ **b.** $\cos\theta = \sin(90° - \theta)$

c. $\sin\theta = \dfrac{1}{\csc\theta}$ **d.** $\tan\theta = \dfrac{1}{\cot\theta}$

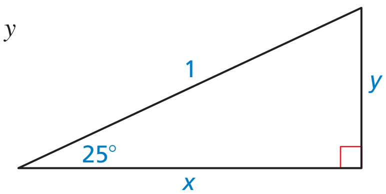
Use the definitions of the trigonometric functions to complete each trigonometric identity.

e. $(\sin\theta)^2 + (\cos\theta)^2 =$ ▭ **f.** $(\sec\theta)^2 - (\tan\theta)^2 =$ ▭

Communicate Your Answer

3. How can you find a trigonometric function of an acute angle θ?

4. Use a calculator to find the lengths x and y of the legs of the right triangle shown.



9.1 Lesson

Core Vocabulary

sine, *p. 462*
cosine, *p. 462*
tangent, *p. 462*
cosecant, *p. 462*
secant, *p. 462*
cotangent, *p. 462*

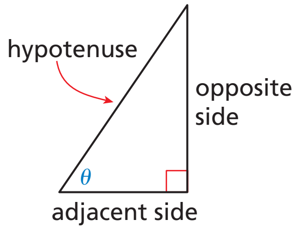
Previous
right triangle
hypotenuse
acute angle
Pythagorean Theorem
reciprocal
complementary angles

What You Will Learn

- Evaluate trigonometric functions of acute angles.
- Find unknown side lengths and angle measures of right triangles.
- Use trigonometric functions to solve real-life problems.

The Six Trigonometric Functions

Consider a right triangle that has an acute angle θ (the Greek letter *theta*). The three sides of the triangle are the *hypotenuse*, the side *opposite* θ, and the side *adjacent* to θ.



Ratios of a right triangle's side lengths are used to define the six trigonometric functions: **sine**, **cosine**, **tangent**, **cosecant**, **secant**, and **cotangent**. These six functions are abbreviated sin, cos, tan, csc, sec, and cot, respectively.

Core Concept

Right Triangle Definitions of Trigonometric Functions

Let θ be an acute angle of a right triangle. The six trigonometric functions of θ are defined as shown.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

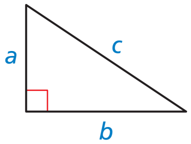
$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

The abbreviations *opp.*, *adj.*, and *hyp.* are often used to represent the side lengths of the right triangle. Note that the ratios in the second row are reciprocals of the ratios in the first row.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

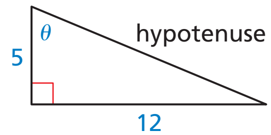
REMEMBER

The Pythagorean Theorem states that $a^2 + b^2 = c^2$ for a right triangle with hypotenuse of length c and legs of lengths a and b.



EXAMPLE 1 Evaluating Trigonometric Functions

Evaluate the six trigonometric functions of the angle θ.



SOLUTION

From the Pythagorean Theorem, the length of the hypotenuse is

$$\begin{aligned} \text{hyp.} &= \sqrt{5^2 + 12^2} \\ &= \sqrt{169} \\ &= 13. \end{aligned}$$

Using adj. $= 5$, opp. $= 12$, and hyp. $= 13$, the values of the six trigonometric functions of θ are:

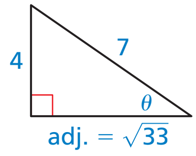
$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} = \frac{12}{13} \qquad \cos\theta = \frac{\text{adj.}}{\text{hyp.}} = \frac{5}{13} \qquad \tan\theta = \frac{\text{opp.}}{\text{adj.}} = \frac{12}{5}$$

$$\csc\theta = \frac{\text{hyp.}}{\text{opp.}} = \frac{13}{12} \qquad \sec\theta = \frac{\text{hyp.}}{\text{adj.}} = \frac{13}{5} \qquad \cot\theta = \frac{\text{adj.}}{\text{opp.}} = \frac{5}{12}$$

EXAMPLE 2 Evaluating Trigonometric Functions

In a right triangle, θ is an acute angle and $\sin\theta = \frac{4}{7}$. Evaluate the other five trigonometric functions of θ.

SOLUTION



Step 1 Draw a right triangle with acute angle θ such that the leg opposite θ has length 4 and the hypotenuse has length 7.

Step 2 Find the length of the adjacent side. By the Pythagorean Theorem, the length of the other leg is

$$\text{adj.} = \sqrt{7^2 - 4^2} = \sqrt{33}.$$

Step 3 Find the values of the remaining five trigonometric functions.

Because $\sin\theta = \frac{4}{7}$, $\csc\theta = \frac{\text{hyp.}}{\text{opp.}} = \frac{7}{4}$. The other values are:

$$\cos\theta = \frac{\text{adj.}}{\text{hyp.}} = \frac{\sqrt{33}}{7} \qquad \tan\theta = \frac{\text{opp.}}{\text{adj.}} = \frac{4}{\sqrt{33}} = \frac{4\sqrt{33}}{33}$$

$$\sec\theta = \frac{\text{hyp.}}{\text{adj.}} = \frac{7}{\sqrt{33}} = \frac{7\sqrt{33}}{33} \qquad \cot\theta = \frac{\text{adj.}}{\text{opp.}} = \frac{\sqrt{33}}{4}$$

Monitoring Progress

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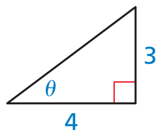
Evaluate the six trigonometric functions of the angle θ.

1.



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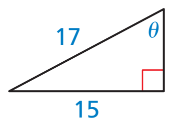

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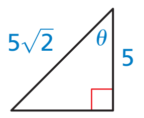


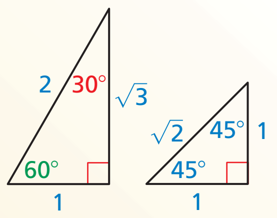
4. In a right triangle, θ is an acute angle and $\cos\theta = \frac{7}{10}$. Evaluate the other five trigonometric functions of θ.

The angles 30°, 45°, and 60° occur frequently in trigonometry. You can use the trigonometric values for these angles to find unknown side lengths in special right triangles.

Core Concept

Trigonometric Values for Special Angles

The table gives the values of the six trigonometric functions for the angles 30°, 45°, and 60°. You can obtain these values from the triangles shown.

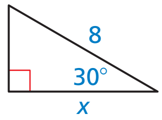


θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Finding Side Lengths and Angle Measures

EXAMPLE 3 Finding an Unknown Side Length

Find the value of x for the right triangle.



SOLUTION

Write an equation using a trigonometric function that involves the ratio of x and 8. Solve the equation for x.

$$\cos 30° = \frac{\text{adj.}}{\text{hyp.}} \qquad \text{Write trigonometric equation.}$$

$$\frac{\sqrt{3}}{2} = \frac{x}{8} \qquad \text{Substitute.}$$

$$4\sqrt{3} = x \qquad \text{Multiply each side by 8.}$$

▶ The length of the side is $x = 4\sqrt{3} \approx 6.93$.

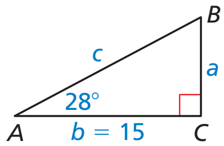
READING

Throughout this chapter, a capital letter is used to denote both an angle of a triangle and its measure. The same letter in lowercase is used to denote the length of the side opposite that angle.

Finding all unknown side lengths and angle measures of a triangle is called *solving the triangle*. Solving right triangles that have acute angles other than 30°, 45°, and 60° may require the use of a calculator. Be sure the calculator is set in *degree* mode.

EXAMPLE 4 Using a Calculator to Solve a Right Triangle

Solve $\triangle ABC$.



SOLUTION

Because the triangle is a right triangle, A and B are complementary angles. So, $B = 90° - 28° = 62°$.

Next, write two equations using trigonometric functions, one that involves the ratio of a and 15, and one that involves c and 15. Solve the first equation for a and the second equation for c.

$\tan 28° = \frac{\text{opp.}}{\text{adj.}}$	Write trigonometric equation.	$\sec 28° = \frac{\text{hyp.}}{\text{adj.}}$
$\tan 28° = \frac{a}{15}$	Substitute.	$\sec 28° = \frac{c}{15}$
$15(\tan 28°) = a$	Solve for the variable.	$15\left(\frac{1}{\cos 28°}\right) = c$
$7.98 \approx a$	Use a calculator.	$16.99 \approx c$

▶ So, $B = 62°$, $a \approx 7.98$, and $c \approx 16.99$.

Monitoring Progress

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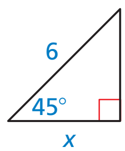
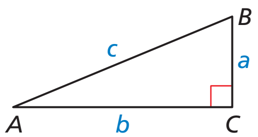
5. Find the value of x for the right triangle shown.





Solve $\triangle ABC$ using the diagram at the left and the given measurements.

6. $B = 45°, c = 5$

7. $A = 32°, b = 10$

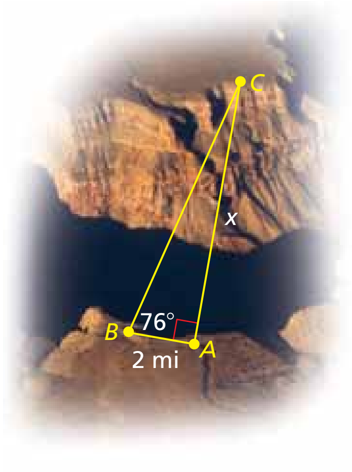
8. $A = 71°, c = 20$

9. $B = 60°, a = 7$

Solving Real-Life Problems

EXAMPLE 5 Using Indirect Measurement

You are hiking near a canyon. While standing at A, you measure an angle of 90º between B and C, as shown. You then walk to B and measure an angle of 76° between A and C. The distance between A and B is about 2 miles. How wide is the canyon between A and C?



FINDING AN ENTRY POINT

The tangent function is used to find the unknown distance because it involves the ratio of x and 2.

SOLUTION

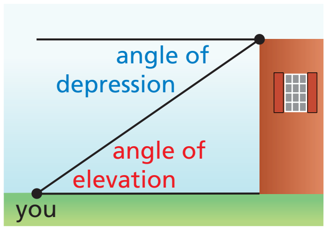
$\tan 76° = \frac{x}{2}$ Write trigonometric equation.

$2(\tan 76°) = x$ Multiply each side by 2.

$8.0 \approx x$ Use a calculator.

▶ The width is about 8.0 miles.

If you look at a point above you, such as the top of a building, the angle that your line of sight makes with a line parallel to the ground is called the *angle of elevation*. At the top of the building, the angle between a line parallel to the ground and your line of sight is called the *angle of depression*. These two angles have the same measure.

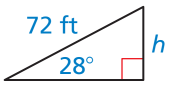


EXAMPLE 6 Using an Angle of Elevation

A parasailer is attached to a boat with a rope 72 feet long. The angle of elevation from the boat to the parasailer is 28°. Estimate the parasailer's height above the boat.

SOLUTION

Step 1 Draw a diagram that represents the situation.



Step 2 Write and solve an equation to find the height h.

$\sin 28° = \frac{h}{72}$ Write trigonometric equation.

$72(\sin 28°) = h$ Multiply each side by 72.

$33.8 \approx h$ Use a calculator.

▶ The height of the parasailer above the boat is about 33.8 feet.

Monitoring Progress

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10. In Example 5, find the distance between B and C.

11. **WHAT IF?** In Example 6, estimate the height of the parasailer above the boat when the angle of elevation is 38°.

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Vocabulary and Core Concept Check

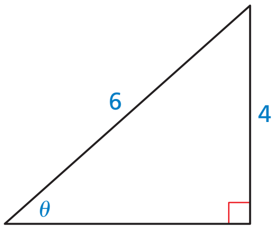
1. **COMPLETE THE SENTENCE** In a right triangle, the two trigonometric functions of θ that are defined using the lengths of the hypotenuse and the side adjacent to θ are ___________ and ___________.

2. **VOCABULARY** Compare an angle of elevation to an angle of depression.

3. **WRITING** Explain what it means to solve a right triangle.

4. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find “both” answers.

   - What is the cosecant of θ?
   - What is $\dfrac{1}{\sin\,\theta}$?
   - What is the ratio of the side opposite θ to the hypotenuse?
   - What is the ratio of the hypotenuse to the side opposite θ?



Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, evaluate the six trigonometric functions of the angle θ. *(See Example 1.)*

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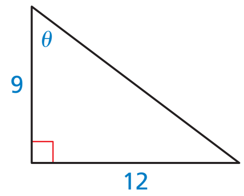

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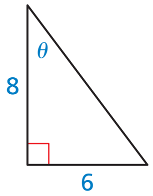

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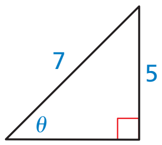

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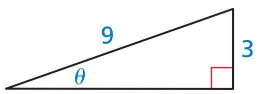

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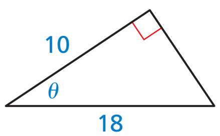

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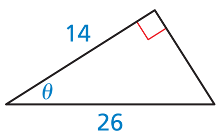


11. **REASONING** Let θ be an acute angle of a right triangle. Use the two trigonometric functions $\tan\,\theta = \dfrac{4}{9}$ and $\sec\,\theta = \dfrac{\sqrt{97}}{9}$ to sketch and label the right triangle. Then evaluate the other four trigonometric functions of θ.

12. **ANALYZING RELATIONSHIPS** Evaluate the six trigonometric functions of the $90° - \theta$ angle in Exercises 5–10. Describe the relationships you notice.

In Exercises 13–18, let θ be an acute angle of a right triangle. Evaluate the other five trigonometric functions of θ. *(See Example 2.)*

13. $\sin\,\theta = \frac{7}{11}$

14. $\cos\,\theta = \frac{5}{12}$

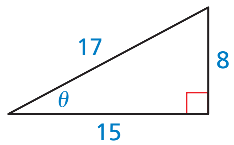
15. $\tan\,\theta = \frac{7}{6}$

16. $\csc\,\theta = \frac{15}{8}$

17. $\sec\,\theta = \frac{14}{9}$

18. $\cot\,\theta = \frac{16}{11}$

19. **ERROR ANALYSIS** Describe and correct the error in finding $\sin\,\theta$ of the triangle below.



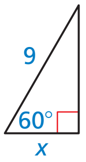
✗ $\sin\,\theta = \dfrac{\text{opp.}}{\text{hyp.}} = \dfrac{15}{17}$

20. **ERROR ANALYSIS** Describe and correct the error in finding csc θ, given that θ is an acute angle of a right triangle and $\cos\theta = \frac{7}{11}$.

✗ $\csc\theta = \dfrac{1}{\cos\theta} = \dfrac{11}{7}$

In Exercises 21–26, find the value of *x* for the right triangle. *(See Example 3.)*

21.



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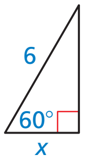

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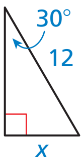

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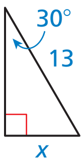

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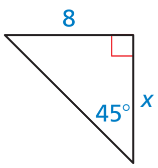

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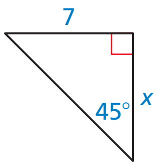


USING TOOLS **In Exercises 27–32, evaluate the trigonometric function using a calculator. Round your answer to four decimal places.**

27. cos 14°

28. tan 31°

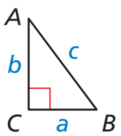
29. csc 59°

30. sin 23°

31. cot 6°

32. sec 11°

In Exercises 33–40, solve $\triangle ABC$ using the diagram and the given measurements. *(See Example 4.)*



33. $B = 36°$, $a = 23$

34. $A = 27°$, $b = 9$

35. $A = 55°$, $a = 17$

36. $B = 16°$, $b = 14$

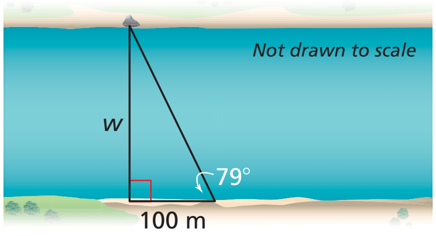
37. $A = 43°$, $b = 31$

38. $B = 31°$, $a = 23$

39. $B = 72°$, $c = 12.8$

40. $A = 64°$, $a = 7.4$

41. **MODELING WITH MATHEMATICS** To measure the width of a river, you plant a stake on one side of the river, directly across from a boulder. You then walk 100 meters to the right of the stake and measure a 79° angle between the stake and the boulder. What is the width *w* of the river? *(See Example 5.)*

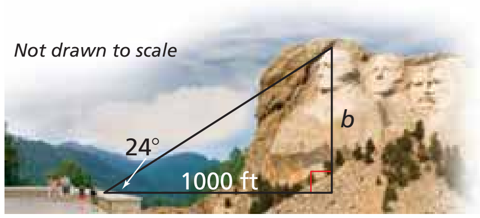


42. **MODELING WITH MATHEMATICS** Katoomba Scenic Railway in Australia is the steepest railway in the world. The railway makes an angle of about 52° with the ground. The railway extends horizontally about 458 feet. What is the height of the railway?

43. **MODELING WITH MATHEMATICS** A person whose eye level is 1.5 meters above the ground is standing 75 meters from the base of the Jin Mao Building in Shanghai, China. The person estimates the angle of elevation to the top of the building is about 80°. What is the approximate height of the building? *(See Example 6.)*

44. **MODELING WITH MATHEMATICS** The Duquesne Incline in Pittsburgh, Pennsylvania, has an angle of elevation of 30°. The track has a length of about 800 feet. Find the height of the incline.

45. **MODELING WITH MATHEMATICS** You are standing on the Grand View Terrace viewing platform at Mount Rushmore, 1000 feet from the base of the monument.

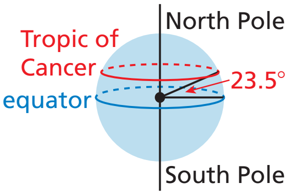


a. You look up at the top of Mount Rushmore at an angle of 24°. How high is the top of the monument from where you are standing? Assume your eye level is 5.5 feet above the platform.

b. The elevation of the Grand View Terrace is 5280 feet. Use your answer in part (a) to find the elevation of the top of Mount Rushmore.

46. **WRITING** Write a real-life problem that can be solved using a right triangle. Then solve your problem.

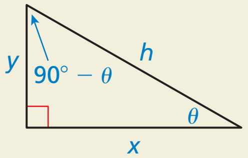
47. **MATHEMATICAL CONNECTIONS** The Tropic of Cancer is the circle of latitude farthest north of the equator where the Sun can appear directly overhead. It lies 23.5° north of the equator, as shown.



a. Find the circumference of the Tropic of Cancer using 3960 miles as the approximate radius of Earth.

b. What is the distance between two points on the Tropic of Cancer that lie directly across from each other?

48. **HOW DO YOU SEE IT?** Use the figure to answer each question.

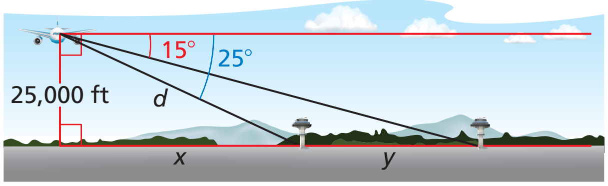


a. Which side is adjacent to θ?

b. Which side is opposite of θ?

c. Does $\cos \theta = \sin(90° - \theta)$? Explain.

49. **PROBLEM SOLVING** A passenger in an airplane sees two towns directly to the left of the plane.



a. What is the distance d from the airplane to the first town?

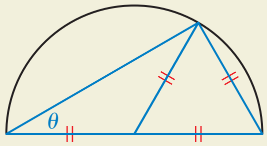
b. What is the horizontal distance x from the airplane to the first town?

c. What is the distance y between the two towns? Explain the process you used to find your answer.

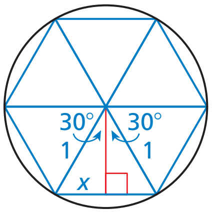
50. **PROBLEM SOLVING** You measure the angle of elevation from the ground to the top of a building as 32°. When you move 50 meters closer to the building, the angle of elevation is 53°. What is the height of the building?

51. **MAKING AN ARGUMENT** Your friend claims it is possible to draw a right triangle so the values of the cosine function of the acute angles are equal. Is your friend correct? Explain your reasoning.

52. **THOUGHT PROVOKING** Consider a semicircle with a radius of 1 unit, as shown below. Write the values of the six trigonometric functions of the angle θ. Explain your reasoning.



53. **CRITICAL THINKING** A procedure for approximating π based on the work of Archimedes is to inscribe a regular hexagon in a circle.



a. Use the diagram to solve for x. What is the perimeter of the hexagon?

b. Show that a regular n-sided polygon inscribed in a circle of radius 1 has a perimeter of $2n \cdot \sin\left(\frac{180}{n}\right)^{\circ}$.

c. Use the result from part (b) to find an expression in terms of n that approximates π. Then evaluate the expression when $n = 50$.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Perform the indicated conversion. *(Skills Review Handbook)*

54. 5 years to seconds

55. 12 pints to gallons

56. 5.6 meters to millimeters

Find the circumference and area of the circle with the given radius or diameter. *(Skills Review Handbook)*

57. $r = 6$ centimeters

58. $r = 11$ inches

59. $d = 14$ feet

9.2 Angles and Radian Measure

Essential Question

How can you find the measure of an angle in radians?

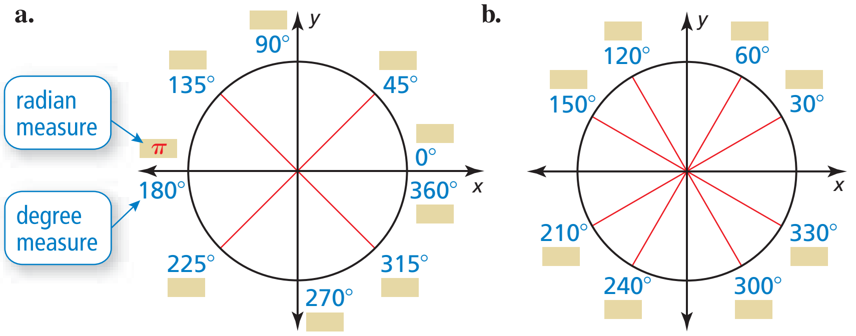
Let the vertex of an angle be at the origin, with one side of the angle on the positive x-axis. The *radian measure* of the angle is a measure of the intercepted arc length on a circle of radius 1. To convert between degree and radian measure, use the fact that

$$\frac{\pi \text{ radians}}{180^\circ} = 1.$$

EXPLORATION 1 Writing Radian Measures of Angles

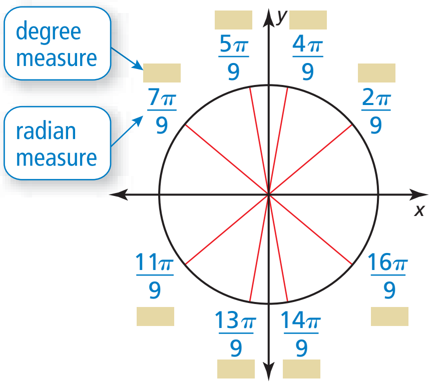
Work with a partner. Write the radian measure of each angle with the given degree measure. Explain your reasoning.

a. **b.**



EXPLORATION 2 Writing Degree Measures of Angles

Work with a partner. Write the degree measure of each angle with the given radian measure. Explain your reasoning.



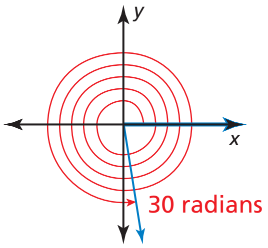
REASONING ABSTRACTLY

To be proficient in math, you need to make sense of quantities and their relationships in problem situations.

Communicate Your Answer

3. How can you find the measure of an angle in radians?

4. The figure shows an angle whose measure is 30 radians. What is the measure of the angle in degrees? How many times greater is 30 radians than 30 degrees? Justify your answers.



9.2 Lesson

Core Vocabulary

initial side, *p. 470*
terminal side, *p. 470*
standard position, *p. 470*
coterminal, *p. 471*
radian, *p. 471*
sector, *p. 472*
central angle, *p. 472*

Previous
radius of a circle
circumference of a circle

What You Will Learn

- Draw angles in standard position.
- Find coterminal angles.
- Use radian measure.

Drawing Angles in Standard Position

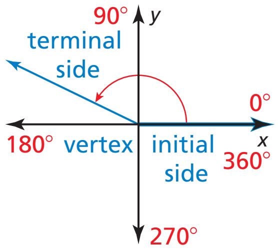
In this lesson, you will expand your study of angles to include angles with measures that can be any real numbers.

Core Concept

Angles in Standard Position

In a coordinate plane, an angle can be formed by fixing one ray, called the **initial side**, and rotating the other ray, called the **terminal side**, about the vertex.

An angle is in **standard position** when its vertex is at the origin and its initial side lies on the positive x-axis.



The measure of an angle is positive when the rotation of its terminal side is counterclockwise and negative when the rotation is clockwise. The terminal side of an angle can rotate more than 360°.

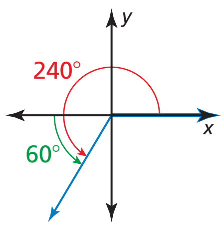
EXAMPLE 1 Drawing Angles in Standard Position

Draw an angle with the given measure in standard position.

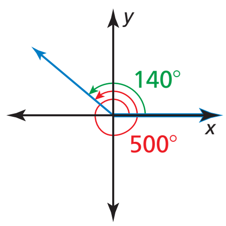
a. 240° **b.** 500° **c.** −50°

SOLUTION

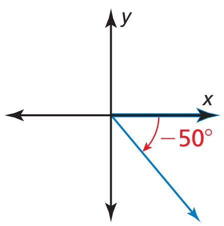
a. Because 240° is 60° more than 180°, the terminal side is 60° counterclockwise past the negative x-axis.



b. Because 500° is 140° more than 360°, the terminal side makes one complete rotation 360° counterclockwise plus 140° more.



c. Because −50° is negative, the terminal side is 50° clockwise from the positive x-axis.



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Draw an angle with the given measure in standard position.

1. 65° **2.** 300° **3.** −120° **4.** −450°

Finding Coterminal Angles

In Example 1(b), the angles 500° and 140° are **coterminal** because their terminal sides coincide. An angle coterminal with a given angle can be found by adding or subtracting multiples of 360°.

STUDY TIP

If two angles differ by a multiple of 360°, then the angles are coterminal.

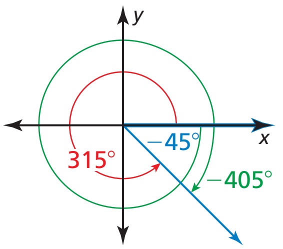
EXAMPLE 2 Finding Coterminal Angles

Find one positive angle and one negative angle that are coterminal with (a) −45° and (b) 395°.

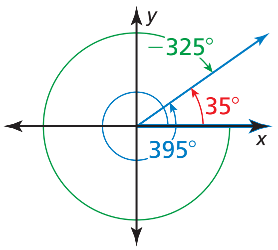
SOLUTION

There are many such angles, depending on what multiple of 360° is added or subtracted.

a. $-45° + 360° = 315°$
$-45° - 360° = -405°$



b. $395° - 360° = 35°$
$395° - 2(360°) = -325°$



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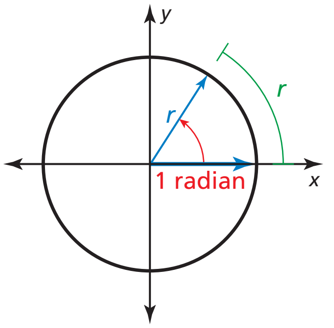
Find one positive angle and one negative angle that are coterminal with the given angle.

5. 80° **6.** 230° **7.** 740° **8.** −135°

Using Radian Measure

Angles can also be measured in *radians*. To define a radian, consider a circle with radius r centered at the origin, as shown. One **radian** is the measure of an angle in standard position whose terminal side intercepts an arc of length r.



Because the circumference of a circle is $2\pi r$, there are 2π radians in a full circle. So, degree measure and radian measure are related by the equation $360° = 2\pi$ radians, or $180° = \pi$ radians.

STUDY TIP

Notice that 1 radian is approximately equal to 57.3°.

$180° = \pi$ radians

$\frac{180°}{\pi} = 1$ radian

$57.3° \approx 1$ radian

Core Concept

Converting Between Degrees and Radians

Degrees to radians

Multiply degree measure by

$$\frac{\pi \text{ radians}}{180°}.$$

Radians to degrees

Multiply radian measure by

$$\frac{180°}{\pi \text{ radians}}.$$

EXAMPLE 3 Convert Between Degrees and Radians

Convert the degree measure to radians or the radian measure to degrees.

a. 120° **b.** $-\frac{\pi}{12}$

READING

The unit "radians" is often omitted. For instance, the measure $-\frac{\pi}{12}$ radians may be written simply as $-\frac{\pi}{12}$.

SOLUTION

a. $120^\circ = 120 \cancel{\text{degrees}}\left(\frac{\pi \text{ radians}}{180 \cancel{\text{degrees}}}\right)$

$= \frac{2\pi}{3}$

b. $-\frac{\pi}{12} = \left(-\frac{\pi}{12} \cancel{\text{radians}}\right)\left(\frac{180^\circ}{\pi \cancel{\text{radians}}}\right)$

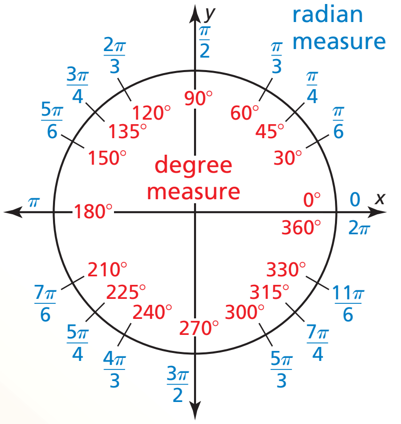
$= -15^\circ$

Concept Summary

Degree and Radian Measures of Special Angles

The diagram shows equivalent degree and radian measures for special angles from 0° to 360° (0 radians to 2π radians).

You may find it helpful to memorize the equivalent degree and radian measures of special angles in the first quadrant and for $90^\circ = \frac{\pi}{2}$ radians. All other special angles shown are multiples of these angles.



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Convert the degree measure to radians or the radian measure to degrees.

9. 135° **10.** -40° **11.** $\frac{5\pi}{4}$ **12.** -6.28

A **sector** is a region of a circle that is bounded by two radii and an arc of the circle. The **central angle** θ of a sector is the angle formed by the two radii. There are simple formulas for the arc length and area of a sector when the central angle is measured in radians.

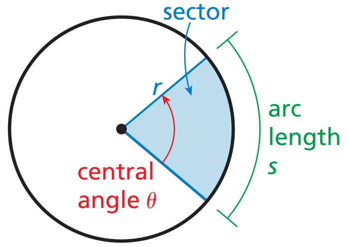
Core Concept

Arc Length and Area of a Sector

The arc length s and area A of a sector with radius r and central angle θ (measured in radians) are as follows.

Arc length: $s = r\theta$

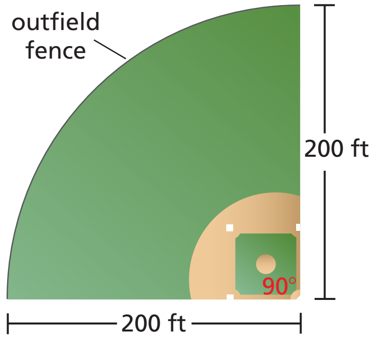
Area: $A = \frac{1}{2}r^2\theta$



EXAMPLE 4 Modeling with Mathematics

A softball field forms a sector with the dimensions shown. Find the length of the outfield fence and the area of the field.

SOLUTION



1. **Understand the Problem** You are given the dimensions of a softball field. You are asked to find the length of the outfield fence and the area of the field.

2. **Make a Plan** Find the measure of the central angle in radians. Then use the arc length and area of a sector formulas.

3. **Solve the Problem**

Step 1 Convert the measure of the central angle to radians.

$$90^\circ = 90 \cancel{\text{ degrees}}\left(\frac{\pi \text{ radians}}{180 \cancel{\text{ degrees}}}\right)$$

$$= \frac{\pi}{2} \text{ radians}$$

COMMON ERROR

You must write the measure of an angle in radians when using these formulas for the arc length and area of a sector.

Step 2 Find the arc length and the area of the sector.

Arc length: $s = r\theta$

$$= 200\left(\frac{\pi}{2}\right)$$

$$= 100\pi$$

$$\approx 314$$

Area: $A = \frac{1}{2}r^2\theta$

$$= \frac{1}{2}(200)^2\left(\frac{\pi}{2}\right)$$

$$= 10{,}000\pi$$

$$\approx 31{,}416$$

ANOTHER WAY

Because the central angle is 90°, the sector represents $\frac{1}{4}$ of a circle with a radius of 200 feet. So,

$$s = \frac{1}{4} \cdot 2\pi r = \frac{1}{4} \cdot 2\pi(200)$$

$$= 100\pi$$

and

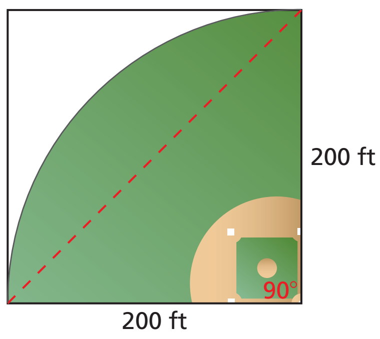
$$A = \frac{1}{4} \cdot \pi r^2 = \frac{1}{4} \cdot \pi(200)^2$$

$$= 10{,}000\pi.$$

▶ The length of the outfield fence is about 314 feet. The area of the field is about 31,416 square feet.

4. **Look Back** To check the area of the field, consider the square formed using the two 200-foot sides.

By drawing the diagonal, you can see that the area of the field is less than the area of the square but greater than one-half of the area of the square.



$$\underbrace{\frac{1}{2}(200)^2}_{\frac{1}{2}\,\cdot\,\text{(area of square)}} \stackrel{?}{<} 31{,}416 \stackrel{?}{<} \underbrace{200^2}_{\text{area of square}}$$

$$20{,}000 < 31{,}416 < 40{,}000 \checkmark$$

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13. **WHAT IF?** In Example 4, the outfield fence is 220 feet from home plate. Estimate the length of the outfield fence and the area of the field.

9.2 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** An angle is in standard position when its vertex is at the __________ and its __________ lies on the positive x-axis.

2. **WRITING** Explain how the sign of an angle measure determines its direction of rotation.

3. **VOCABULARY** In your own words, define a radian.

4. **WHICH ONE DOESN'T BELONG?** Which angle does *not* belong with the other three? Explain your reasoning.

$-90°$	$450°$	$90°$	$-270°$

Monitoring Progress and Modeling with Mathematics

In Exercises 5–8, draw an angle with the given measure in standard position. *(See Example 1.)*

5. $110°$
6. $450°$
7. $-900°$
8. $-10°$

In Exercises 9–12, find one positive angle and one negative angle that are coterminal with the given angle. *(See Example 2.)*

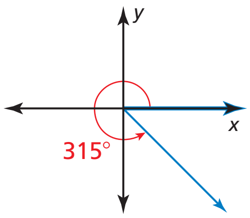
9. $70°$
10. $255°$
11. $-125°$
12. $-800°$

In Exercises 13–20, convert the degree measure to radians or the radian measure to degrees. *(See Example 3.)*

13. $40°$
14. $315°$
15. $-260°$
16. $-500°$
17. $\dfrac{\pi}{9}$
18. $\dfrac{3\pi}{4}$
19. -5
20. 12

21. **WRITING** The terminal side of an angle in standard position rotates one-sixth of a revolution counterclockwise from the positive x-axis. Describe how to find the measure of the angle in both degree and radian measures.

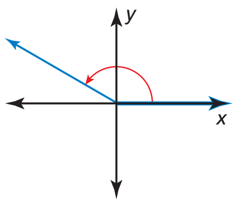
22. **OPEN-ENDED** Using radian measure, give one positive angle and one negative angle that are coterminal with the angle shown. Justify your answers.



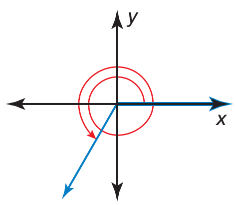
ANALYZING RELATIONSHIPS In Exercises 23–26, match the angle measure with the angle.

23. $600°$
24. $-\dfrac{9\pi}{4}$
25. $\dfrac{5\pi}{6}$
26. $-240°$

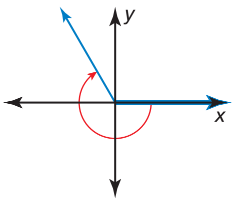
A.



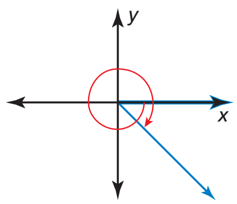
B.



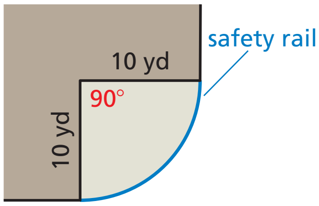
C.



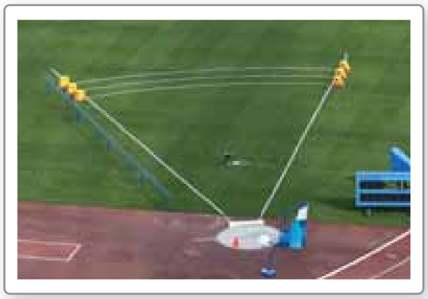
D.



27. **MODELING WITH MATHEMATICS** The observation deck of a building forms a sector with the dimensions shown. Find the length of the safety rail and the area of the deck. *(See Example 4.)*



28. **MODELING WITH MATHEMATICS** In the men's shot put event at the 2012 Summer Olympic Games, the length of the winning shot was 21.89 meters. A shot put must land within a sector having a central angle of 34.92° to be considered fair.



a. The officials draw an arc across the fair landing area, marking the farthest throw. Find the length of the arc.

b. All fair throws in the 2012 Olympics landed within a sector bounded by the arc in part (a). What is the area of this sector?

29. **ERROR ANALYSIS** Describe and correct the error in converting the degree measure to radians.

✗

$$24° = 24\ \cancel{\text{degrees}}\left(\frac{180\ \cancel{\text{degrees}}}{\pi\ \text{radians}}\right)$$

$$= \frac{4320}{\pi}\ \text{radians}$$

$$\approx 1375.1\ \text{radians}$$

30. **ERROR ANALYSIS** Describe and correct the error in finding the area of a sector with a radius of 6 centimeters and a central angle of 40°.

✗

$$A = \frac{1}{2}(6)^2(40) = 720\ \text{cm}^2$$

31. **PROBLEM SOLVING** When a CD player reads information from the outer edge of a CD, the CD spins about 200 revolutions per minute. At that speed, through what angle does a point on the CD spin in one minute? Give your answer in both degree and radian measures.

32. **PROBLEM SOLVING** You work every Saturday from 9:00 A.M. to 5:00 P.M. Draw a diagram that shows the rotation completed by the hour hand of a clock during this time. Find the measure of the angle generated by the hour hand in both degrees and radians. Compare this angle with the angle generated by the minute hand from 9:00 A.M. to 5:00 P.M.

USING TOOLS **In Exercises 33–38, use a calculator to evaluate the trigonometric function.**

33. $\cos \frac{4\pi}{3}$

34. $\sin \frac{7\pi}{8}$

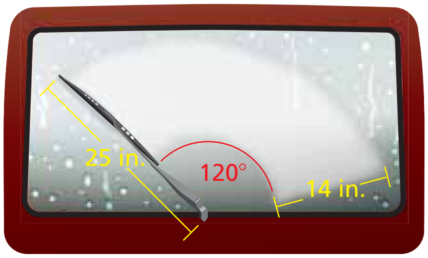
35. $\csc \frac{10\pi}{11}$

36. $\cot\left(-\frac{6\pi}{5}\right)$

37. $\cot(-14)$

38. $\cos 6$

39. **MODELING WITH MATHEMATICS** The rear windshield wiper of a car rotates 120°, as shown. Find the area cleared by the wiper.

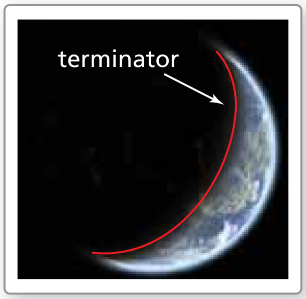


40. **MODELING WITH MATHEMATICS** A scientist performed an experiment to study the effects of gravitational force on humans. In order for humans to experience twice Earth's gravity, they were placed in a centrifuge 58 feet long and spun at a rate of about 15 revolutions per minute.

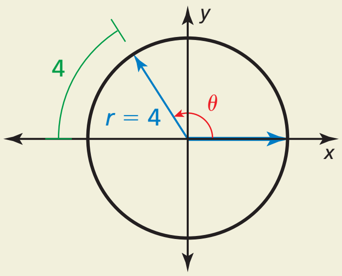
a. Through how many radians did the people rotate each second?

b. Find the length of the arc through which the people rotated each second.

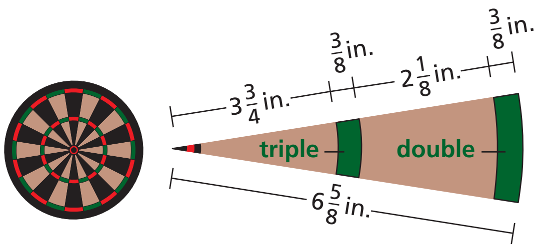
41. **REASONING** In astronomy, the *terminator* is the day-night line on a planet that divides the planet into daytime and nighttime regions. The terminator moves across the surface of a planet as the planet rotates. It takes about 4 hours for Earth's terminator to move across the continental United States. Through what angle has Earth rotated during this time? Give your answer in both degree and radian measures.



42. **HOW DO YOU SEE IT?** Use the graph to find the measure of θ. Explain your reasoning.



43. **MODELING WITH MATHEMATICS** A dartboard is divided into 20 sectors. Each sector is worth a point value from 1 to 20 and has shaded regions that double or triple this value. A sector is shown below. Find the areas of the entire sector, the double region, and the triple region.



44. **THOUGHT PROVOKING** π is an irrational number, which means that it cannot be written as the ratio of two whole numbers. π can, however, be written exactly as a *continued fraction*, as follows.

$$3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}}}$$

Show how to use this continued fraction to obtain a decimal approximation for π.

45. **MAKING AN ARGUMENT** Your friend claims that when the arc length of a sector equals the radius, the area can be given by $A = \frac{s^2}{2}$. Is your friend correct? Explain.

46. **PROBLEM SOLVING** A spiral staircase has 15 steps. Each step is a sector with a radius of 42 inches and a central angle of $\frac{\pi}{8}$.

a. What is the length of the arc formed by the outer edge of a step?

b. Through what angle would you rotate by climbing the stairs?

c. How many square inches of carpeting would you need to cover the 15 steps?

47. **MULTIPLE REPRESENTATIONS** There are 60 *minutes* in 1 degree of arc, and 60 *seconds* in 1 minute of arc. The notation $50° \, 30' \, 10''$ represents an angle with a measure of 50 degrees, 30 minutes, and 10 seconds.

a. Write the angle measure 70.55° using the notation above.

b. Write the angle measure $110° \, 45' \, 30''$ to the nearest hundredth of a degree. Justify your answer.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find the distance between the two points. *(Skills Review Handbook)*

48. (1, 4), (3, 6)

49. (−7, −13), (10, 8)

50. (−3, 9), (−3, 16)

51. (2, 12), (8, −5)

52. (−14, −22), (−20, −32)

53. (4, 16), (−1, 34)

9.3 Trigonometric Functions of Any Angle

Essential Question

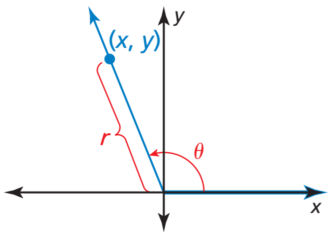
Essential Question How can you use the unit circle to define the trigonometric functions of any angle?

Let θ be an angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$. The six trigonometric functions of θ are defined as shown.

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y},\ y \neq 0$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x},\ x \neq 0$$

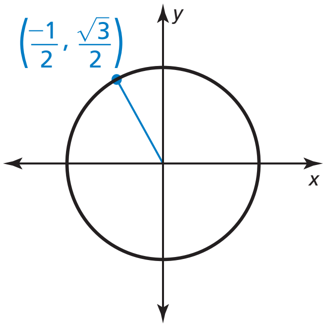
$$\tan\theta = \frac{y}{x},\ x \neq 0 \qquad \cot\theta = \frac{x}{y},\ y \neq 0$$



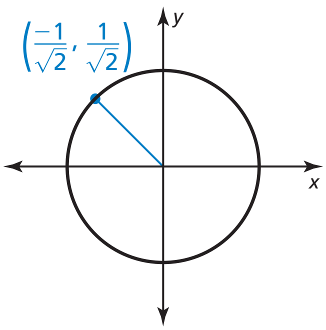
EXPLORATION 1 **Writing Trigonometric Functions**

Work with a partner. Find the sine, cosine, and tangent of the angle θ in standard position whose terminal side intersects the unit circle at the point (x, y) shown.

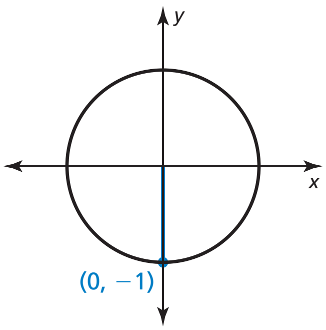
a.



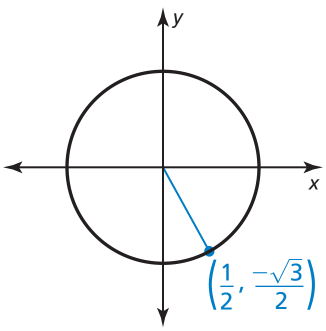
b.



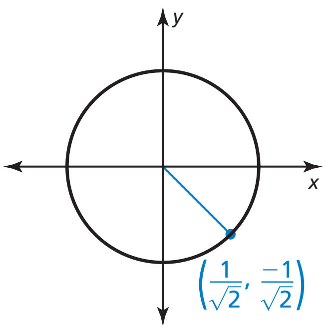
c.



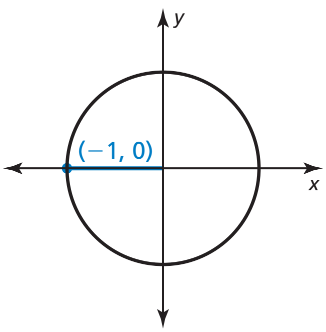
d.



e.



f.



CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to understand and use stated assumptions, definitions, and previously established results.

Communicate Your Answer

2. How can you use the unit circle to define the trigonometric functions of any angle?
3. For which angles are each function undefined? Explain your reasoning.

   a. tangent **b.** cotangent **c.** secant **d.** cosecant

9.3 Lesson

Core Vocabulary

unit circle, *p. 479*
quadrantal angle, *p. 479*
reference angle, *p. 480*

Previous
circle
radius
Pythagorean Theorem

What You Will Learn

- Evaluate trigonometric functions of any angle.
- Find and use reference angles to evaluate trigonometric functions.

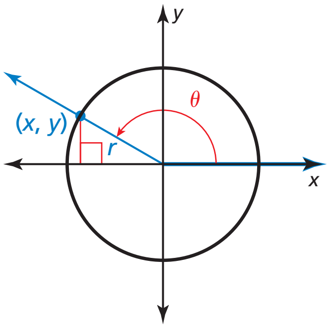
Trigonometric Functions of Any Angle

You can generalize the right-triangle definitions of trigonometric functions so that they apply to any angle in standard position.

Core Concept

General Definitions of Trigonometric Functions

Let θ be an angle in standard position, and let (x, y) be the point where the terminal side of θ intersects the circle $x^2 + y^2 = r^2$. The six trigonometric functions of θ are defined as shown.



$\sin\theta = \dfrac{y}{r}$ $\qquad$ $\csc\theta = \dfrac{r}{y},\ y \neq 0$

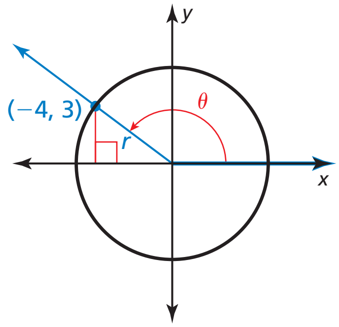
$\cos\theta = \dfrac{x}{r}$ $\qquad$ $\sec\theta = \dfrac{r}{x},\ x \neq 0$

$\tan\theta = \dfrac{y}{x},\ x \neq 0$ $\qquad$ $\cot\theta = \dfrac{x}{y},\ y \neq 0$

These functions are sometimes called *circular functions*.

EXAMPLE 1 Evaluating Trigonometric Functions Given a Point

Let $(-4, 3)$ be a point on the terminal side of an angle θ in standard position. Evaluate the six trigonometric functions of θ.



SOLUTION

Use the Pythagorean Theorem to find the length of r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-4)^2 + 3^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Using $x = -4$, $y = 3$, and $r = 5$, the values of the six trigonometric functions of θ are:

$\sin\theta = \dfrac{y}{r} = \dfrac{3}{5}$ $\qquad$ $\csc\theta = \dfrac{r}{y} = \dfrac{5}{3}$

$\cos\theta = \dfrac{x}{r} = -\dfrac{4}{5}$ $\qquad$ $\sec\theta = \dfrac{r}{x} = -\dfrac{5}{4}$

$\tan\theta = \dfrac{y}{x} = -\dfrac{3}{4}$ $\qquad$ $\cot\theta = \dfrac{x}{y} = -\dfrac{4}{3}$

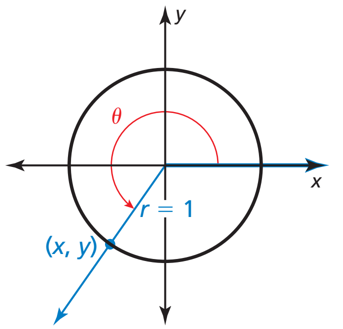
Core Concept

The Unit Circle

The circle $x^2 + y^2 = 1$, which has center (0, 0) and radius 1, is called the **unit circle**. The values of $\sin\theta$ and $\cos\theta$ are simply the y-coordinate and x-coordinate, respectively, of the point where the terminal side of θ intersects the unit circle.

$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y$$

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x$$



It is convenient to use the unit circle to find trigonometric functions of **quadrantal angles**. A quadrantal angle is an angle in standard position whose terminal side lies on an axis. The measure of a quadrantal angle is always a multiple of 90º, or $\frac{\pi}{2}$ radians.

ANOTHER WAY

The general circle $x^2 + y^2 = r^2$ can also be used to find the six trigonometric functions of θ. The terminal side of θ intersects the circle at $(0, -r)$. So,

$$\sin\theta = \frac{y}{r} = \frac{-r}{r} = -1.$$

The other functions can be evaluated similarly.

EXAMPLE 2 Using the Unit Circle

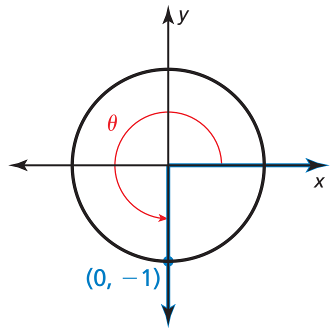
Use the unit circle to evaluate the six trigonometric functions of $\theta = 270º$.

SOLUTION

Step 1 Draw a unit circle with the angle $\theta = 270º$ in standard position.

Step 2 Identify the point where the terminal side of θ intersects the unit circle. The terminal side of θ intersects the unit circle at $(0, -1)$.

Step 3 Find the values of the six trigonometric functions. Let $x = 0$ and $y = -1$ to evaluate the trigonometric functions.



$$\sin\theta = \frac{y}{r} = \frac{-1}{1} = -1 \qquad \csc\theta = \frac{r}{y} = \frac{1}{-1} = -1$$

$$\cos\theta = \frac{x}{r} = \frac{0}{1} = 0 \qquad \sec\theta = \frac{r}{x} = \cancel{\frac{1}{0}} \quad \text{undefined}$$

$$\tan\theta = \frac{y}{x} = \cancel{\frac{-1}{0}} \quad \text{undefined} \qquad \cot\theta = \frac{x}{y} = \frac{0}{-1} = 0$$

Monitoring Progress

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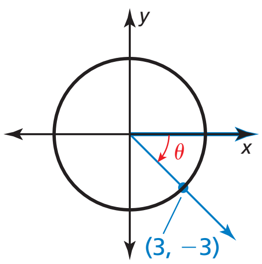
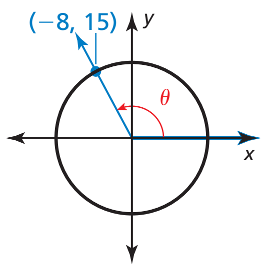
Evaluate the six trigonometric functions of θ.

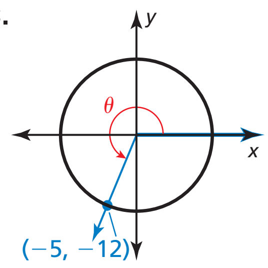
1.



2.



3.



4. Use the unit circle to evaluate the six trigonometric functions of $\theta = 180º$.

Reference Angles

READING

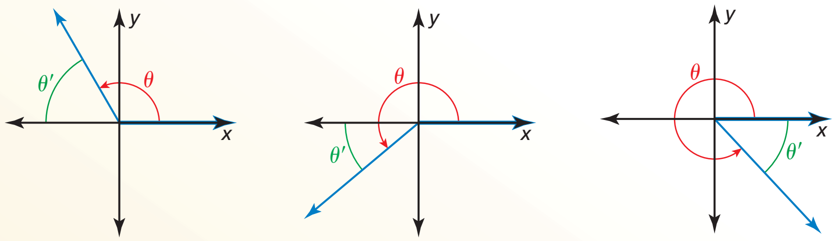
The symbol θ' is read as "theta prime."

Core Concept

Reference Angle Relationships

Let θ be an angle in standard position. The **reference angle** for θ is the acute angle θ' formed by the terminal side of θ and the x-axis. The relationship between θ and θ' is shown below for nonquadrantal angles θ such that $90° < \theta < 360°$ or, in radians, $\frac{\pi}{2} < \theta < 2\pi$.

Quadrant II	Quadrant III	Quadrant IV



Degrees: $\theta' = 180° - \theta$	Degrees: $\theta' = \theta - 180°$	Degrees: $\theta' = 360° - \theta$
Radians: $\theta' = \pi - \theta$	Radians: $\theta' = \theta - \pi$	Radians: $\theta' = 2\pi - \theta$

EXAMPLE 3 **Finding Reference Angles**

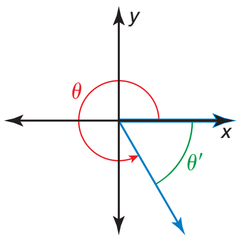
Find the reference angle θ' for (a) $\theta = \frac{5\pi}{3}$ and (b) $\theta = -130°$.

SOLUTION

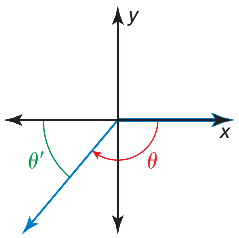
a. The terminal side of θ lies in Quadrant IV. So,

$\theta' = 2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$. The figure at the right shows

$\theta = \frac{5\pi}{3}$ and $\theta' = \frac{\pi}{3}$.



b. Note that θ is coterminal with 230°, whose terminal side lies in Quadrant III. So, $\theta' = 230° - 180° = 50°$. The figure at the left shows $\theta = -130°$ and $\theta' = 50°$.



Reference angles allow you to evaluate a trigonometric function for any angle θ. The sign of the trigonometric function value depends on the quadrant in which θ lies.

Core Concept

Evaluating Trigonometric Functions

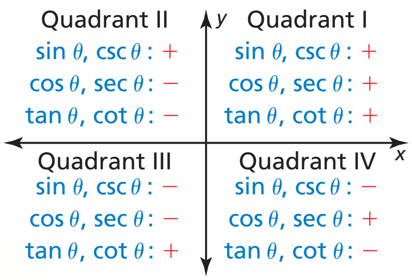
Use these steps to evaluate a trigonometric function for any angle θ:

Step 1 Find the reference angle θ'.

Step 2 Evaluate the trigonometric function for θ'.

Step 3 Determine the sign of the trigonometric function value from the quadrant in which θ lies.

Signs of Function Values



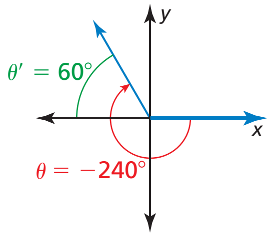
EXAMPLE 4 Using Reference Angles to Evaluate Functions

Evaluate (a) $\tan(-240^\circ)$ and (b) $\csc \frac{17\pi}{6}$.

SOLUTION

a. The angle -240° is coterminal with 120°. The reference angle is $\theta' = 180^\circ - 120^\circ = 60^\circ$. The tangent function is negative in Quadrant II, so

$$\tan(-240^\circ) = -\tan 60^\circ = -\sqrt{3}.$$

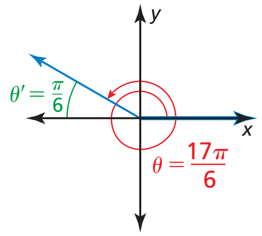


b. The angle $\frac{17\pi}{6}$ is coterminal with $\frac{5\pi}{6}$. The reference angle is

$$\theta' = \pi - \frac{5\pi}{6} = \frac{\pi}{6}.$$

The cosecant function is positive in Quadrant II, so

$$\csc \frac{17\pi}{6} = \csc \frac{\pi}{6} = 2.$$



INTERPRETING MODELS

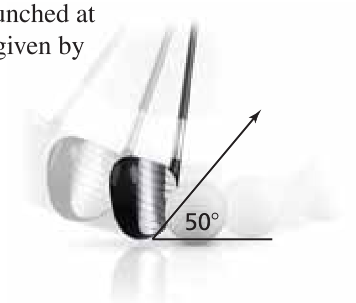
This model neglects air resistance and assumes that the projectile's starting and ending heights are the same.

EXAMPLE 5 Solving a Real-Life Problem

The horizontal distance d (in feet) traveled by a projectile launched at an angle θ and with an initial speed v (in feet per second) is given by

$$d = \frac{v^2}{32} \sin 2\theta. \qquad \text{Model for horizontal distance}$$

Estimate the horizontal distance traveled by a golf ball that is hit at an angle of 50° with an initial speed of 105 feet per second.



SOLUTION

Note that the golf ball is launched at an angle of $\theta = 50^\circ$ with initial speed of $v = 105$ feet per second.

$$d = \frac{v^2}{32} \sin 2\theta \qquad \text{Write model for horizontal distance.}$$

$$= \frac{105^2}{32} \sin(2 \cdot 50^\circ) \qquad \text{Substitute 105 for } v \text{ and } 50^\circ \text{ for } \theta.$$

$$\approx 339 \qquad \text{Use a calculator.}$$

▶ The golf ball travels a horizontal distance of about 339 feet.

Monitoring Progress

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Sketch the angle. Then find its reference angle.

5. 210° **6.** -260° **7.** $\frac{-7\pi}{9}$ **8.** $\frac{15\pi}{4}$

Evaluate the function without using a calculator.

9. $\cos(-210^\circ)$ **10.** $\sec \frac{11\pi}{4}$

11. Use the model given in Example 5 to estimate the horizontal distance traveled by a track and field long jumper who jumps at an angle of 20° and with an initial speed of 27 feet per second.

9.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

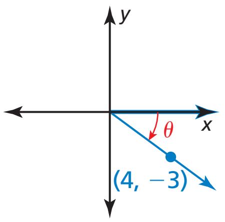
1. **COMPLETE THE SENTENCE** A(n) ___________ is an angle in standard position whose terminal side lies on an axis.

2. **WRITING** Given an angle θ in standard position with its terminal side in Quadrant III, explain how you can use a reference angle to find $\cos\theta$.

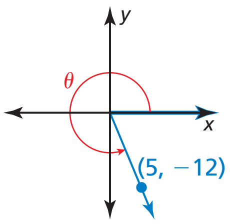
Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, evaluate the six trigonometric functions of θ. *(See Example 1.)*

3.



4.



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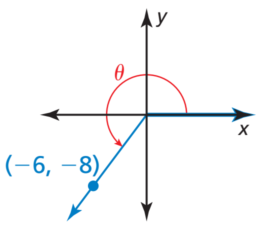

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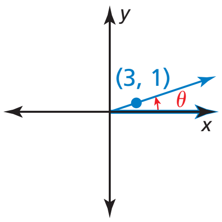

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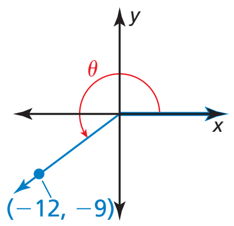

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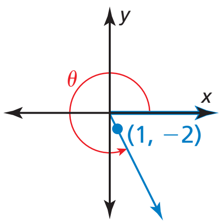


In Exercises 9–14, use the unit circle to evaluate the six trigonometric functions of θ. *(See Example 2.)*

9. $\theta = 0°$

10. $\theta = 540°$

11. $\theta = \frac{\pi}{2}$

12. $\theta = \frac{7\pi}{2}$

13. $\theta = -270°$

14. $\theta = -2\pi$

In Exercises 15–22, sketch the angle. Then find its reference angle. *(See Example 3.)*

15. $-100°$

16. $150°$

17. $320°$

18. $-370°$

19. $\frac{15\pi}{4}$

20. $\frac{8\pi}{3}$

21. $-\frac{5\pi}{6}$

22. $-\frac{13\pi}{6}$

23. ERROR ANALYSIS Let $(-3, 2)$ be a point on the terminal side of an angle θ in standard position. Describe and correct the error in finding $\tan\theta$.

$$\tan\theta = \frac{x}{y} = -\frac{3}{2}$$

24. ERROR ANALYSIS Describe and correct the error in finding a reference angle θ' for $\theta = 650°$.

θ is coterminal with 290°, whose terminal side lies in Quadrant IV.

So, $\theta' = 290° - 270° = 20°$.

In Exercises 25–32, evaluate the function without using a calculator. *(See Example 4.)*

25. $\sec 135°$

26. $\tan 240°$

27. $\sin(-150°)$

28. $\csc(-420°)$

29. $\tan\left(-\frac{3\pi}{4}\right)$

30. $\cot\left(\frac{-8\pi}{3}\right)$

31. $\cos\frac{7\pi}{4}$

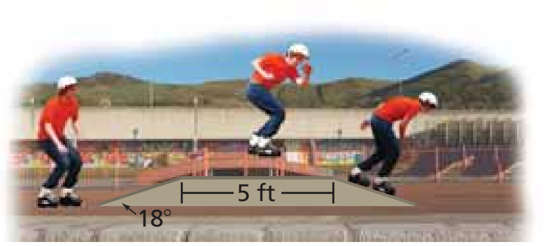
32. $\sec\frac{11\pi}{6}$

In Exercises 33–36, use the model for horizontal distance given in Example 5.

33. You kick a football at an angle of 60° with an initial speed of 49 feet per second. Estimate the horizontal distance traveled by the football. *(See Example 5.)*

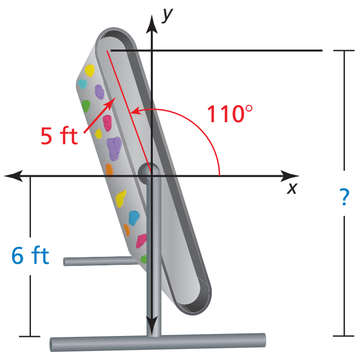
34. The "frogbot" is a robot designed for exploring rough terrain on other planets. It can jump at a 45° angle with an initial speed of 14 feet per second. Estimate the horizontal distance the frogbot can jump on Earth.

35. At what speed must the in-line skater launch himself off the ramp in order to land on the other side of the ramp?



36. To win a javelin throwing competition, your last throw must travel a horizontal distance of at least 100 feet. You release the javelin at a 40° angle with an initial speed of 71 feet per second. Do you win the competition? Justify your answer.

37. MODELING WITH MATHEMATICS A rock climber is using a rock climbing treadmill that is 10 feet long. The climber begins by lying horizontally on the treadmill, which is then rotated about its midpoint by 110° so that the rock climber is climbing toward the top. If the midpoint of the treadmill is 6 feet above the ground, how high above the ground is the top of the treadmill?



38. REASONING A Ferris wheel has a radius of 75 feet. You board a car at the bottom of the Ferris wheel, which is 10 feet above the ground, and rotate 255° counterclockwise before the ride temporarily stops. How high above the ground are you when the ride stops? If the radius of the Ferris wheel is doubled, is your height above the ground doubled? Explain your reasoning.

39. DRAWING CONCLUSIONS A sprinkler at ground level is used to water a garden. The water leaving the sprinkler has an initial speed of 25 feet per second.

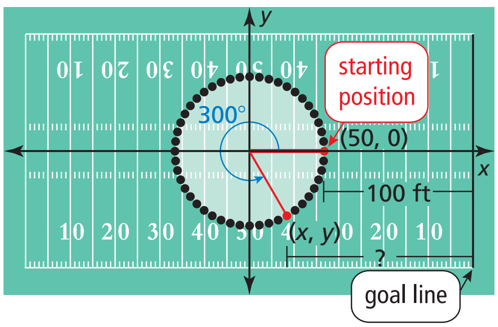
a. Use the model for horizontal distance given in Example 5 to complete the table.

Angle of sprinkler, θ	Horizontal distance water travels, d
30°	
35°	
40°	
45°	
50°	
55°	
60°	

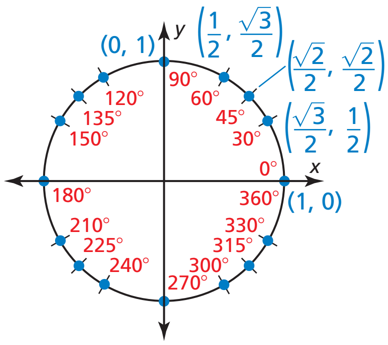
b. Which value of θ appears to maximize the horizontal distance traveled by the water? Use the model for horizontal distance and the unit circle to explain why your answer makes sense.

c. Compare the horizontal distance traveled by the water when $\theta = (45 - k)°$ with the distance when $\theta = (45 + k)°$, for $0 < k < 45$.

40. MODELING WITH MATHEMATICS Your school's marching band is performing at halftime during a football game. In the last formation, the band members form a circle 100 feet wide in the center of the field. You start at a point on the circle 100 feet from the goal line, march 300° around the circle, and then walk toward the goal line to exit the field. How far from the goal line are you at the point where you leave the circle?



41. **ANALYZING RELATIONSHIPS** Use symmetry and the given information to label the coordinates of the other points corresponding to special angles on the unit circle.



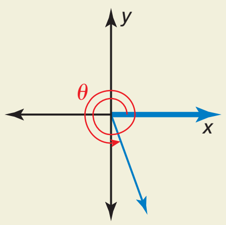
42. **THOUGHT PROVOKING** Use the interactive unit circle tool at *BigIdeasMath.com* to describe all values of θ for each situation.

a. $\sin\theta > 0$, $\cos\theta < 0$, and $\tan\theta > 0$

b. $\sin\theta > 0$, $\cos\theta < 0$, and $\tan\theta < 0$

43. **CRITICAL THINKING** Write $\tan\theta$ as the ratio of two other trigonometric functions. Use this ratio to explain why $\tan 90°$ is undefined but $\cot 90° = 0$.

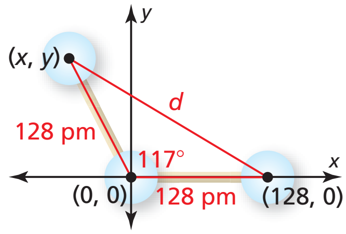
44. **HOW DO YOU SEE IT?** Determine whether each of the six trigonometric functions of θ is *positive*, *negative*, or *zero*. Explain your reasoning.



45. **USING STRUCTURE** A line with slope m passes through the origin. An angle θ in standard position has a terminal side that coincides with the line. Use a trigonometric function to relate the slope of the line to the angle.

46. **MAKING AN ARGUMENT** Your friend claims that the only solution to the trigonometric equation $\tan\theta = \sqrt{3}$ is $\theta = 60°$. Is your friend correct? Explain your reasoning.

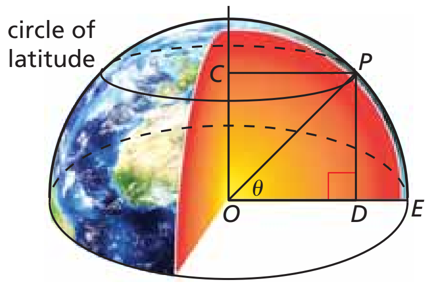
47. **PROBLEM SOLVING** When two atoms in a molecule are bonded to a common atom, chemists are interested in both the bond angle and the lengths of the bonds. An ozone molecule is made up of two oxygen atoms bonded to a third oxygen atom, as shown.



a. In the diagram, coordinates are given in picometers (pm). (*Note*: 1 pm $= 10^{-12}$ m) Find the coordinates (x, y) of the center of the oxygen atom in Quadrant II.

b. Find the distance d (in picometers) between the centers of the two unbonded oxygen atoms.

48. **MATHEMATICAL CONNECTIONS** The latitude of a point on Earth is the degree measure of the shortest arc from that point to the equator. For example, the latitude of point P in the diagram equals the degree measure of arc PE. At what latitude θ is the circumference of the circle of latitude at P half the distance around the equator?



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find all real zeros of the polynomial function. *(Section 4.6)*

49. $f(x) = x^4 + 2x^3 + x^2 + 8x - 12$

50. $f(x) = x^5 + 4x^4 - 14x^3 - 14x^2 - 15x - 18$

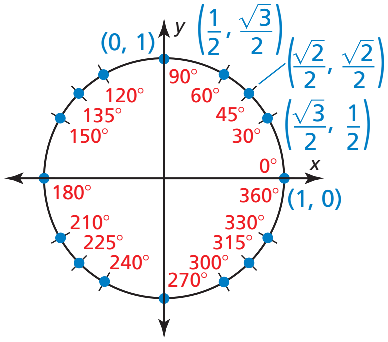
Graph the function. *(Section 4.8)*

51. $f(x) = 2(x + 3)^2(x - 1)$

52. $f(x) = \frac{1}{3}(x - 4)(x + 5)(x + 9)$

53. $f(x) = x^2(x + 1)^3(x - 2)$

41. **ANALYZING RELATIONSHIPS** Use symmetry and the given information to label the coordinates of the other points corresponding to special angles on the unit circle.



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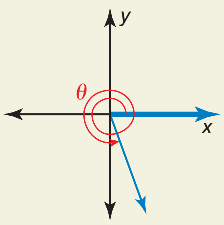

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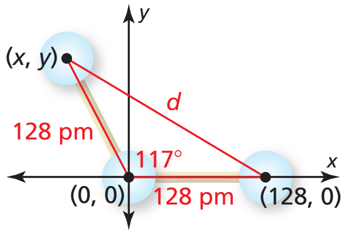

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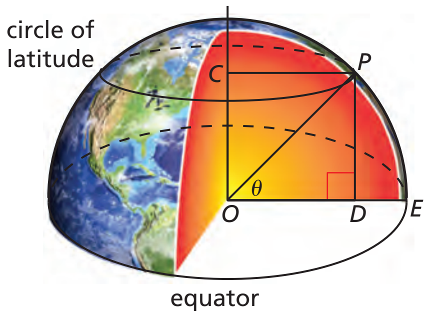

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find all real zeros of the polynomial function. *(Section 4.6)*

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9.4 Lesson

Core Vocabulary

amplitude, *p. 486*
periodic function, *p. 486*
cycle, *p. 486*
period, *p. 486*
phase shift, *p. 488*
midline, *p. 488*

Previous
transformations
x-intercept

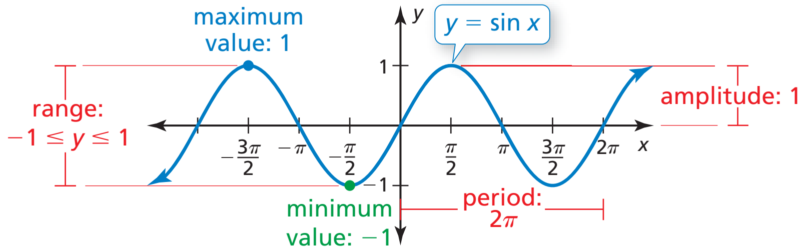
What You Will Learn

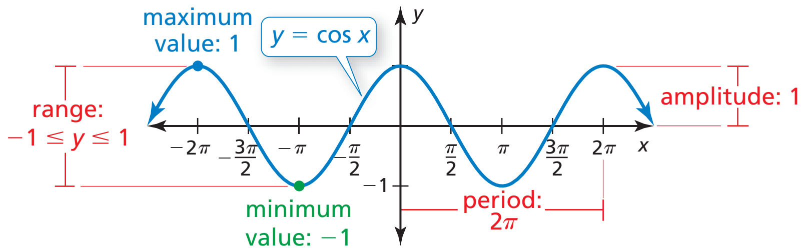
- Explore characteristics of sine and cosine functions.
- Stretch and shrink graphs of sine and cosine functions.
- Translate graphs of sine and cosine functions.
- Reflect graphs of sine and cosine functions.

Exploring Characteristics of Sine and Cosine Functions

In this lesson, you will learn to graph sine and cosine functions. The graphs of sine and cosine functions are related to the graphs of the parent functions $y = \sin x$ and $y = \cos x$, which are shown below.

x	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \sin x$	0	1	0	-1	0	1	0	-1	0
$y = \cos x$	1	0	-1	0	1	0	-1	0	1





Core Concept

Characteristics of $y = \sin x$ and $y = \cos x$

- The domain of each function is all real numbers.
- The range of each function is $-1 \leq y \leq 1$. So, the minimum value of each function is -1 and the maximum value is 1.
- The **amplitude** of the graph of each function is one-half of the difference of the maximum value and the minimum value, or $\frac{1}{2}[1 - (-1)] = 1$.
- Each function is **periodic**, which means that its graph has a repeating pattern. The shortest repeating portion of the graph is called a **cycle**. The horizontal length of each cycle is called the **period**. Each graph shown above has a period of 2π.
- The x-intercepts for $y = \sin x$ occur when $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.
- The x-intercepts for $y = \cos x$ occur when $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$.

Stretching and Shrinking Sine and Cosine Functions

The graphs of $y = a \sin bx$ and $y = a \cos bx$ represent transformations of their parent functions. The value of a indicates a vertical stretch ($a > 1$) or a vertical shrink ($0 < a < 1$) and changes the amplitude of the graph. The value of b indicates a horizontal stretch ($0 < b < 1$) or a horizontal shrink ($b > 1$) and changes the period of the graph.

REMEMBER

The graph of $y = a \cdot f(x)$ is a vertical stretch or shrink of the graph of $y = f(x)$ by a factor of a.

The graph of $y = f(bx)$ is a horizontal stretch or shrink of the graph of $y = f(x)$ by a factor of $\frac{1}{b}$.

$$y = a \sin bx$$
$$y = a \cos bx$$

vertical stretch or shrink by a factor of a

horizontal stretch or shrink by a factor of $\frac{1}{b}$

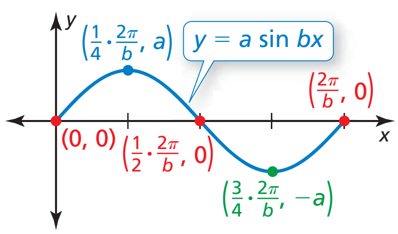
Core Concept

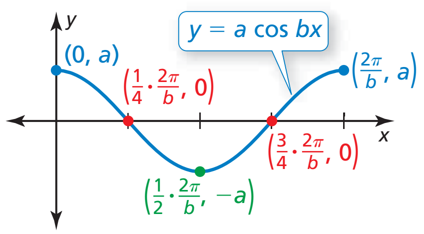
Amplitude and Period

The amplitude and period of the graphs of $y = a \sin bx$ and $y = a \cos bx$, where a and b are nonzero real numbers, are as follows:

$$\text{Amplitude} = |a| \qquad\qquad \text{Period} = \frac{2\pi}{|b|}$$

Each graph below shows five key points that partition the interval $0 \leq x \leq \frac{2\pi}{b}$ into four equal parts. You can use these points to sketch the graphs of $y = a \sin bx$ and $y = a \cos bx$. The x-intercepts, maximum, and minimum occur at these points.



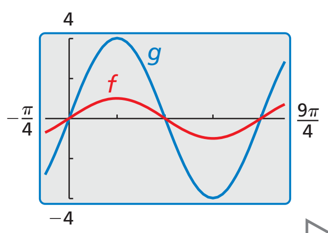


EXAMPLE 1 **Graphing a Sine Function**

Identify the amplitude and period of $g(x) = 4 \sin x$. Then graph the function and describe the graph of g as a transformation of the graph of $f(x) = \sin x$.

REMEMBER

A vertical stretch of a graph does not change its x-intercept(s). So, it makes sense that the x-intercepts of $g(x) = 4 \sin x$ and $f(x) = \sin x$ are the same.



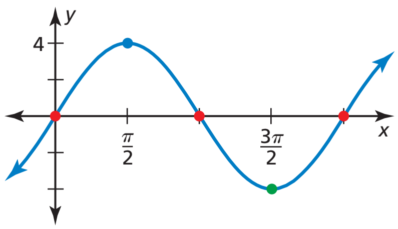
SOLUTION

The function is of the form $g(x) = a \sin bx$ where $a = 4$ and $b = 1$. So, the amplitude is $a = 4$ and the period is $\frac{2\pi}{b} = \frac{2\pi}{1} = 2\pi$.

Intercepts: $(0, 0)$; $\left(\frac{1}{2} \cdot 2\pi, 0\right) = (\pi, 0)$; $(2\pi, 0)$

Maximum: $\left(\frac{1}{4} \cdot 2\pi, 4\right) = \left(\frac{\pi}{2}, 4\right)$

Minimum: $\left(\frac{3}{4} \cdot 2\pi, -4\right) = \left(\frac{3\pi}{2}, -4\right)$



▶ The graph of g is a vertical stretch by a factor of 4 of the graph of f.

EXAMPLE 2 **Graphing a Cosine Function**

Identify the amplitude and period of $g(x) = \frac{1}{2}\cos 2\pi x$. Then graph the function and describe the graph of g as a transformation of the graph of $f(x) = \cos x$.

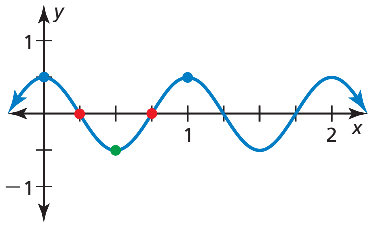
SOLUTION

The function is of the form $g(x) = a \cos bx$ where $a = \frac{1}{2}$ and $b = 2\pi$. So, the amplitude is $a = \frac{1}{2}$ and the period is $\frac{2\pi}{b} = \frac{2\pi}{2\pi} = 1$.

Intercepts: $\left(\frac{1}{4} \cdot 1, 0\right) = \left(\frac{1}{4}, 0\right)$; $\left(\frac{3}{4} \cdot 1, 0\right) = \left(\frac{3}{4}, 0\right)$

Maximums: $\left(0, \frac{1}{2}\right)$; $\left(1, \frac{1}{2}\right)$

Minimum: $\left(\frac{1}{2} \cdot 1, -\frac{1}{2}\right) = \left(\frac{1}{2}, -\frac{1}{2}\right)$



STUDY TIP

After you have drawn one complete cycle of the graph in Example 2 on the interval $0 \le x \le 1$, you can extend the graph by repeating the cycle as many times as desired to the left and right of $0 \le x \le 1$.

▶ The graph of g is a vertical shrink by a factor of $\frac{1}{2}$ and a horizontal shrink by a factor of $\frac{1}{2\pi}$ of the graph of f.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Identify the amplitude and period of the function. Then graph the function and describe the graph of g as a transformation of the graph of its parent function.

1. $g(x) = \frac{1}{4}\sin x$ **2.** $g(x) = \cos 2x$ **3.** $g(x) = 2\sin \pi x$ **4.** $g(x) = \frac{1}{3}\cos \frac{1}{2}x$

Translating Sine and Cosine Functions

REMEMBER

The graph of $y = f(x) + k$ is a vertical translation of the graph of $y = f(x)$.

The graph of $y = f(x - h)$ is a horizontal translation of the graph of $y = f(x)$.

The graphs of $y = a \sin b(x - h) + k$ and $y = a \cos b(x - h) + k$ represent translations of $y = a \sin bx$ and $y = a \cos bx$. The value of k indicates a translation up ($k > 0$) or down ($k < 0$). The value of h indicates a translation left ($h < 0$) or right ($h > 0$). A horizontal translation of a periodic function is called a **phase shift**.

Core Concept

Graphing $y = a \sin b(x - h) + k$ and $y = a \cos b(x - h) + k$

To graph $y = a \sin b(x - h) + k$ or $y = a \cos b(x - h) + k$ where $a > 0$ and $b > 0$, follow these steps:

Step 1 Identify the amplitude a, the period $\frac{2\pi}{b}$, the horizontal shift h, and the vertical shift k of the graph.

Step 2 Draw the horizontal line $y = k$, called the **midline** of the graph.

Step 3 Find the five key points by translating the key points of $y = a \sin bx$ or $y = a \cos bx$ horizontally h units and vertically k units.

Step 4 Draw the graph through the five translated key points.

EXAMPLE 3 Graphing a Vertical Translation

Graph $g(x) = 2 \sin 4x + 3$.

LOOKING FOR STRUCTURE

The graph of g is a translation 3 units up of the graph of $f(x) = 2 \sin 4x$. So, add 3 to the y-coordinates of the five key points of f.

SOLUTION

Step 1 Identify the amplitude, period, horizontal shift, and vertical shift.

Amplitude: $a = 2$ Horizontal shift: $h = 0$

Period: $\dfrac{2\pi}{b} = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$ Vertical shift: $k = 3$

Step 2 Draw the midline of the graph, $y = 3$.

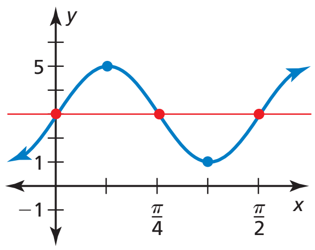
Step 3 Find the five key points.

On $y = k$: $(0, 0 + 3) = (0, 3)$; $\left(\dfrac{\pi}{4}, 0 + 3\right) = \left(\dfrac{\pi}{4}, 3\right)$; $\left(\dfrac{\pi}{2}, 0 + 3\right) = \left(\dfrac{\pi}{2}, 3\right)$

Maximum: $\left(\dfrac{\pi}{8}, 2 + 3\right) = \left(\dfrac{\pi}{8}, 5\right)$

Minimum: $\left(\dfrac{3\pi}{8}, -2 + 3\right) = \left(\dfrac{3\pi}{8}, 1\right)$

Step 4 Draw the graph through the key points.



EXAMPLE 4 Graphing a Horizontal Translation

Graph $g(x) = 5 \cos \dfrac{1}{2}(x - 3\pi)$.

LOOKING FOR STRUCTURE

The graph of g is a translation 3π units right of the graph of $f(x) = 5 \cos \frac{1}{2}x$. So, add 3π to the x-coordinates of the five key points of f.

SOLUTION

Step 1 Identify the amplitude, period, horizontal shift, and vertical shift.

Amplitude: $a = 5$ Horizontal shift: $h = 3\pi$

Period: $\dfrac{2\pi}{b} = \dfrac{2\pi}{\frac{1}{2}} = 4\pi$ Vertical shift: $k = 0$

Step 2 Draw the midline of the graph. Because $k = 0$, the midline is the x-axis.

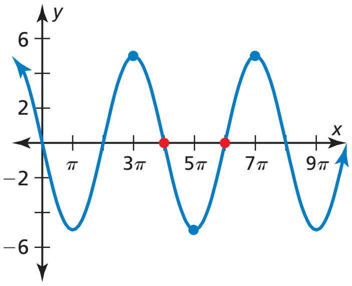
Step 3 Find the five key points.

On $y = k$: $(\pi + 3\pi, 0) = (4\pi, 0)$;
$(3\pi + 3\pi, 0) = (6\pi, 0)$

Maximums: $(0 + 3\pi, 5) = (3\pi, 5)$;
$(4\pi + 3\pi, 5) = (7\pi, 5)$

Minimum: $(2\pi + 3\pi, -5) = (5\pi, -5)$

Step 4 Draw the graph through the key points.



Monitoring Progress

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Graph the function.

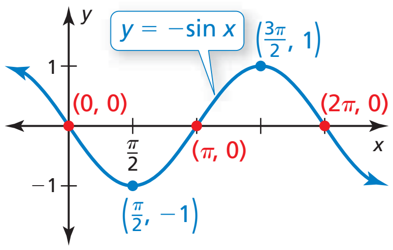
5. $g(x) = \cos x + 4$

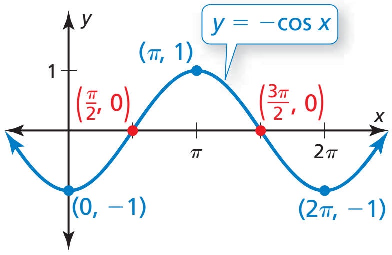
6. $g(x) = \dfrac{1}{2} \sin\left(x - \dfrac{\pi}{2}\right)$

7. $g(x) = \sin(x + \pi) - 1$

Reflecting Sine and Cosine Functions

You have graphed functions of the form $y = a \sin b(x - h) + k$ and $y = a \cos b(x - h) + k$, where $a > 0$ and $b > 0$. To see what happens when $a < 0$, consider the graphs of $y = -\sin x$ and $y = -\cos x$.





REMEMBER

This result makes sense because the graph of $y = -f(x)$ is a reflection in the x-axis of the graph of $y = f(x)$.

The graphs are reflections of the graphs of $y = \sin x$ and $y = \cos x$ in the x-axis. In general, when $a < 0$, the graphs of $y = a \sin b(x - h) + k$ and $y = a \cos b(x - h) + k$ are reflections of the graphs of $y = |a| \sin b(x - h) + k$ and $y = |a| \cos b(x - h) + k$, respectively, in the midline $y = k$.

EXAMPLE 5 Graphing a Reflection

Graph $g(x) = -2 \sin \frac{2}{3}\left(x - \frac{\pi}{2}\right)$.

SOLUTION

Step 1 Identify the amplitude, period, horizontal shift, and vertical shift.

Amplitude: $|a| = |-2| = 2$ Horizontal shift: $h = \frac{\pi}{2}$

Period: $\frac{2\pi}{b} = \frac{2\pi}{\frac{2}{3}} = 3\pi$ Vertical shift: $k = 0$

Step 2 Draw the midline of the graph. Because $k = 0$, the midline is the x-axis.

Step 3 Find the five key points of $f(x) = |-2| \sin \frac{2}{3}\left(x - \frac{\pi}{2}\right)$.

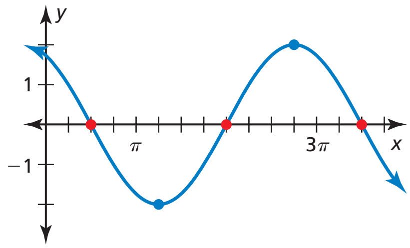
On $y = k$: $\left(0 + \frac{\pi}{2}, 0\right) = \left(\frac{\pi}{2}, 0\right)$; $\left(\frac{3\pi}{2} + \frac{\pi}{2}, 0\right) = (2\pi, 0)$; $\left(3\pi + \frac{\pi}{2}, 0\right) = \left(\frac{7\pi}{2}, 0\right)$

Maximum: $\left(\frac{3\pi}{4} + \frac{\pi}{2}, 2\right) = \left(\frac{5\pi}{4}, 2\right)$ Minimum: $\left(\frac{9\pi}{4} + \frac{\pi}{2}, -2\right) = \left(\frac{11\pi}{4}, -2\right)$

STUDY TIP

In Example 5, the maximum value and minimum value of f are the minimum value and maximum value, respectively, of g.

Step 4 Reflect the graph. Because $a < 0$, the graph is reflected in the midline $y = 0$. So, $\left(\frac{5\pi}{4}, 2\right)$ becomes $\left(\frac{5\pi}{4}, -2\right)$ and $\left(\frac{11\pi}{4}, -2\right)$ becomes $\left(\frac{11\pi}{4}, 2\right)$.



Step 5 Draw the graph through the key points.

Monitoring Progress

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Graph the function.

8. $g(x) = -\cos\left(x + \frac{\pi}{2}\right)$ **9.** $g(x) = -3 \sin \frac{1}{2}x + 2$ **10.** $g(x) = -2 \cos 4x - 1$

9.4 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The shortest repeating portion of the graph of a periodic function is called a(n) __________.

2. **WRITING** Compare the amplitudes and periods of the functions $y = \frac{1}{2}\cos x$ and $y = 3\cos 2x$.

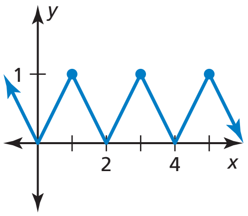
3. **VOCABULARY** What is a phase shift? Give an example of a sine function that has a phase shift.

4. **VOCABULARY** What is the midline of the graph of the function $y = 2\sin 3(x + 1) - 2$?

Monitoring Progress and Modeling with Mathematics

USING STRUCTURE In Exercises 5–8, determine whether the graph represents a periodic function. If so, identify the period.

5.



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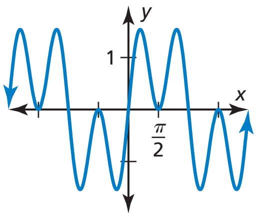

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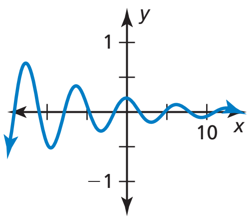

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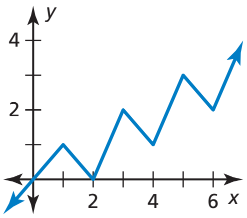
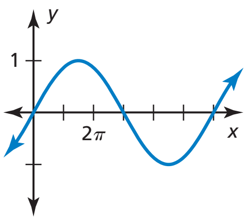


In Exercises 9–12, identify the amplitude and period of the graph of the function.

9.



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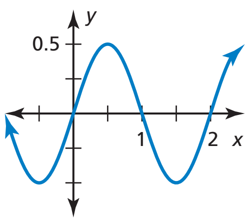

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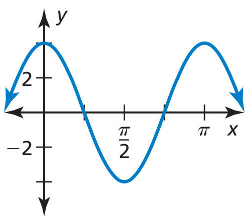

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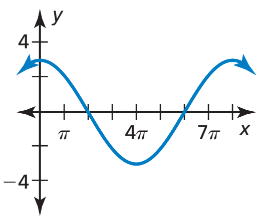


In Exercises 13–20, identify the amplitude and period of the function. Then graph the function and describe the graph of *g* as a transformation of the graph of its parent function. *(See Examples 1 and 2.)*

13. $g(x) = 3\sin x$

14. $g(x) = 2\sin x$

15. $g(x) = \cos 3x$

16. $g(x) = \cos 4x$

17. $g(x) = \sin 2\pi x$

18. $g(x) = 3\sin 2x$

19. $g(x) = \frac{1}{3}\cos 4x$

20. $g(x) = \frac{1}{2}\cos 4\pi x$

21. ANALYZING EQUATIONS Which functions have an amplitude of 4 and a period of 2?

Ⓐ $y = 4\cos 2x$

Ⓑ $y = -4\sin \pi x$

Ⓒ $y = 2\sin 4x$

Ⓓ $y = 4\cos \pi x$

22. WRITING EQUATIONS Write an equation of the form $y = a\sin bx$, where $a > 0$ and $b > 0$, so that the graph has the given amplitude and period.

a. amplitude: 1
period: 5

b. amplitude: 10
period: 4

c. amplitude: 2
period: 2π

d. amplitude: $\frac{1}{2}$
period: 3π

23. MODELING WITH MATHEMATICS The motion of a pendulum can be modeled by the function $d = 4\cos 8\pi t$, where d is the horizontal displacement (in inches) of the pendulum relative to its position at rest and t is the time (in seconds). Find and interpret the period and amplitude in the context of this situation. Then graph the function.

24. **MODELING WITH MATHEMATICS** A buoy bobs up and down as waves go past. The vertical displacement y (in feet) of the buoy with respect to sea level can be modeled by $y = 1.75 \cos \frac{\pi}{3}t$, where t is the time (in seconds). Find and interpret the period and amplitude in the context of the problem. Then graph the function.

In Exercises 25–34, graph the function. *(See Examples 3 and 4.)*

25. $g(x) = \sin x + 2$

26. $g(x) = \cos x - 4$

27. $g(x) = \cos\left(x - \frac{\pi}{2}\right)$

28. $g(x) = \sin\left(x + \frac{\pi}{4}\right)$

29. $g(x) = 2 \cos x - 1$

30. $g(x) = 3 \sin x + 1$

31. $g(x) = \sin 2(x + \pi)$

32. $g(x) = \cos 2(x - \pi)$

33. $g(x) = \sin \frac{1}{2}(x + 2\pi) + 3$

34. $g(x) = \cos \frac{1}{2}(x - 3\pi) - 5$

35. **ERROR ANALYSIS** Describe and correct the error in finding the period of the function $y = \sin \frac{2}{3}x$.

Period: $\dfrac{|b|}{2\pi} = \dfrac{\left|\frac{2}{3}\right|}{2\pi} = \dfrac{1}{3\pi}$

36. **ERROR ANALYSIS** Describe and correct the error in determining the point where the maximum value of the function $y = 2 \sin\left(x - \frac{\pi}{2}\right)$ occurs.

Maximum:

$\left(\left(\frac{1}{4} \cdot 2\pi\right) - \frac{\pi}{2}, 2\right) = \left(\frac{\pi}{2} - \frac{\pi}{2}, 2\right)$

$= (0, 2)$

USING STRUCTURE **In Exercises 37–40, describe the transformation of the graph of f represented by the function g.**

37. $f(x) = \cos x$, $g(x) = 2 \cos\left(x - \frac{\pi}{2}\right) + 1$

38. $f(x) = \sin x$, $g(x) = 3 \sin\left(x + \frac{\pi}{4}\right) - 2$

39. $f(x) = \sin x$, $g(x) = \sin 3(x + 3\pi) - 5$

40. $f(x) = \cos x$, $g(x) = \cos 6(x - \pi) + 9$

In Exercises 41–48, graph the function. *(See Example 5.)*

41. $g(x) = -\cos x + 3$

42. $g(x) = -\sin x - 5$

43. $g(x) = -\sin \frac{1}{2}x - 2$

44. $g(x) = -\cos 2x + 1$

45. $g(x) = -\sin(x - \pi) + 4$

46. $g(x) = -\cos(x + \pi) - 2$

47. $g(x) = -4 \cos\left(x + \frac{\pi}{4}\right) - 1$

48. $g(x) = -5 \sin\left(x - \frac{\pi}{2}\right) + 3$

49. **USING EQUATIONS** Which of the following is a point where the maximum value of the graph of $y = -4 \cos\left(x - \frac{\pi}{2}\right)$ occurs?

Ⓐ $\left(-\frac{\pi}{2}, 4\right)$

Ⓑ $\left(\frac{\pi}{2}, 4\right)$

Ⓒ $(0, 4)$

Ⓓ $(\pi, 4)$

50. **ANALYZING RELATIONSHIPS** Match each function with its graph. Explain your reasoning.

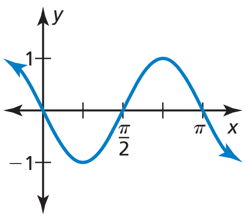
a. $y = 3 + \sin x$

b. $y = -3 + \cos x$

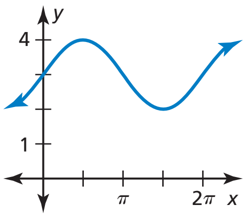
c. $y = \sin 2\left(x - \frac{\pi}{2}\right)$

d. $y = \cos 2\left(x - \frac{\pi}{2}\right)$

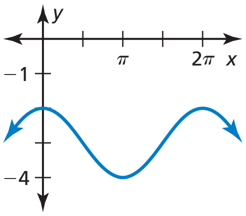
A.



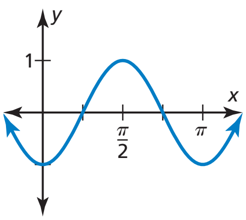
B.



C.



D.



WRITING EQUATIONS **In Exercises 51–54, write a rule for g that represents the indicated transformations of the graph of f.**

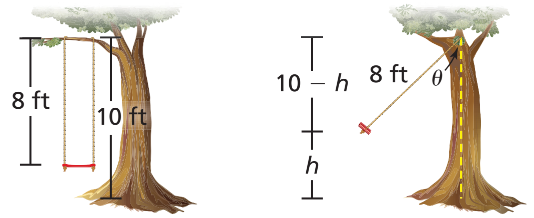
51. $f(x) = 3 \sin x$; translation 2 units up and π units right

52. $f(x) = \cos 2\pi x$; translation 4 units down and 3 units left

53. $f(x) = \frac{1}{3} \cos \pi x$; translation 1 unit down, followed by a reflection in the line $y = -1$

54. $f(x) = \frac{1}{2} \sin 6x$; translation $\frac{3}{2}$ units down and 1 unit right, followed by a reflection in the line $y = -\frac{3}{2}$

55. **MODELING WITH MATHEMATICS** The height h (in feet) of a swing above the ground can be modeled by the function $h = -8 \cos \theta + 10$, where the pivot is 10 feet above the ground, the rope is 8 feet long, and θ is the angle that the rope makes with the vertical. Graph the function. What is the height of the swing when θ is 45°?



Front view   Side view

56. **DRAWING A CONCLUSION** In a particular region, the population L (in thousands) of lynx (the predator) and the population H (in thousands) of hares (the prey) can be modeled by the equations

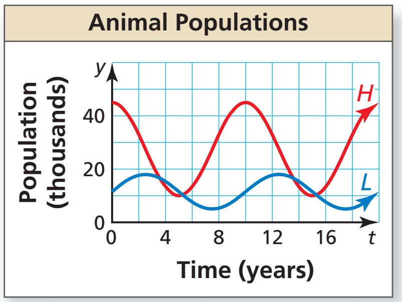
$$L = 11.5 + 6.5 \sin \frac{\pi}{5} t$$

$$H = 27.5 + 17.5 \cos \frac{\pi}{5} t$$

where t is the time in years.

a. Determine the ratio of hares to lynx when $t = 0$, 2.5, 5, and 7.5 years.

b. Use the figure to explain how the changes in the two populations appear to be related.

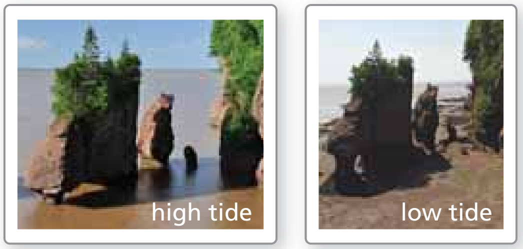


57. **USING TOOLS** The average wind speed s (in miles per hour) in the Boston Harbor can be approximated by

$$s = 3.38 \sin \frac{\pi}{180}(t + 3) + 11.6$$

where t is the time in days and $t = 0$ represents January 1. Use a graphing calculator to graph the function. On which days of the year is the average wind speed 10 miles per hour? Explain your reasoning.

58. **USING TOOLS** The water depth d (in feet) for the Bay of Fundy can be modeled by $d = 35 - 28 \cos \frac{\pi}{6.2} t$, where t is the time in hours and $t = 0$ represents midnight. Use a graphing calculator to graph the function. At what time(s) is the water depth 7 feet? Explain.



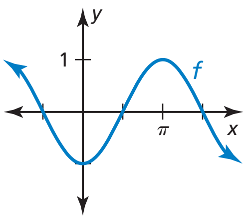
59. **MULTIPLE REPRESENTATIONS** Find the average rate of change of each function over the interval $0 < x < \pi$.

a. $y = 2 \cos x$

b.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$f(x) = -\cos x$	-1	0	1	0	-1

c.



60. **REASONING** Consider the functions $y = \sin(-x)$ and $y = \cos(-x)$.

a. Construct a table of values for each equation using the quadrantal angles in the interval $-2\pi \le x \le 2\pi$.

b. Graph each function.

c. Describe the transformations of the graphs of the parent functions.

61. **MODELING WITH MATHEMATICS** You are riding a Ferris wheel that turns for 180 seconds. Your height h (in feet) above the ground at any time t (in seconds) can be modeled by the equation

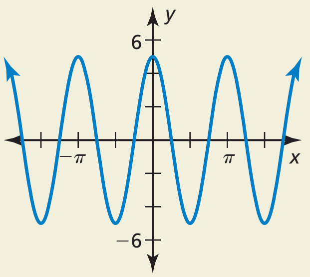
$$h = 85 \sin \frac{\pi}{20}(t - 10) + 90.$$

a. Graph the function.

b. How many cycles does the Ferris wheel make in 180 seconds?

c. What are your maximum and minimum heights?

62. **HOW DO YOU SEE IT?** Use the graph to answer each question.



a. Does the graph represent a function of the form $f(x) = a \sin bx$ or $f(x) = a \cos bx$? Explain.

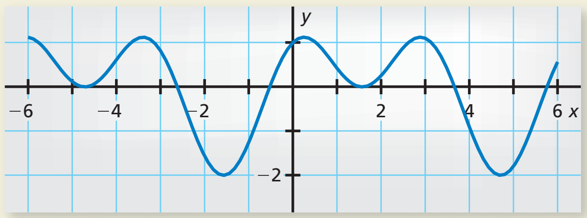
b. Identify the maximum value, minimum value, period, and amplitude of the function.

63. **FINDING A PATTERN** Write an expression in terms of the integer n that represents all the x-intercepts of the graph of the function $y = \cos 2x$. Justify your answer.

64. **MAKING AN ARGUMENT** Your friend states that for functions of the form $y = a \sin bx$ and $y = a \cos bx$, the values of a and b affect the x-intercepts of the graph of the function. Is your friend correct? Explain.

65. **CRITICAL THINKING** Describe a transformation of the graph of $f(x) = \sin x$ that results in the graph of $g(x) = \cos x$.

66. **THOUGHT PROVOKING** Use a graphing calculator to find a function of the form $y = \sin b_1 x + \cos b_2 x$ whose graph matches that shown below.



67. **PROBLEM SOLVING** For a person at rest, the blood pressure P (in millimeters of mercury) at time t (in seconds) is given by the function

$$P = 100 - 20 \cos \frac{8\pi}{3}t.$$

Graph the function. One cycle is equivalent to one heartbeat. What is the pulse rate (in heartbeats per minute) of the person?

68. **PROBLEM SOLVING** The motion of a spring can be modeled by $y = A \cos kt$, where y is the vertical displacement (in feet) of the spring relative to its position at rest, A is the initial displacement (in feet), k is a constant that measures the elasticity of the spring, and t is the time (in seconds).

a. You have a spring whose motion can be modeled by the function $y = 0.2 \cos 6t$. Find the initial displacement and the period of the spring. Then graph the function.

b. When a damping force is applied to the spring, the motion of the spring can be modeled by the function $y = 0.2e^{-4.5t} \cos 4t$. Graph this function. What effect does damping have on the motion?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the rational expression, if possible. *(Section 7.3)*

69. $\dfrac{x^2 + x - 6}{x + 3}$

70. $\dfrac{x^3 - 2x^2 - 24x}{x^2 - 2x - 24}$

71. $\dfrac{x^2 - 4x - 5}{x^2 + 4x - 5}$

72. $\dfrac{x^2 - 16}{x^2 + x - 20}$

Find the least common multiple of the expressions. *(Section 7.4)*

73. $2x$, $2(x - 5)$

74. $x^2 - 4$, $x + 2$

75. $x^2 + 8x + 12$, $x + 6$

9.1–9.4 What Did You Learn?

Core Vocabulary

sine, *p. 462*
cosine, *p. 462*
tangent, *p. 462*
cosecant, *p. 462*
secant, *p. 462*
cotangent, *p. 462*
initial side, *p. 470*
terminal side, *p. 470*
standard position, *p. 470*
coterminal, *p. 471*
radian, *p. 471*
sector, *p. 472*
central angle, *p. 472*
unit circle, *p. 479*
quadrantal angle, *p. 479*
reference angle, *p. 480*
amplitude, *p. 486*
periodic function, *p. 486*
cycle, *p. 486*
period, *p. 486*
phase shift, *p. 488*
midline, *p. 488*

Core Concepts

Section 9.1

Right Triangle Definitions of Trigonometric Functions, *p. 462*
Trigonometric Values for Special Angles, *p. 463*

Section 9.2

Angles in Standard Position, *p. 470*
Converting Between Degrees and Radians, *p. 471*
Degree and Radian Measures of Special Angles, *p. 472*
Arc Length and Area of a Sector, *p. 472*

Section 9.3

General Definitions of Trigonometric Functions, *p. 478*
The Unit Circle, *p. 479*
Reference Angle Relationships, *p. 480*
Evaluating Trigonometric Functions, *p. 480*

Section 9.4

Characteristics of $y = \sin x$ and $y = \cos x$, *p. 486*
Amplitude and Period, *p. 487*
Graphing $y = a \sin b(x - h) + k$ and $y = a \cos b(x - h) + k$, *p. 488*

Mathematical Practices

1. Make a conjecture about the horizontal distances traveled in part (c) of Exercise 39 on page 483.

2. Explain why the quantities in part (a) of Exercise 56 on page 493 make sense in the context of the situation.

Study Skills

Form a Final Exam Study Group

Form a study group several weeks before the final exam. The intent of this group is to review what you have already learned while continuing to learn new material.