8 Sequences and Series

8.1 Defining and Using Sequences and Series

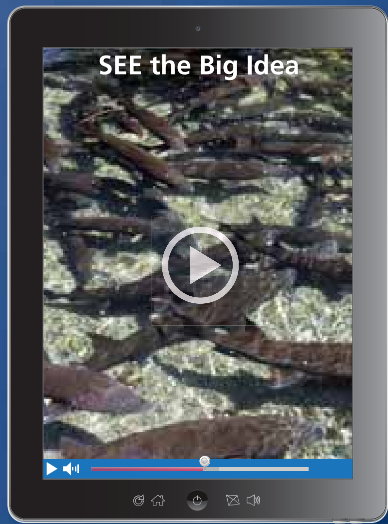
8.2 Analyzing Arithmetic Sequences and Series

8.3 Analyzing Geometric Sequences and Series

8.4 Finding Sums of Infinite Geometric Series

8.5 Using Recursive Rules with Sequences

Tree Farm ***(p. 449)***



Fish Population ***(p. 445)***

Skydiving ***(p. 431)***

Marching Band ***(p. 423)***

Museum Skylight ***(p. 416)***

Maintaining Mathematical Proficiency

Evaluating Functions

Example 1 Evaluate the function $y = 2x^2 - 10$ for the values $x = 0, 1, 2, 3,$ and 4.

Input, x	$2x^2 - 10$	Output, y
0	$2(0)^2 - 10$	-10
1	$2(1)^2 - 10$	-8
2	$2(2)^2 - 10$	-2
3	$2(3)^2 - 10$	8
4	$2(4)^2 - 10$	22

Copy and complete the table to evaluate the function.

1. $y = 3 - 2^x$

x	y
1	
2	
3	

2. $y = 5x^2 + 1$

x	y
2	
3	
4	

3. $y = -4x + 24$

x	y
5	
10	
15	

Solving Equations

Example 2 Solve the equation $45 = 5(3)^x$.

$45 = 5(3)^x$ Write original equation.

$\frac{45}{5} = \frac{5(3)^x}{5}$ Divide each side by 5.

$9 = 3^x$ Simplify.

$\log_3 9 = \log_3 3^x$ Take $\log_3$ of each side.

$2 = x$ Simplify.

Solve the equation. Check your solution(s).

4. $7x + 3 = 31$

5. $\frac{1}{16} = 4\left(\frac{1}{2}\right)^x$

6. $216 = 3(x + 6)$

7. $2^x + 16 = 144$

8. $\frac{1}{4}x - 8 = 17$

9. $8\left(\frac{3}{4}\right)^x = \frac{27}{8}$

10. ABSTRACT REASONING The graph of the exponential decay function $f(x) = b^x$ has an asymptote $y = 0$. How is the graph of f different from a scatter plot consisting of the points $(1, b^1), (2, b^1 + b^2), (3, b^1 + b^2 + b^3), \ldots$? How is the graph of f similar?

Mathematical Practices

Mathematically proficient students consider the available tools when solving a mathematical problem.

Using Appropriate Tools Strategically

Core Concept

Using a Spreadsheet

To use a spreadsheet, it is common to write one cell as a function of another cell. For instance, in the spreadsheet shown, the cells in column A starting with cell A2 contain functions of the cell in the preceding row. Also, the cells in column B contain functions of the cells in the same row in column A.

	A	B
1	1	0
2	2	2
3	3	4
4	4	6
5	5	8
6	6	10
7	7	12
8	8	14

A2 = A1+1

B1 = 2*A1−2

EXAMPLE 1 Using a Spreadsheet

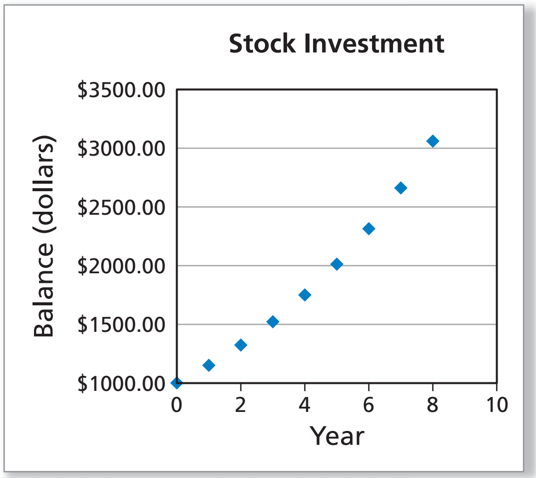
You deposit $1000 in stocks that earn 15% interest compounded annually. Use a spreadsheet to find the balance at the end of each year for 8 years. Describe the type of growth.

SOLUTION

You can enter the given information into a spreadsheet and generate the graph shown. From the formula in the spreadsheet, you can see that the growth pattern is exponential. The graph also appears to be exponential.

	A	B
1	Year	Balance
2	0	$1000.00
3	1	$1150.00
4	2	$1322.50
5	3	$1520.88
6	4	$1749.01
7	5	$2011.36
8	6	$2313.06
9	7	$2660.02
10	8	$3059.02

B3 = B2*1.15



Monitoring Progress

Use a spreadsheet to help you answer the question.

1. A pilot flies a plane at a speed of 500 miles per hour for 4 hours. Find the total distance flown at 30-minute intervals. Describe the pattern.
2. A population of 60 rabbits increases by 25% each year for 8 years. Find the population at the end of each year. Describe the type of growth.
3. An endangered population has 500 members. The population declines by 10% each decade for 80 years. Find the population at the end of each decade. Describe the type of decline.
4. The top eight runners finishing a race receive cash prizes. First place receives $200, second place receives $175, third place receives $150, and so on. Find the fifth through eighth place prizes. Describe the type of decline.

8.1 Defining and Using Sequences and Series

Essential Question

How can you write a rule for the *n*th term of a sequence?

A **sequence** is an ordered list of numbers. There can be a limited number or an infinite number of *terms* of a sequence.

$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ Terms of a sequence

Here is an example.

$1, 4, 7, 10, \ldots, 3n - 2, \ldots$

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to reason inductively about data.

EXPLORATION 1 Writing Rules for Sequences

Work with a partner. Match each sequence with its graph. The horizontal axes represent n, the position of each term in the sequence. Then write a rule for the nth term of the sequence, and use the rule to find a_{10}.

a. $1, 2.5, 4, 5.5, 7, \ldots$

b. $8, 6.5, 5, 3.5, 2, \ldots$

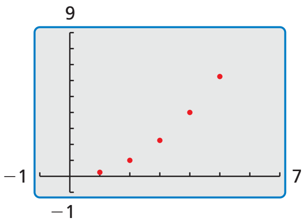
c. $\frac{1}{4}, \frac{4}{4}, \frac{9}{4}, \frac{16}{4}, \frac{25}{4}, \ldots$

d. $\frac{25}{4}, \frac{16}{4}, \frac{9}{4}, \frac{4}{4}, \frac{1}{4}, \ldots$

e. $\frac{1}{2}, 1, 2, 4, 8, \ldots$

f. $8, 4, 2, 1, \frac{1}{2}, \ldots$

A.



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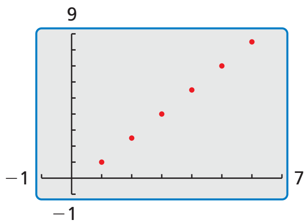

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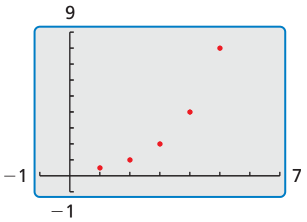

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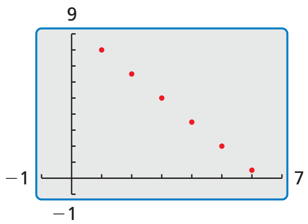
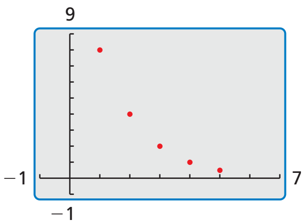
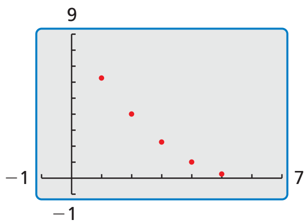


E.



F.



Communicate Your Answer

2. How can you write a rule for the *n*th term of a sequence?

3. What do you notice about the relationship between the terms in (a) an arithmetic sequence and (b) a geometric sequence? Justify your answers.

8.1 Lesson

Core Vocabulary

sequence, *p. 410*
terms of a sequence, *p. 410*
series, *p. 412*
summation notation, *p. 412*
sigma notation, *p. 412*

Previous
domain
range

What You Will Learn

- Use sequence notation to write terms of sequences.
- Write a rule for the nth term of a sequence.
- Sum the terms of a sequence to obtain a series and use summation notation.

Writing Terms of Sequences

Core Concept

Sequences

A **sequence** is an ordered list of numbers. A *finite sequence* is a function that has a limited number of terms and whose domain is the finite set $\{1, 2, 3, \ldots, n\}$. The values in the range are called the **terms** of the sequence.

Domain:	1	2	3	4	. . .	n	Relative position of each term
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		$\downarrow$	
Range:	a_1	a_2	a_3	a_4	. . .	a_n	Terms of the sequence

An *infinite sequence* is a function that continues without stopping and whose domain is the set of positive integers. Here are examples of a finite sequence and an infinite sequence.

Finite sequence: 2, 4, 6, 8 **Infinite sequence:** 2, 4, 6, 8, . . .

A sequence can be specified by an equation, or *rule*. For example, both sequences above can be described by the rule $a_n = 2n$ or $f(n) = 2n$.

The domain of a sequence may begin with 0 instead of 1. When this is the case, the domain of a finite sequence is the set $\{0, 1, 2, 3, \ldots, n\}$ and the domain of an infinite sequence becomes the set of nonnegative integers. Unless otherwise indicated, assume the domain of a sequence begins with 1.

EXAMPLE 1 Writing the Terms of Sequences

Write the first six terms of (a) $a_n = 2n + 5$ and (b) $f(n) = (-3)^{n-1}$.

SOLUTION

a. $a_1 = 2(1) + 5 = 7$	1st term	**b.** $f(1) = (-3)^{1-1} = 1$
$a_2 = 2(2) + 5 = 9$	2nd term	$f(2) = (-3)^{2-1} = -3$
$a_3 = 2(3) + 5 = 11$	3rd term	$f(3) = (-3)^{3-1} = 9$
$a_4 = 2(4) + 5 = 13$	4th term	$f(4) = (-3)^{4-1} = -27$
$a_5 = 2(5) + 5 = 15$	5th term	$f(5) = (-3)^{5-1} = 81$
$a_6 = 2(6) + 5 = 17$	6th term	$f(6) = (-3)^{6-1} = -243$

Monitoring Progress

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Write the first six terms of the sequence.

1. $a_n = n + 4$ **2.** $f(n) = (-2)^{n-1}$ **3.** $a_n = \dfrac{n}{n+1}$

Writing Rules for Sequences

When the terms of a sequence have a recognizable pattern, you may be able to write a rule for the nth term of the sequence.

STUDY TIP

When you are given only the first several terms of a sequence, there may be more than one rule for the nth term. For instance, the sequence 2, 4, 8, . . . can be given by $a_n = 2^n$ or $a_n = n^2 - n + 2$.

EXAMPLE 2 Writing Rules for Sequences

Describe the pattern, write the next term, and write a rule for the nth term of the sequences (a) $-1, -8, -27, -64, \ldots$ and (b) $0, 2, 6, 12, \ldots$.

SOLUTION

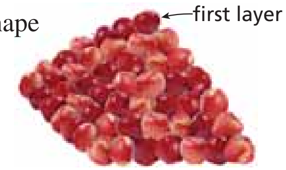
a. You can write the terms as $(-1)^3, (-2)^3, (-3)^3, (-4)^3, \ldots$. The next term is $a_5 = (-5)^3 = -125$. A rule for the nth term is $a_n = (-n)^3$.

b. You can write the terms as $0(1), 1(2), 2(3), 3(4), \ldots$. The next term is $f(5) = 4(5) = 20$. A rule for the nth term is $f(n) = (n - 1)n$.

To graph a sequence, let the horizontal axis represent the position numbers (the domain) and the vertical axis represent the terms (the range).

EXAMPLE 3 Solving a Real-Life Problem

You work in a grocery store and are stacking apples in the shape of a square pyramid with seven layers. Write a rule for the number of apples in each layer. Then graph the sequence.



SOLUTION

Step 1 Make a table showing the number of fruit in the first three layers. Let a_n represent the number of apples in layer n.

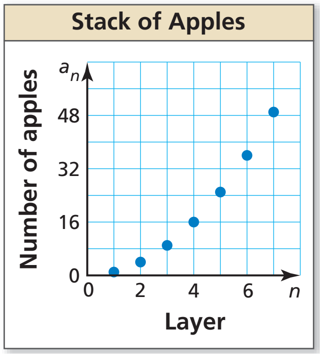
Layer, n	1	2	3
Number of apples, a_n	 $1 = 1^2$	 $4 = 2^2$	 $9 = 3^2$

COMMON ERROR

Although the plotted points in Example 3 follow a curve, do not draw the curve because the sequence is defined only for integer values of n, specifically $n = 1, 2, 3, 4, 5, 6,$ and 7.

Step 2 Write a rule for the number of apples in each layer. From the table, you can see that $a_n = n^2$.

Step 3 Plot the points (1, 1), (2, 4), (3, 9), (4, 16), (5, 25), (6, 36), and (7, 49). The graph is shown at the right.



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Describe the pattern, write the next term, graph the first five terms, and write a rule for the nth term of the sequence.

4. 3, 5, 7, 9, . . .

5. 3, 8, 15, 24, . . .

6. $1, -2, 4, -8, \ldots$

7. 2, 5, 10, 17, . . .

8. **WHAT IF?** In Example 3, suppose there are nine layers of apples. How many apples are in the ninth layer?

Writing Rules for Series

Core Concept

Series and Summation Notation

When the terms of a sequence are added together, the resulting expression is a **series**. A series can be finite or infinite.

Finite series: $2 + 4 + 6 + 8$

Infinite series: $2 + 4 + 6 + 8 + \cdots$

You can use **summation notation** to write a series. For example, the two series above can be written in summation notation as follows:

Finite series: $2 + 4 + 6 + 8 = \sum_{i=1}^{4} 2i$

Infinite series: $2 + 4 + 6 + 8 + \cdots = \sum_{i=1}^{\infty} 2i$

For both series, the *index of summation* is i and the *lower limit of summation* is 1. The *upper limit of summation* is 4 for the finite series and ∞ (infinity) for the infinite series. Summation notation is also called **sigma notation** because it uses the uppercase Greek letter *sigma*, written Σ.

READING

When written in summation notation, this series is read as "the sum of $2i$ for values of i from 1 to 4."

EXAMPLE 4 Writing Series Using Summation Notation

Write each series using summation notation.

a. $25 + 50 + 75 + \cdots + 250$

b. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$

SOLUTION

a. Notice that the first term is 25(1), the second is 25(2), the third is 25(3), and the last is 25(10). So, the terms of the series can be written as:

$$a_i = 25i, \text{ where } i = 1, 2, 3, \ldots, 10$$

The lower limit of summation is 1 and the upper limit of summation is 10.

▶ The summation notation for the series is $\sum_{i=1}^{10} 25i$.

b. Notice that for each term, the denominator of the fraction is 1 more than the numerator. So, the terms of the series can be written as:

$$a_i = \frac{i}{i+1}, \text{ where } i = 1, 2, 3, 4, \ldots$$

The lower limit of summation is 1 and the upper limit of summation is infinity.

▶ The summation notation for the series is $\sum_{i=1}^{\infty} \frac{i}{i+1}$.

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Write the series using summation notation.

9. $5 + 10 + 15 + \cdots + 100$

10. $\frac{1}{2} + \frac{4}{5} + \frac{9}{10} + \frac{16}{17} + \cdots$

11. $6 + 36 + 216 + 1296 + \cdots$

12. $5 + 6 + 7 + \cdots + 12$

COMMON ERROR

Be sure to use the correct lower and upper limits of summation when finding the sum of a series.

The index of summation for a series does not have to be i—any letter can be used. Also, the index does not have to begin at 1. For instance, the index begins at 4 in the next example.

EXAMPLE 5 Finding the Sum of a Series

Find the sum $\sum_{k=4}^{8}(3 + k^2)$.

SOLUTION

$$\begin{aligned}\sum_{k=4}^{8}(3 + k^2) &= (3 + 4^2) + (3 + 5^2) + (3 + 6^2) + (3 + 7^2) + (3 + 8^2)\\ &= 19 + 28 + 39 + 52 + 67\\ &= 205\end{aligned}$$

For series with many terms, finding the sum by adding the terms can be tedious. Below are formulas you can use to find the sums of three special types of series.

Core Concept

Formulas for Special Series

Sum of n terms of 1: $\sum_{i=1}^{n} 1 = n$

Sum of first n positive integers: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Sum of squares of first n positive integers: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

EXAMPLE 6 Using a Formula for a Sum

How many apples are in the stack in Example 3?

SOLUTION

From Example 3, you know that the ith term of the series is given by $a_i = i^2$, where $i = 1, 2, 3, \ldots, 7$. Using summation notation and the third formula listed above, you can find the total number of apples as follows:

$$1^2 + 2^2 + \cdots + 7^2 = \sum_{i=1}^{7} i^2 = \frac{7(7+1)(2 \cdot 7 + 1)}{6} = \frac{7(8)(15)}{6} = 140$$

▶ There are 140 apples in the stack. Check this by adding the number of apples in each of the seven layers.

Monitoring Progress

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Find the sum.

13. $\sum_{i=1}^{5} 8i$

14. $\sum_{k=3}^{7} (k^2 - 1)$

15. $\sum_{i=1}^{34} 1$

16. $\sum_{k=1}^{6} k$

17. WHAT IF? Suppose there are nine layers in the apple stack in Example 3. How many apples are in the stack?

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Vocabulary and Core Concept Check

1. **VOCABULARY** What is another name for summation notation?

2. **COMPLETE THE SENTENCE** In a sequence, the numbers are called __________ of the sequence.

3. **WRITING** Compare sequences and series.

4. **WHICH ONE DOESN'T BELONG?** Which does *not* belong with the other three? Explain your reasoning.

$\sum_{i=1}^{6} i^2$	91	$1 + 4 + 9 + 16 + 25 + 36$	$\sum_{i=0}^{5} i^2$

Monitoring Progress and Modeling with Mathematics

In Exercises 5–14, write the first six terms of the sequence. *(See Example 1.)*

5. $a_n = n + 2$

6. $a_n = 6 - n$

7. $a_n = n^2$

8. $f(n) = n^3 + 2$

9. $f(n) = 4^{n-1}$

10. $a_n = -n^2$

11. $a_n = n^2 - 5$

12. $a_n = (n + 3)^2$

13. $f(n) = \dfrac{2n}{n + 2}$

14. $f(n) = \dfrac{n}{2n - 1}$

In Exercises 15–26, describe the pattern, write the next term, and write a rule for the *n*th term of the sequence. *(See Example 2.)*

15. 1, 6, 11, 16, . . .

16. 1, 2, 4, 8, . . .

17. 3.1, 3.8, 4.5, 5.2, . . .

18. 9, 16.8, 24.6, 32.4, . . .

19. 5.8, 4.2, 2.6, 1, −0.6 . . .

20. −4, 8, −12, 16, . . .

21. $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \ldots$

22. $\frac{1}{10}, \frac{3}{20}, \frac{5}{30}, \frac{7}{40}, \ldots$

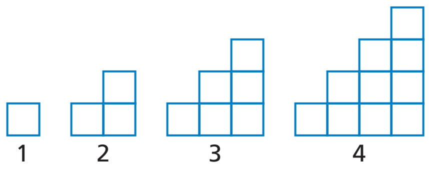
23. $\frac{2}{3}, \frac{2}{6}, \frac{2}{9}, \frac{2}{12}, \ldots$

24. $\frac{2}{3}, \frac{4}{4}, \frac{6}{5}, \frac{8}{6}, \ldots$

25. 2, 9, 28, 65, . . .

26. 1.2, 4.2, 9.2, 16.2, . . .

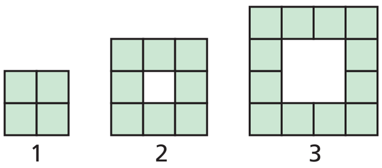
27. **FINDING A PATTERN** Which rule gives the total number of squares in the *n*th figure of the pattern shown? Justify your answer.



Ⓐ $a_n = 3n - 3$ Ⓑ $a_n = 4n - 5$

Ⓒ $a_n = n$ Ⓓ $a_n = \dfrac{n(n+1)}{2}$

28. **FINDING A PATTERN** Which rule gives the total number of green squares in the *n*th figure of the pattern shown? Justify your answer.



Ⓐ $a_n = n^2 - 1$ Ⓑ $a_n = \dfrac{n^2}{2}$

Ⓒ $a_n = 4n$ Ⓓ $a_n = 2n + 1$

29. **MODELING WITH MATHEMATICS** Rectangular tables are placed together along their short edges, as shown in the diagram. Write a rule for the number of people that can be seated around n tables arranged in this manner. Then graph the sequence. *(See Example 3.)*

30. **MODELING WITH MATHEMATICS** An employee at a construction company earns \$33,000 for the first year of employment. Employees at the company receive raises of \$2400 each year. Write a rule for the salary of the employee each year. Then graph the sequence.

In Exercises 31–38, write the series using summation notation. *(See Example 4.)*

31. $7 + 10 + 13 + 16 + 19$

32. $5 + 11 + 17 + 23 + 29$

33. $4 + 7 + 12 + 19 + \cdots$

34. $-1 + 2 + 7 + 14 + \cdots$

35. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$

36. $\frac{1}{4} + \frac{2}{5} + \frac{3}{6} + \frac{4}{7} + \cdots$

37. $-3 + 4 - 5 + 6 - 7$

38. $-2 + 4 - 8 + 16 - 32$

In Exercises 39–50, find the sum. *(See Examples 5 and 6.)*

39. $\sum_{i=1}^{6} 2i$

40. $\sum_{i=1}^{5} 7i$

41. $\sum_{n=0}^{4} n^3$

42. $\sum_{k=1}^{4} 3k^2$

43. $\sum_{k=3}^{6} (5k - 2)$

44. $\sum_{n=1}^{5} (n^2 - 1)$

45. $\sum_{i=2}^{8} \frac{2}{i}$

46. $\sum_{k=4}^{6} \frac{k}{k + 1}$

47. $\sum_{i=1}^{35} 1$

48. $\sum_{n=1}^{16} n$

49. $\sum_{i=10}^{25} i$

50. $\sum_{n=1}^{18} n^2$

ERROR ANALYSIS In Exercises 51 and 52, describe and correct the error in finding the sum of the series.

51.
$$\sum_{n=1}^{10} (3n - 5) = -2 + 1 + 4 + 7 + 10$$
$$= 20$$

52.
$$\sum_{i=2}^{4} i^2 = \frac{4(4 + 1)(2 \cdot 4 + 1)}{6}$$
$$= \frac{180}{6}$$
$$= 30$$

53. **PROBLEM SOLVING** You want to save \$500 for a school trip. You begin by saving a penny on the first day. You save an additional penny each day after that. For example, you will save two pennies on the second day, three pennies on the third day, and so on.

  a. How much money will you have saved after 100 days?

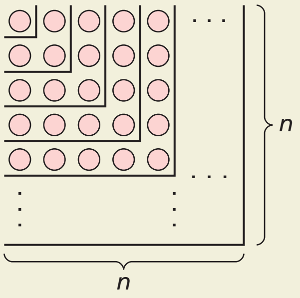
  b. Use a series to determine how many days it takes you to save \$500.

54. **MODELING WITH MATHEMATICS** You begin an exercise program. The first week you do 25 push-ups. Each week you do 10 more push-ups than the previous week. How many push-ups will you do in the ninth week? Justify your answer.

55. **MODELING WITH MATHEMATICS** For a display at a sports store, you are stacking soccer balls in a pyramid whose base is an equilateral triangle with five layers. Write a rule for the number of soccer balls in each layer. Then graph the sequence.

56. HOW DO YOU SEE IT? Use the diagram to determine the sum of the series. Explain your reasoning.

$$1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1) = ?$$



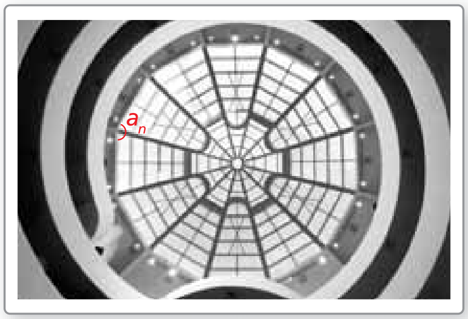
57. MAKING AN ARGUMENT You use a calculator to evaluate $\sum_{i=3}^{1659} i$ because the lower limit of summation is 3, not 1. Your friend claims there is a way to use the formula for the sum of the first n positive integers. Is your friend correct? Explain.

58. MATHEMATICAL CONNECTIONS A *regular* polygon has equal angle measures and equal side lengths. For a regular n-sided polygon ($n \geq 3$), the measure a_n of an interior angle is given by $a_n = \frac{180(n - 2)}{n}$.

a. Write the first five terms of the sequence.

b. Write a rule for the sequence giving the sum T_n of the measures of the interior angles in each regular n-sided polygon.

c. Use your rule in part (b) to find the sum of the interior angle measures in the Guggenheim Museum skylight, which is a regular dodecagon.



Guggenheim Museum Skylight

59. USING STRUCTURE Determine whether each statement is true. If so, provide a proof. If not, provide a counterexample.

a. $\sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i$

b. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

c. $\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i$

d. $\sum_{i=1}^{n} (a_i)^c = \left(\sum_{i=1}^{n} a_i \right)^c$

60. THOUGHT PROVOKING In this section, you learned the following formulas.

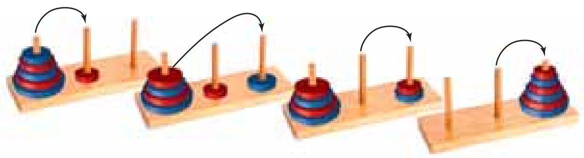
$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}$$

Write a formula for the sum of the cubes of the first n positive integers.

61. MODELING WITH MATHEMATICS In the puzzle called the Tower of Hanoi, the object is to use a series of moves to take the rings from one peg and stack them in order on another peg. A move consists of moving exactly one ring, and no ring may be placed on top of a smaller ring. The minimum number a_n of moves required to move n rings is 1 for 1 ring, 3 for 2 rings, 7 for 3 rings, 15 for 4 rings, and 31 for 5 rings.



Step 1 Step 2 Step 3 . . . End

a. Write a rule for the sequence.

b. What is the minimum number of moves required to move 6 rings? 7 rings? 8 rings?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the system. Check your solution. *(Section 1.4)*

62. $2x - y - 3z = 6$
$x + y + 4z = -1$
$3x - 2z = 8$

63. $2x - 2y + z = 5$
$-2x + 3y + 2z = -1$
$x - 4y + 5z = 4$

64. $2x - 3y + z = 4$
$x - 2z = 1$
$y + z = 2$

8.2 Analyzing Arithmetic Sequences and Series

Essential Question

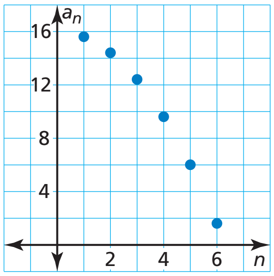
How can you recognize an arithmetic sequence from its graph?

In an **arithmetic sequence**, the difference of consecutive terms, called the *common difference*, is constant. For example, in the arithmetic sequence 1, 4, 7, 10, . . . , the common difference is 3.

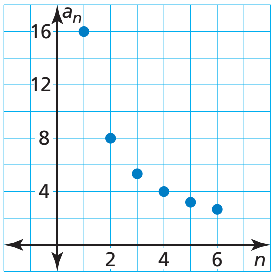
EXPLORATION 1 Recognizing Graphs of Arithmetic Sequences

Work with a partner. Determine whether each graph shows an arithmetic sequence. If it does, then write a rule for the nth term of the sequence, and use a spreadsheet to find the sum of the first 20 terms. What do you notice about the graph of an arithmetic sequence?

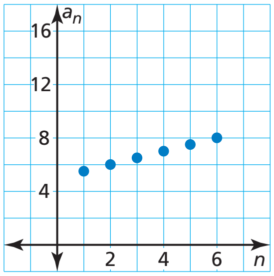
a.



b.



c.



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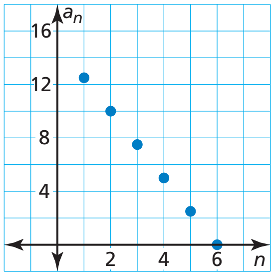

EXPLORATION 2 Finding the Sum of an Arithmetic Sequence

Work with a partner. A teacher of German mathematician Carl Friedrich Gauss (1777–1855) asked him to find the sum of all the whole numbers from 1 through 100. To the astonishment of his teacher, Gauss came up with the answer after only a few moments. Here is what Gauss did:

$$\begin{array}{rcrcrcccr}
1 & + & 2 & + & 3 & + & \cdots & + & 100 \\
100 & + & 99 & + & 98 & + & \cdots & + & 1 \\
\hline
101 & + & 101 & + & 101 & + & \cdots & + & 101
\end{array}
\qquad \frac{100 \times 101}{2} = 5050$$

Explain Gauss's thought process. Then write a formula for the sum S_n of the first n terms of an arithmetic sequence. Verify your formula by finding the sums of the first 20 terms of the arithmetic sequences in Exploration 1. Compare your answers to those you obtained using a spreadsheet.

REASONING ABSTRACTLY

To be proficient in math, you need to make sense of quantities and their relationships in problem situations.

Communicate Your Answer

3. How can you recognize an arithmetic sequence from its graph?

4. Find the sum of the terms of each arithmetic sequence.

a. 1, 4, 7, 10, . . . , 301 **b.** 1, 2, 3, 4, . . . , 1000 **c.** 2, 4, 6, 8, . . . , 800

8.2 Lesson

Core Vocabulary

arithmetic sequence, *p. 418*
common difference, *p. 418*
arithmetic series, *p. 420*

Previous
linear function
mean

What You Will Learn

- Identify arithmetic sequences.
- Write rules for arithmetic sequences.
- Find sums of finite arithmetic series.

Identifying Arithmetic Sequences

In an **arithmetic sequence**, the difference of consecutive terms is constant. This constant difference is called the **common difference** and is denoted by d.

EXAMPLE 1 Identifying Arithmetic Sequences

Tell whether each sequence is arithmetic.

a. $-9, -2, 5, 12, 19, \ldots$ **b.** $23, 15, 9, 5, 3, \ldots$

SOLUTION

Find the differences of consecutive terms.

a. $a_2 - a_1 = -2 - (-9) = 7$

$a_3 - a_2 = 5 - (-2) = 7$

$a_4 - a_3 = 12 - 5 = 7$

$a_5 - a_4 = 19 - 12 = 7$

▶ Each difference is 7, so the sequence is arithmetic.

b. $a_2 - a_1 = 15 - 23 = -8$

$a_3 - a_2 = 9 - 15 = -6$

$a_4 - a_3 = 5 - 9 = -4$

$a_5 - a_4 = 3 - 5 = -2$

▶ The differences are not constant, so the sequence is not arithmetic.

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Tell whether the sequence is arithmetic. Explain your reasoning.

1. $2, 5, 8, 11, 14, \ldots$ **2.** $15, 9, 3, -3, -9, \ldots$ **3.** $8, 4, 2, 1, \frac{1}{2}, \ldots$

Writing Rules for Arithmetic Sequences

Core Concept

Rule for an Arithmetic Sequence

Algebra The nth term of an arithmetic sequence with first term a_1 and common difference d is given by:

$$a_n = a_1 + (n - 1)d$$

Example The nth term of an arithmetic sequence with a first term of 3 and a common difference of 2 is given by:

$$a_n = 3 + (n - 1)2, \text{ or } a_n = 2n + 1$$

EXAMPLE 2 Writing a Rule for the nth Term

Write a rule for the nth term of each sequence. Then find a_{15}.

a. 3, 8, 13, 18, . . . **b.** 55, 47, 39, 31, . . .

COMMON ERROR

In the general rule for an arithmetic sequence, note that the common difference d is multiplied by $n - 1$, not n.

SOLUTION

a. The sequence is arithmetic with first term $a_1 = 3$, and common difference $d = 8 - 3 = 5$. So, a rule for the nth term is

$$\begin{aligned} a_n &= a_1 + (n-1)d && \text{Write general rule.} \\ &= 3 + (n-1)5 && \text{Substitute 3 for } a_1 \text{ and 5 for } d. \\ &= 5n - 2. && \text{Simplify.} \end{aligned}$$

▶ A rule is $a_n = 5n - 2$, and the 15th term is $a_{15} = 5(15) - 2 = 73$.

b. The sequence is arithmetic with first term $a_1 = 55$, and common difference $d = 47 - 55 = -8$. So, a rule for the nth term is

$$\begin{aligned} a_n &= a_1 + (n-1)d && \text{Write general rule.} \\ &= 55 + (n-1)(-8) && \text{Substitute 55 for } a_1 \text{ and } -8 \text{ for } d. \\ &= -8n + 63. && \text{Simplify.} \end{aligned}$$

▶ A rule is $a_n = -8n + 63$, and the 15th term is $a_{15} = -8(15) + 63 = -57$.

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4. Write a rule for the nth term of the sequence 7, 11, 15, 19, ... Then find a_{15}.

EXAMPLE 3 Writing a Rule Given a Term and Common Difference

One term of an arithmetic sequence is $a_{19} = -45$. The common difference is $d = -3$. Write a rule for the nth term. Then graph the first six terms of the sequence.

SOLUTION

Step 1 Use the general rule to find the first term.

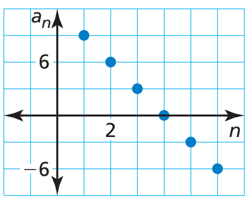
$$\begin{aligned} a_n &= a_1 + (n-1)d && \text{Write general rule.} \\ a_{19} &= a_1 + (19-1)d && \text{Substitute 19 for } n. \\ -45 &= a_1 + 18(-3) && \text{Substitute } -45 \text{ for } a_{19} \text{ and } -3 \text{ for } d. \\ 9 &= a_1 && \text{Solve for } a_1. \end{aligned}$$

Step 2 Write a rule for the nth term.

$$\begin{aligned} a_n &= a_1 + (n-1)d && \text{Write general rule.} \\ &= 9 + (n-1)(-3) && \text{Substitute 9 for } a_1 \text{ and } -3 \text{ for } d. \\ &= -3n + 12 && \text{Simplify.} \end{aligned}$$

Step 3 Use the rule to create a table of values for the sequence. Then plot the points.

n	1	2	3	4	5	6
a_n	9	6	3	0	−3	−6



ANALYZING RELATIONSHIPS

Notice that the points lie on a line. This is true for any arithmetic sequence. So, an arithmetic sequence is a linear function whose domain is a subset of the integers. You can also use function notation to write sequences:

$$f(n) = -3n + 12.$$

EXAMPLE 4 **Writing a Rule Given Two Terms**

Two terms of an arithmetic sequence are $a_7 = 17$ and $a_{26} = 93$. Write a rule for the nth term.

SOLUTION

Step 1 Write a system of equations using $a_n = a_1 + (n - 1)d$. Substitute 26 for n to write Equation 1. Substitute 7 for n to write Equation 2.

$a_{26} = a_1 + (26 - 1)d$	➡	$93 = a_1 + 25d$	Equation 1
$a_7 = a_1 + (7 - 1)d$	➡	$17 = a_1 + 6d$	Equation 2

Step 2 Solve the system.

$76 = 19d$	Subtract.
$4 = d$	Solve for d.
$93 = a_1 + 25(4)$	Substitute for d in Equation 1.
$-7 = a_1$	Solve for a_1.

Step 3 Write a rule for a_n.

$a_n = a_1 + (n - 1)d$	Write general rule.
$= -7 + (n - 1)4$	Substitute for a_1 and d.
$= 4n - 11$	Simplify.

Check

Use the rule to verify that the 7th term is 17 and the 26th term is 93.

$a_7 = 4(7) - 11 = 17$ ✓

$a_{26} = 4(26) - 11 = 93$ ✓

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Write a rule for the *n*th term of the sequence. Then graph the first six terms of the sequence.

5. $a_{11} = 50, d = 7$

6. $a_7 = 71, a_{16} = 26$

Finding Sums of Finite Arithmetic Series

The expression formed by adding the terms of an arithmetic sequence is called an **arithmetic series**. The sum of the first n terms of an arithmetic series is denoted by S_n. To find a rule for S_n, you can write S_n in two different ways and add the results.

$$\begin{aligned} S_n &= a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + a_n \\ S_n &= a_n + (a_n - d) + (a_n - 2d) + \cdots + a_1 \\ \hline 2S_n &= \underbrace{(a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n)}_{(a_1 + a_n) \text{ is added } n \text{ times.}} \end{aligned}$$

You can conclude that $2S_n = n(a_1 + a_n)$, which leads to the following result.

Core Concept

The Sum of a Finite Arithmetic Series

The sum of the first n terms of an arithmetic series is

$$S_n = n\left(\frac{a_1 + a_n}{2}\right).$$

In words, S_n is the mean of the first and nth terms, multiplied by the number of terms.

EXAMPLE 5 Finding the Sum of an Arithmetic Series

Find the sum $\sum_{i=1}^{20}(3i + 7)$.

SOLUTION

Step 1 Find the first and last terms.

$$a_1 = 3(1) + 7 = 10 \qquad \text{Identify first term.}$$

$$a_{20} = 3(20) + 7 = 67 \qquad \text{Identify last term.}$$

Step 2 Find the sum.

$$S_{20} = 20\left(\frac{a_1 + a_{20}}{2}\right) \qquad \text{Write rule for } S_{20}.$$

$$= 20\left(\frac{10 + 67}{2}\right) \qquad \text{Substitute 10 for } a_1 \text{ and 67 for } a_{20}.$$

$$= 770 \qquad \text{Simplify.}$$

STUDY TIP

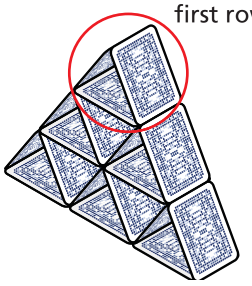
This sum is actually a *partial* sum. You cannot find the complete sum of an infinite arithmetic series because its terms continue indefinitely.

EXAMPLE 6 Solving a Real-Life Problem

You are making a house of cards similar to the one shown.

a. Write a rule for the number of cards in the nth row when the top row is row 1.

b. How many cards do you need to make a house of cards with 12 rows?



SOLUTION

a. Starting with the top row, the number of cards in the rows are 3, 6, 9, 12, ... These numbers form an arithmetic sequence with a first term of 3 and a common difference of 3. So, a rule for the sequence is:

$$a_n = a_1 + (n - 1)d \qquad \text{Write general rule.}$$

$$= 3 + (n - 1)(3) \qquad \text{Substitute 3 for } a_1 \text{ and 3 for } d.$$

$$= 3n \qquad \text{Simplify.}$$

b. Find the sum of an arithmetic series with first term $a_1 = 3$ and last term $a_{12} = 3(12) = 36$.

$$S_{12} = 12\left(\frac{a_1 + a_{12}}{2}\right) = 12\left(\frac{3 + 36}{2}\right) = 234$$

So, you need 234 cards to make a house of cards with 12 rows.

Check

Use a graphing calculator to check the sum.

```
sum(seq(3X,X,1,1
2))
                 234
```

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Find the sum.

7. $\sum_{i=1}^{10} 9i$

8. $\sum_{k=1}^{12}(7k + 2)$

9. $\sum_{n=1}^{20}(-4n + 6)$

10. **WHAT IF?** In Example 6, how many cards do you need to make a house of cards with eight rows?

8.2 Exercises

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Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The constant difference between consecutive terms of an arithmetic sequence is called the ________________.

2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

What sequence consists of all the positive odd numbers?
What sequence starts with 1 and has a common difference of 2?
What sequence has an nth term of $a_n = 1 + (n - 1)2$?
What sequence has an nth term of $a_n = 2n + 1$?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, tell whether the sequence is arithmetic. Explain your reasoning. *(See Example 1.)*

3. $1, -1, -3, -5, -7, \ldots$
4. $12, 6, 0, -6, -12, \ldots$
5. $5, 8, 13, 20, 29, \ldots$
6. $3, 5, 9, 15, 23, \ldots$
7. $36, 18, 9, \frac{9}{2}, \frac{9}{4}, \ldots$
8. $81, 27, 9, 3, 1, \ldots$
9. $\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \ldots$
10. $\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \ldots$

11. **WRITING EQUATIONS** Write a rule for the arithmetic sequence with the given description.
    a. The first term is -3 and each term is 6 less than the previous term.
    b. The first term is 7 and each term is 5 more than the previous term.

12. **WRITING** Compare the terms of an arithmetic sequence when $d > 0$ to when $d < 0$.

In Exercises 13–20, write a rule for the nth term of the sequence. Then find a_{20}. *(See Example 2.)*

13. $12, 20, 28, 36, \ldots$
14. $7, 12, 17, 22, \ldots$
15. $51, 48, 45, 42, \ldots$
16. $86, 79, 72, 65, \ldots$
17. $-1, -\frac{1}{3}, \frac{1}{3}, 1, \ldots$
18. $-2, -\frac{5}{4}, -\frac{1}{2}, \frac{1}{4}, \ldots$
19. $2.3, 1.5, 0.7, -0.1, \ldots$
20. $11.7, 10.8, 9.9, 9, \ldots$

ERROR ANALYSIS **In Exercises 21 and 22, describe and correct the error in writing a rule for the nth term of the arithmetic sequence $22, 9, -4, -17, -30, \ldots$.**

21. ✗
Use $a_1 = 22$ and $d = -13$.
$a_n = a_1 + nd$
$a_n = 22 + n(-13)$
$a_n = 22 - 13n$

22. ✗
The first term is 22 and the common difference is -13.
$a_n = -13 + (n - 1)(22)$
$a_n = -35 + 22n$

In Exercises 23–28, write a rule for the nth term of the sequence. Then graph the first six terms of the sequence. *(See Example 3.)*

23. $a_{11} = 43, d = 5$
24. $a_{13} = 42, d = 4$
25. $a_{20} = -27, d = -2$
26. $a_{15} = -35, d = -3$
27. $a_{17} = -5, d = -\frac{1}{2}$
28. $a_{21} = -25, d = -\frac{3}{2}$

29. **USING EQUATIONS** One term of an arithmetic sequence is $a_8 = -13$. The common difference is -8. What is a rule for the nth term of the sequence?
    Ⓐ $a_n = 51 + 8n$
    Ⓑ $a_n = 35 + 8n$
    Ⓒ $a_n = 51 - 8n$
    Ⓓ $a_n = 35 - 8n$

30. **FINDING A PATTERN** One term of an arithmetic sequence is $a_{12} = 43$. The common difference is 6. What is another term of the sequence?

Ⓐ $a_3 = -11$ Ⓑ $a_4 = -53$

Ⓒ $a_5 = 13$ Ⓓ $a_6 = -47$

In Exercises 31–38, write a rule for the *n*th term of the arithmetic sequence. *(See Example 4.)*

31. $a_5 = 41, a_{10} = 96$

32. $a_7 = 58, a_{11} = 94$

33. $a_6 = -8, a_{15} = -62$

34. $a_8 = -15, a_{17} = -78$

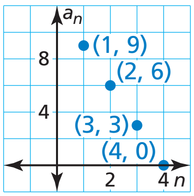
35. $a_{18} = -59, a_{21} = -71$

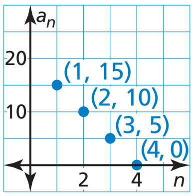
36. $a_{12} = -38, a_{19} = -73$

37. $a_8 = 12, a_{16} = 22$

38. $a_{12} = 9, a_{27} = 15$

WRITING EQUATIONS **In Exercises 39–44, write a rule for the sequence with the given terms.**

39.


40.


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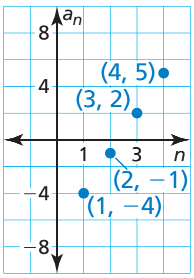
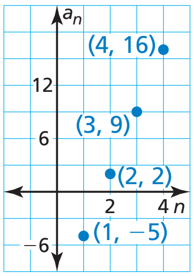


42.


43.

n	4	5	6	7	8
a_n	25	29	33	37	41

44.

n	4	5	6	7	8
a_n	31	39	47	55	63

45. **WRITING** Compare the graph of $a_n = 3n + 1$, where n is a positive integer, with the graph of $f(x) = 3x + 1$, where x is a real number.

46. **DRAWING CONCLUSIONS** Describe how doubling each term in an arithmetic sequence changes the common difference of the sequence. Justify your answer.

In Exercises 47–52, find the sum. *(See Example 5.)*

47. $\sum_{i=1}^{20}(2i - 3)$ **48.** $\sum_{i=1}^{26}(4i + 7)$

49. $\sum_{i=1}^{33}(6 - 2i)$ **50.** $\sum_{i=1}^{31}(-3 - 4i)$

51. $\sum_{i=1}^{41}(-2.3 + 0.1i)$ **52.** $\sum_{i=1}^{39}(-4.1 + 0.4i)$

NUMBER SENSE **In Exercises 53 and 54, find the sum of the arithmetic sequence.**

53. The first 19 terms of the sequence 9, 2, −5, −12, ...

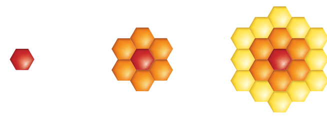
54. The first 22 terms of the sequence 17, 9, 1, −7, ...

55. **MODELING WITH MATHEMATICS** A marching band is arranged in rows. The first row has three band members, and each row after the first has two more band members than the row before it. *(See Example 6.)*

a. Write a rule for the number of band members in the *n*th row.

b. How many band members are in a formation with seven rows?

56. **MODELING WITH MATHEMATICS** Domestic bees make their honeycomb by starting with a single hexagonal cell, then forming ring after ring of hexagonal cells around the initial cell, as shown. The number of cells in successive rings forms an arithmetic sequence.

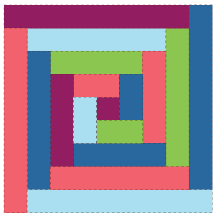


Initial cell 1 ring 2 rings

a. Write a rule for the number of cells in the *n*th ring.

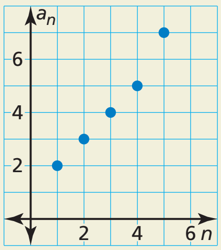
b. How many cells are in the honeycomb after the ninth ring is formed?

57. MATHEMATICAL CONNECTIONS A quilt is made up of strips of cloth, starting with an inner square surrounded by rectangles to form successively larger squares. The inner square and all rectangles have a width of 1 foot. Write an expression using summation notation that gives the sum of the areas of all the strips of cloth used to make the quilt shown. Then evaluate the expression.

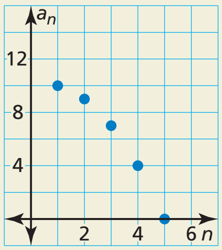


58. HOW DO YOU SEE IT? Which graph(s) represents an arithmetic sequence? Explain your reasoning.

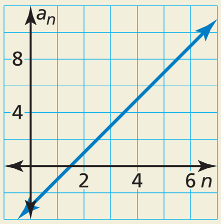
a.



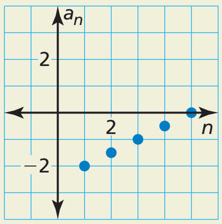
b.



c.



d.



59. MAKING AN ARGUMENT Your friend believes the sum of a series doubles when the common difference of an arithmetic series is doubled and the first term and number of terms in the series remain unchanged. Is your friend correct? Explain your reasoning.

60. THOUGHT PROVOKING In number theory, the *Dirichlet Prime Number Theorem* states that if a and b are relatively prime, then the arithmetic sequence

$$a, a + b, a + 2b, a + 3b, \ldots$$

contains infinitely many prime numbers. Find the first 10 primes in the sequence when $a = 3$ and $b = 4$.

61. REASONING Find the sum of the positive odd integers less than 300. Explain your reasoning.

62. USING EQUATIONS Find the value of n.

a. $\sum_{i=1}^{n}(3i + 5) = 544$ **b.** $\sum_{i=1}^{n}(-4i - 1) = -1127$

c. $\sum_{i=5}^{n}(7 + 12i) = 455$ **d.** $\sum_{i=3}^{n}(-3 - 4i) = -507$

63. ABSTRACT REASONING A theater has n rows of seats, and each row has d more seats than the row in front of it. There are x seats in the last (nth) row and a total of y seats in the entire theater. How many seats are in the front row of the theater? Write your answer in terms of n, x, and y.

64. CRITICAL THINKING The expressions $3 - x$, x, and $1 - 3x$ are the first three terms in an arithmetic sequence. Find the value of x and the next term in the sequence.

65. CRITICAL THINKING One of the major sources of our knowledge of Egyptian mathematics is the Ahmes papyrus, which is a scroll copied in 1650 B.C. by an Egyptian scribe. The following problem is from the Ahmes papyrus.

Divide 10 hekats of barley among 10 men so that the common difference is $\frac{1}{8}$ of a hekat of barley.

Use what you know about arithmetic sequences and series to determine what portion of a hekat each man should receive.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the expression. *(Section 5.2)*

66. $\dfrac{7}{7^{1/3}}$

67. $\dfrac{3^{-2}}{3^{-4}}$

68. $\left(\dfrac{9}{49}\right)^{1/2}$

69. $(5^{1/2} \cdot 5^{1/4})$

Tell whether the function represents *exponential growth* or *exponential decay*. Then graph the function. *(Section 6.2)*

70. $y = 2e^x$

71. $y = e^{-3x}$

72. $y = 3e^{-x}$

73. $y = e^{0.25x}$

8.3 Analyzing Geometric Sequences and Series

Essential Question

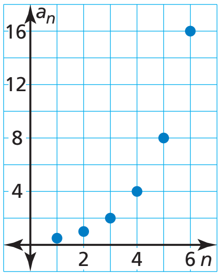
How can you recognize a geometric sequence from its graph?

In a **geometric sequence**, the ratio of any term to the previous term, called the *common ratio*, is constant. For example, in the geometric sequence 1, 2, 4, 8, . . . , the common ratio is 2.

EXPLORATION 1 Recognizing Graphs of Geometric Sequences

Work with a partner. Determine whether each graph shows a geometric sequence. If it does, then write a rule for the nth term of the sequence and use a spreadsheet to find the sum of the first 20 terms. What do you notice about the graph of a geometric sequence?

a.



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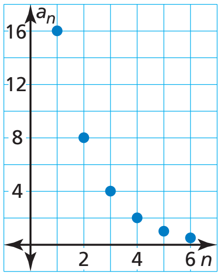

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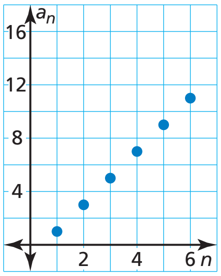

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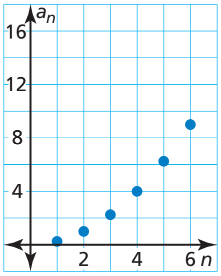

EXPLORATION 2 Finding the Sum of a Geometric Sequence

Work with a partner. You can write the nth term of a geometric sequence with first term a_1 and common ratio r as

$$a_n = a_1 r^{n-1}.$$

So, you can write the sum S_n of the first n terms of a geometric sequence as

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1}.$$

Rewrite this formula by finding the difference $S_n - rS_n$ and solving for S_n. Then verify your rewritten formula by finding the sums of the first 20 terms of the geometric sequences in Exploration 1. Compare your answers to those you obtained using a spreadsheet.

LOOKING FOR REGULARITY IN REPEATED REASONING

To be proficient in math, you need to notice when calculations are repeated, and look both for general methods and for shortcuts.

Communicate Your Answer

3. How can you recognize a geometric sequence from its graph?

4. Find the sum of the terms of each geometric sequence.

a. 1, 2, 4, 8, . . . , 8192 **b.** 0.1, 0.01, 0.001, 0.0001, . . . , 10^{-10}

8.3 Lesson

Core Vocabulary

geometric sequence, *p. 426*
common ratio, *p. 426*
geometric series, *p. 428*

Previous
exponential function
properties of exponents

What You Will Learn

- Identify geometric sequences.
- Write rules for geometric sequences.
- Find sums of finite geometric series.

Identifying Geometric Sequences

In a **geometric sequence**, the ratio of any term to the previous term is constant. This constant ratio is called the **common ratio** and is denoted by r.

EXAMPLE 1 Identifying Geometric Sequences

Tell whether each sequence is geometric.

a. 6, 12, 20, 30, 42, . . .

b. 256, 64, 16, 4, 1, . . .

SOLUTION

Find the ratios of consecutive terms.

a. $\frac{a_2}{a_1} = \frac{12}{6} = 2$ $\qquad \frac{a_3}{a_2} = \frac{20}{12} = \frac{5}{3}$ $\qquad \frac{a_4}{a_3} = \frac{30}{20} = \frac{3}{2}$ $\qquad \frac{a_5}{a_4} = \frac{42}{30} = \frac{7}{5}$

- The ratios are not constant, so the sequence is not geometric.

b. $\frac{a_2}{a_1} = \frac{64}{256} = \frac{1}{4}$ $\qquad \frac{a_3}{a_2} = \frac{16}{64} = \frac{1}{4}$ $\qquad \frac{a_4}{a_3} = \frac{4}{16} = \frac{1}{4}$ $\qquad \frac{a_5}{a_4} = \frac{1}{4}$

- Each ratio is $\frac{1}{4}$, so the sequence is geometric.

Monitoring Progress

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Tell whether the sequence is geometric. Explain your reasoning.

1. $27, 9, 3, 1, \frac{1}{3}, \ldots$ **2.** $2, 6, 24, 120, 720, \ldots$ **3.** $-1, 2, -4, 8, -16, \ldots$

Writing Rules for Geometric Sequences

Core Concept

Rule for a Geometric Sequence

Algebra The nth term of a geometric sequence with first term a_1 and common ratio r is given by:

$$a_n = a_1 r^{n-1}$$

Example The nth term of a geometric sequence with a first term of 2 and a common ratio of 3 is given by:

$$a_n = 2(3)^{n-1}$$

EXAMPLE 2 Writing a Rule for the *n*th Term

Write a rule for the *n*th term of each sequence. Then find a_8.

a. 5, 15, 45, 135, . . .   **b.** 88, −44, 22, −11, . . .

COMMON ERROR

In the general rule for a geometric sequence, note that the exponent is $n - 1$, not n.

SOLUTION

a. The sequence is geometric with first term $a_1 = 5$ and common ratio $r = \frac{15}{5} = 3$. So, a rule for the *n*th term is

$$a_n = a_1 r^{n-1} \qquad \text{Write general rule.}$$

$$= 5(3)^{n-1}. \qquad \text{Substitute 5 for } a_1 \text{ and 3 for } r.$$

▶ A rule is $a_n = 5(3)^{n-1}$, and the 8th term is $a_8 = 5(3)^{8-1} = 10{,}935$.

b. The sequence is geometric with first term $a_1 = 88$ and common ratio $r = \frac{-44}{88} = -\frac{1}{2}$. So, a rule for the *n*th term is

$$a_n = a_1 r^{n-1} \qquad \text{Write general rule.}$$

$$= 88\left(-\frac{1}{2}\right)^{n-1}. \qquad \text{Substitute 88 for } a_1 \text{ and } -\frac{1}{2} \text{ for } r.$$

▶ A rule is $a_n = 88\left(-\frac{1}{2}\right)^{n-1}$, and the 8th term is $a_8 = 88\left(-\frac{1}{2}\right)^{8-1} = -\frac{11}{16}$.

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4. Write a rule for the *n*th term of the sequence 3, 15, 75, 375, ... Then find a_9.

EXAMPLE 3 Writing a Rule Given a Term and Common Ratio

One term of a geometric sequence is $a_4 = 12$. The common ratio is $r = 2$. Write a rule for the *n*th term. Then graph the first six terms of the sequence.

SOLUTION

Step 1 Use the general rule to find the first term.

$$a_n = a_1 r^{n-1} \qquad \text{Write general rule.}$$

$$a_4 = a_1 r^{4-1} \qquad \text{Substitute 4 for } n.$$

$$12 = a_1(2)^3 \qquad \text{Substitute 12 for } a_4 \text{ and 2 for } r.$$

$$1.5 = a_1 \qquad \text{Solve for } a_1.$$

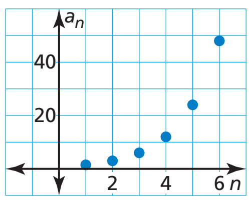
Step 2 Write a rule for the *n*th term.

$$a_n = a_1 r^{n-1} \qquad \text{Write general rule.}$$

$$= 1.5(2)^{n-1} \qquad \text{Substitute 1.5 for } a_1 \text{ and 2 for } r.$$

Step 3 Use the rule to create a table of values for the sequence. Then plot the points.

n	1	2	3	4	5	6
a_n	1.5	3	6	12	24	48



ANALYZING RELATIONSHIPS

Notice that the points lie on an exponential curve because consecutive terms change by equal factors. So, a geometric sequence in which $r > 0$ and $r \neq 1$ is an exponential function whose domain is a subset of the integers.

EXAMPLE 4 Writing a Rule Given Two Terms

Two terms of a geometric sequence are $a_2 = 12$ and $a_5 = -768$. Write a rule for the nth term.

SOLUTION

Step 1 Write a system of equations using $a_n = a_1 r^{n-1}$. Substitute 2 for n to write Equation 1. Substitute 5 for n to write Equation 2.

$a_2 = a_1 r^{2-1}$ ➡ $12 = a_1 r$ Equation 1

$a_5 = a_1 r^{5-1}$ ➡ $-768 = a_1 r^4$ Equation 2

Step 2 Solve the system.

$\frac{12}{r} = a_1$ Solve Equation 1 for a_1.

$-768 = \frac{12}{r}(r^4)$ Substitute for a_1 in Equation 2.

$-768 = 12r^3$ Simplify.

$-4 = r$ Solve for r.

$12 = a_1(-4)$ Substitute for r in Equation 1.

$-3 = a_1$ Solve for a_1.

Step 3 Write a rule for a_n.

$a_n = a_1 r^{n-1}$ Write general rule.

$= -3(-4)^{n-1}$ Substitute for a_1 and r.

Check

Use the rule to verify that the 2nd term is 12 and the 5th term is -768.

$a_2 = -3(-4)^{2-1}$

$= -3(-4) = 12$ ✓

$a_5 = -3(-4)^{5-1}$

$= -3(256) = -768$ ✓

Monitoring Progress

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Write a rule for the nth term of the sequence. Then graph the first six terms of the sequence.

5. $a_6 = -96, r = -2$

6. $a_2 = 12, a_4 = 3$

Finding Sums of Finite Geometric Series

The expression formed by adding the terms of a geometric sequence is called a **geometric series**. The sum of the first n terms of a geometric series is denoted by S_n. You can develop a rule for S_n as follows.

$$\begin{aligned} S_n &= a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1} \\ -rS_n &= \quad\;\; - a_1 r - a_1 r^2 - a_1 r^3 - \cdots - a_1 r^{n-1} - a_1 r^n \\ \hline S_n - rS_n &= a_1 + 0 + 0 + 0 + \cdots + 0 \quad - a_1 r^n \\ S_n(1 - r) &= a_1(1 - r^n) \end{aligned}$$

When $r \neq 1$, you can divide each side of this equation by $1 - r$ to obtain the following rule for S_n.

Core Concept

The Sum of a Finite Geometric Series

The sum of the first n terms of a geometric series with common ratio $r \neq 1$ is

$$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right).$$

EXAMPLE 5 Finding the Sum of a Geometric Series

Find the sum $\sum_{k=1}^{10} 4(3)^{k-1}$.

SOLUTION

Step 1 Find the first term and the common ratio.

$$a_1 = 4(3)^{1-1} = 4 \qquad \text{Identify first term.}$$

$$r = 3 \qquad \text{Identify common ratio.}$$

Step 2 Find the sum.

$$S_{10} = a_1\left(\frac{1-r^{10}}{1-r}\right) \qquad \text{Write rule for } S_{10}.$$

$$= 4\left(\frac{1-3^{10}}{1-3}\right) \qquad \text{Substitute 4 for } a_1 \text{ and 3 for } r.$$

$$= 118{,}096 \qquad \text{Simplify.}$$

Check

Use a graphing calculator to check the sum.

```
sum(seq(4*3^(X-1
),X,1,10))
             118096
```

EXAMPLE 6 Solving a Real-Life Problem

You can calculate the monthly payment M (in dollars) for a loan using the formula

$$M = \frac{L}{\sum_{k=1}^{t}\left(\frac{1}{1+i}\right)^k}$$

where L is the loan amount (in dollars), i is the monthly interest rate (in decimal form), and t is the term (in months). Calculate the monthly payment on a 5-year loan for \$20,000 with an annual interest rate of 6%.

SOLUTION

Step 1 Substitute for L, i, and t. The loan amount is $L = 20{,}000$, the monthly interest rate is $i = \frac{0.06}{12} = 0.005$, and the term is $t = 5(12) = 60$.

$$M = \frac{20{,}000}{\sum_{k=1}^{60}\left(\frac{1}{1+0.005}\right)^k}$$

Step 2 Notice that the denominator is a geometric series with first term $\frac{1}{1.005}$ and common ratio $\frac{1}{1.005}$. Use a calculator to find the monthly payment.

```
1/1.005→R
       .9950248756
R((1-R^60)/(1-R)
)
        51.72556075
20000/Ans
        386.6560306
```

▶ So, the monthly payment is \$386.66.

USING TECHNOLOGY

Storing the value of $\frac{1}{1.005}$ helps minimize mistakes and also assures an accurate answer. Rounding this value to 0.995 results in a monthly payment of \$386.94.

Monitoring Progress

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Find the sum.

7. $\sum_{k=1}^{8} 5^{k-1}$

8. $\sum_{i=1}^{12} 6(-2)^{i-1}$

9. $\sum_{t=1}^{7} -16(0.5)^{t-1}$

10. **WHAT IF?** In Example 6, how does the monthly payment change when the annual interest rate is 5%?

8.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The constant ratio of consecutive terms in a geometric sequence is called the ____________.

2. **WRITING** How can you determine whether a sequence is geometric from its graph?

3. **COMPLETE THE SENTENCE** The nth term of a geometric sequence has the form $a_n =$ ____________.

4. **VOCABULARY** State the rule for the sum of the first n terms of a geometric series.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, tell whether the sequence is geometric. Explain your reasoning. *(See Example 1.)*

5. 96, 48, 24, 12, 6, . . .

6. 729, 243, 81, 27, 9, . . .

7. 2, 4, 6, 8, 10, . . .

8. 5, 20, 35, 50, 65, . . .

9. 0.2, 3.2, −12.8, 51.2, −204.8, . . .

10. 0.3, −1.5, 7.5, −37.5, 187.5, . . .

11. $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \frac{1}{162}, \ldots$

12. $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \frac{1}{1024}, \ldots$

13. **WRITING EQUATIONS** Write a rule for the geometric sequence with the given description.

    a. The first term is −3, and each term is 5 times the previous term.

    b. The first term is 72, and each term is $\frac{1}{3}$ times the previous term.

14. **WRITING** Compare the terms of a geometric sequence when $r > 1$ to when $0 < r < 1$.

In Exercises 15–22, write a rule for the nth term of the sequence. Then find a_7. *(See Example 2.)*

15. 4, 20, 100, 500, . . .

16. 6, 24, 96, 384, . . .

17. 112, 56, 28, 14, . . .

18. 375, 75, 15, 3, . . .

19. $4, 6, 9, \frac{27}{2}, \ldots$

20. $2, \frac{3}{2}, \frac{9}{8}, \frac{27}{32}, \ldots$

21. 1.3, −3.9, 11.7, −35.1, . . .

22. 1.5, −7.5, 37.5, −187.5, . . .

In Exercises 23–30, write a rule for the nth term of the sequence. Then graph the first six terms of the sequence. *(See Example 3.)*

23. $a_3 = 4, r = 2$

24. $a_3 = 27, r = 3$

25. $a_2 = 30, r = \frac{1}{2}$

26. $a_2 = 64, r = \frac{1}{4}$

27. $a_4 = -192, r = 4$

28. $a_4 = -500, r = 5$

29. $a_5 = 3, r = -\frac{1}{3}$

30. $a_5 = 1, r = -\frac{1}{5}$

ERROR ANALYSIS In Exercises 31 and 32, describe and correct the error in writing a rule for the nth term of the geometric sequence for which $a_2 = 48$ and $r = 6$.

31.
$$a_n = a_1 r^n$$
$$48 = a_1 6^2$$
$$\frac{4}{3} = a_1$$
$$a_n = \frac{4}{3}(6)^n$$

32.
$$a_n = r(a_1)^{n-1}$$
$$48 = 6(a_1)^{2-1}$$
$$8 = a_1$$
$$a_n = 6(8)^{n-1}$$

In Exercises 33–40, write a rule for the nth term of the geometric sequence. *(See Example 4.)*

33. $a_2 = 28, a_5 = 1792$

34. $a_1 = 11, a_4 = 88$

35. $a_1 = -6, a_5 = -486$

36. $a_2 = -10, a_6 = -6250$

37. $a_2 = 64, a_4 = 1$

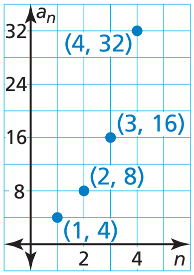
38. $a_1 = 1, a_2 = 49$

39. $a_2 = -72, a_6 = -\frac{1}{18}$

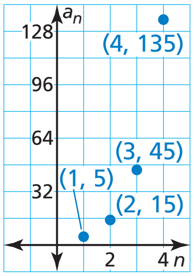
40. $a_2 = -48, a_5 = \frac{3}{4}$

WRITING EQUATIONS **In Exercises 41–46, write a rule for the sequence with the given terms.**

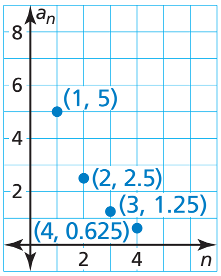
41.



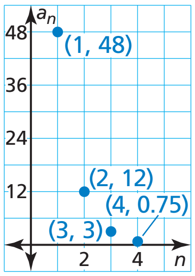
42.



43.



44.



45.

n	2	3	4	5	6
a_n	−12	24	−48	96	−192

46.

n	2	3	4	5	6
a_n	−21	63	−189	567	−1701

In Exercises 47–52, find the sum. *(See Example 5.)*

47. $\sum_{i=1}^{9} 6(7)^{i-1}$

48. $\sum_{i=1}^{10} 7(4)^{i-1}$

49. $\sum_{i=1}^{10} 4\left(\frac{3}{4}\right)^{i-1}$

50. $\sum_{i=1}^{8} 5\left(\frac{1}{3}\right)^{i-1}$

51. $\sum_{i=0}^{8} 8\left(-\frac{2}{3}\right)^{i}$

52. $\sum_{i=0}^{9} 9\left(-\frac{3}{4}\right)^{i}$

NUMBER SENSE **In Exercises 53 and 54, find the sum.**

53. The first 8 terms of the geometric sequence $-12, -48, -192, -768, \ldots$.

54. The first 9 terms of the geometric sequence $-14, -42, -126, -378, \ldots$.

55. WRITING Compare the graph of $a_n = 5(3)^{n-1}$, where n is a positive integer, to the graph of $f(x) = 5 \cdot 3^{x-1}$, where x is a real number.

56. ABSTRACT REASONING Use the rule for the sum of a finite geometric series to write each polynomial as a rational expression.

a. $1 + x + x^2 + x^3 + x^4$

b. $3x + 6x^3 + 12x^5 + 24x^7$

MODELING WITH MATHEMATICS **In Exercises 57 and 58, use the monthly payment formula given in Example 6.**

57. You are buying a new car. You take out a 5-year loan for \$15,000. The annual interest rate of the loan is 4%. Calculate the monthly payment. *(See Example 6.)*

58. You are buying a new house. You take out a 30-year mortgage for \$200,000. The annual interest rate of the loan is 4.5%. Calculate the monthly payment.

59. MODELING WITH MATHEMATICS A regional soccer tournament has 64 participating teams. In the first round of the tournament, 32 games are played. In each successive round, the number of games decreases by a factor of $\frac{1}{2}$.

a. Write a rule for the number of games played in the nth round. For what values of n does the rule make sense? Explain.

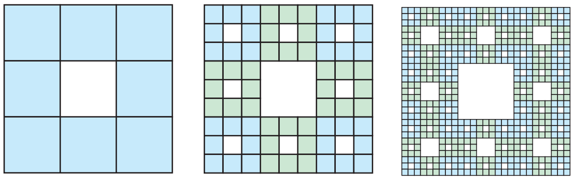
b. Find the total number of games played in the regional soccer tournament.

60. MODELING WITH MATHEMATICS In a skydiving formation with R rings, each ring after the first has twice as many skydivers as the preceding ring. The formation for $R = 2$ is shown.

a. Let a_n be the number of skydivers in the nth ring. Write a rule for a_n.

b. Find the total number of skydivers when there are four rings.

61. **PROBLEM SOLVING** The *Sierpinski carpet* is a fractal created using squares. The process involves removing smaller squares from larger squares. First, divide a large square into nine congruent squares. Then remove the center square. Repeat these steps for each smaller square, as shown below. Assume that each side of the initial square is 1 unit long.



Stage 1 Stage 2 Stage 3

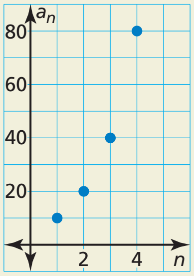
a. Let a_n be the total number of squares removed at the nth stage. Write a rule for a_n. Then find the total number of squares removed through Stage 8.

b. Let b_n be the remaining area of the original square after the nth stage. Write a rule for b_n. Then find the remaining area of the original square after Stage 12.

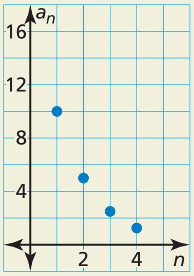
62. **HOW DO YOU SEE IT?** Match each sequence with its graph. Explain your reasoning.

a. $a_n = 10\left(\frac{1}{2}\right)^{n-1}$ **b.** $a_n = 10(2)^{n-1}$

A.

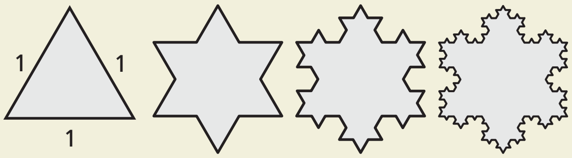


B.



63. **CRITICAL THINKING** On January 1, you deposit $2000 in a retirement account that pays 5% annual interest. You make this deposit each January 1 for the next 30 years. How much money do you have in your account immediately after you make your last deposit?

64. **THOUGHT PROVOKING** The first four iterations of the fractal called the *Koch snowflake* are shown below. Find the perimeter and area of each iteration. Do the perimeters and areas form geometric sequences? Explain your reasoning.



65. **MAKING AN ARGUMENT** You and your friend are comparing two loan options for a $165,000 house. Loan 1 is a 15-year loan with an annual interest rate of 3%. Loan 2 is a 30-year loan with an annual interest rate of 4%. Your friend claims the total amount repaid over the loan will be less for Loan 2. Is your friend correct? Justify your answer.

66. **CRITICAL THINKING** Let L be the amount of a loan (in dollars), i be the monthly interest rate (in decimal form), t be the term (in months), and M be the monthly payment (in dollars).

a. When making monthly payments, you are paying the loan amount plus the interest the loan gathers each month. For a 1-month loan, $t = 1$, the equation for repayment is $L(1 + i) - M = 0$. For a 2-month loan, $t = 2$, the equation is $[L(1 + i) - M](1 + i) - M = 0$. Solve both of these repayment equations for L.

b. Use the pattern in the equations you solved in part (a) to write a repayment equation for a t-month loan. (*Hint*: L is equal to M times a geometric series.) Then solve the equation for M.

c. Use the rule for the sum of a finite geometric series to show that the formula in part (b) is equivalent to

$$M = L\left(\frac{i}{1 - (1 + i)^{-t}}\right).$$

Use this formula to check your answers in Exercises 57 and 58.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Graph the function. State the domain and range. *(Section 7.2)*

67. $f(x) = \dfrac{1}{x - 3}$

68. $g(x) = \dfrac{2}{x} + 3$

69. $h(x) = \dfrac{1}{x - 2} + 1$

70. $p(x) = \dfrac{3}{x + 1} - 2$

8.1–8.3 What Did You Learn?

Core Vocabulary

sequence, *p. 410*
terms of a sequence, *p. 410*
series, *p. 412*
summation notation, *p. 412*
sigma notation, *p. 412*
arithmetic sequence, *p. 418*
common difference, *p. 418*
arithmetic series, *p. 420*
geometric sequence, *p. 426*
common ratio, *p. 426*
geometric series, *p. 428*

Core Concepts

Section 8.1

Sequences, *p. 410*
Series and Summation Notation, *p. 412*
Formulas for Special Series, *p. 413*

Section 8.2

Rule for an Arithmetic Sequence, *p. 418*
The Sum of a Finite Arithmetic Series, *p. 420*

Section 8.3

Rule for a Geometric Sequence, *p. 426*
The Sum of a Finite Geometric Series, *p. 428*

Mathematical Practices

1. Explain how viewing each arrangement as individual tables can be helpful in Exercise 29 on page 415.
2. How can you use tools to find the sum of the arithmetic series in Exercises 53 and 54 on page 423?
3. How did understanding the domain of each function help you to compare the graphs in Exercise 55 on page 431?

Study Skills

Keeping Your Mind Focused

- Before doing homework, review the concept boxes and examples. Talk through the examples out loud.
- Complete homework as though you were also preparing for a quiz. Memorize the different types of problems, formulas, rules, and so on.

8.1–8.3 Quiz

Describe the pattern, write the next term, and write a rule for the *n*th term of the sequence. *(Section 8.1)*

1. 1, 7, 13, 19, . . .

2. $-5, 10, -15, 20, \ldots$

3. $\frac{1}{20}, \frac{2}{30}, \frac{3}{40}, \frac{4}{50}, \ldots$

Write the series using summation notation. Then find the sum of the series. *(Section 8.1)*

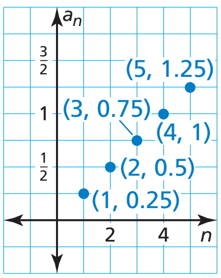
4. $1 + 2 + 3 + 4 + \cdots + 15$

5. $0 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{7}{8}$

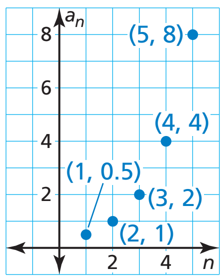
6. $9 + 16 + 25 + \cdots + 100$

Write a rule for the *n*th term of the sequence. *(Sections 8.2 and 8.3)*

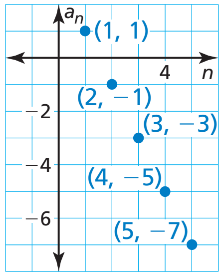
7.



8.



9.



Tell whether the sequence is *arithmetic*, *geometric*, or *neither*. Write a rule for the *n*th term of the sequence. Then find a_9. *(Sections 8.2 and 8.3)*

10. $13, 6, -1, -8, \ldots$

11. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

12. $1, -3, 9, -27, \ldots$

13. One term of an arithmetic sequence is $a_{12} = 19$. The common difference is $d = 7$. Write a rule for the *n*th term. Then graph the first six terms of the sequence. *(Section 8.2)*

14. Two terms of a geometric sequence are $a_6 = -50$ and $a_9 = -6250$. Write a rule for the *n*th term. *(Section 8.3)*

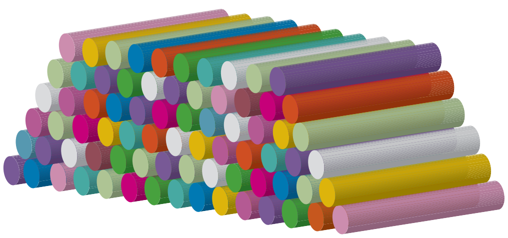
Find the sum. *(Sections 8.2 and 8.3)*

15. $\sum_{n=1}^{9} (3n + 5)$

16. $\sum_{k=1}^{5} 11(-3)^{k-2}$

17. $\sum_{i=1}^{12} -4\left(\frac{1}{2}\right)^{i+3}$

18. Pieces of chalk are stacked in a pile. Part of the pile is shown. The bottom row has 15 pieces of chalk, and the top row has 6 pieces of chalk. Each row has one less piece of chalk than the row below it. How many pieces of chalk are in the pile? *(Section 8.2)*



19. You accept a job as an environmental engineer that pays a salary of \$45,000 in the first year. After the first year, your salary increases by 3.5% per year. *(Section 8.3)*

a. Write a rule giving your salary a_n for your *n*th year of employment.

b. What will your salary be during your fifth year of employment?

c. You work 10 years for the company. What are your total earnings? Justify your answer.

8.4 Finding Sums of Infinite Geometric Series

Essential Question How can you find the sum of an infinite geometric series?

EXPLORATION 1 **Finding Sums of Infinite Geometric Series**

USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technological tools, such as a spreadsheet, to explore and deepen your understanding of concepts.

Work with a partner. Enter each geometric series in a spreadsheet. Then use the spreadsheet to determine whether the infinite geometric series has a finite sum. If it does, find the sum. Explain your reasoning. (The figure shows a partially completed spreadsheet for part (a).)

a. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

b. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$

c. $1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \frac{81}{16} + \cdots$

d. $1 + \frac{5}{4} + \frac{25}{16} + \frac{125}{64} + \frac{625}{256} + \cdots$

e. $1 + \frac{4}{5} + \frac{16}{25} + \frac{64}{125} + \frac{256}{625} + \cdots$

f. $1 + \frac{9}{10} + \frac{81}{100} + \frac{729}{1000} + \frac{6561}{10,000} + \cdots$

	A	B
1	1	1
2	2	0.5
3	3	0.25
4	4	0.125
5	5	0.0625
6	6	0.03125
7	7	
8	8	
9	9	
10	10	
11	11	
12	12	
13	13	
14	14	
15	15	
16	**Sum**	

EXPLORATION 2 **Writing a Conjecture**

Work with a partner. Look back at the infinite geometric series in Exploration 1. Write a conjecture about how you can determine whether the infinite geometric series

$$a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots$$

has a finite sum.

EXPLORATION 3 **Writing a Formula**

Work with a partner. In Lesson 8.3, you learned that the sum of the first n terms of a geometric series with first term a_1 and common ratio $r \neq 1$ is

$$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right).$$

When an infinite geometric series has a finite sum, what happens to r^n as n increases? Explain your reasoning. Write a formula to find the sum of an infinite geometric series. Then verify your formula by checking the sums you obtained in Exploration 1.

Communicate Your Answer

4. How can you find the sum of an infinite geometric series?

5. Find the sum of each infinite geometric series, if it exists.

a. $1 + 0.1 + 0.01 + 0.001 + 0.0001 + \cdots$ **b.** $2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \frac{32}{81} + \cdots$

8.4 Lesson

Core Vocabulary

partial sum, *p. 436*

Previous
repeating decimal
fraction in simplest form
rational number

What You Will Learn

- Find partial sums of infinite geometric series.
- Find sums of infinite geometric series.

Partial Sums of Infinite Geometric Series

The sum S_n of the first n terms of an infinite series is called a **partial sum**. The partial sums of an infinite geometric series may approach a limiting value.

EXAMPLE 1 Finding Partial Sums

Consider the infinite geometric series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots.$$

Find and graph the partial sums S_n for $n = 1, 2, 3, 4$, and 5. Then describe what happens to S_n as n increases.

SOLUTION

Step 1 Find the partial sums.

$$S_1 = \frac{1}{2} = 0.5$$

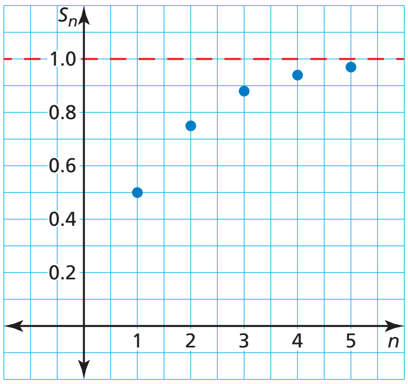
$$S_2 = \frac{1}{2} + \frac{1}{4} = 0.75$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \approx 0.88$$

$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \approx 0.94$$

$$S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \approx 0.97$$

Step 2 Plot the points (1, 0.5), (2, 0.75), (3, 0.88), (4, 0.94), and (5, 0.97). The graph is shown at the right.



From the graph, S_n appears to approach 1 as n increases.

Sums of Infinite Geometric Series

In Example 1, you can understand why S_n approaches 1 as n increases by considering the rule for the sum of a finite geometric series.

$$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right) = \frac{1}{2}\left(\frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}\right) = 1 - \left(\frac{1}{2}\right)^n$$

As n increases, $\left(\frac{1}{2}\right)^n$ approaches 0, so S_n approaches 1. Therefore, 1 is defined to be the sum of the infinite geometric series in Example 1. More generally, as n increases for *any* infinite geometric series with common ratio r between -1 and 1, the value of S_n approaches

$$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right) \approx a_1\left(\frac{1 - 0}{1 - r}\right) = \frac{a_1}{1 - r}.$$

Core Concept

The Sum of an Infinite Geometric Series

The sum of an infinite geometric series with first term a_1 and common ratio r is given by

$$S = \frac{a_1}{1 - r}$$

provided $|r| < 1$. If $|r| \ge 1$, then the series has no sum.

UNDERSTANDING MATHEMATICAL TERMS

Even though a geometric series with a common ratio of $|r| < 1$ has *infinitely* many terms, the series has a *finite* sum.

EXAMPLE 2 Finding Sums of Infinite Geometric Series

Find the sum of each infinite geometric series.

a. $\sum_{i=1}^{\infty} 3(0.7)^{i-1}$ **b.** $1 + 3 + 9 + 27 + \cdots$ **c.** $1 - \frac{3}{4} + \frac{9}{16} - \frac{27}{64} + \cdots$

SOLUTION

a. For this series, $a_1 = 3(0.7)^{1-1} = 3$ and $r = 0.7$. The sum of the series is

$$S = \frac{a_1}{1 - r}$$ Formula for sum of an infinite geometric series

$$= \frac{3}{1 - 0.7}$$ Substitute 3 for a_1 and 0.7 for r.

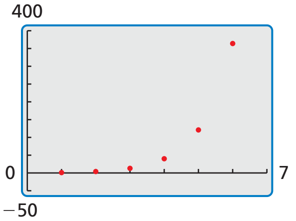
$$= 10.$$ Simplify.

b. For this series, $a_1 = 1$ and $a_2 = 3$. So, the common ratio is $r = \frac{3}{1} = 3$.

Because $|3| \ge 1$, the sum does not exist.

STUDY TIP

For the geometric series in part (b), the graph of the partial sums S_n for $n = 1, 2, 3, 4, 5$, and 6 are shown. From the graph, it appears that as n increases, the partial sums do not approach a fixed number.



c. For this series, $a_1 = 1$ and $a_2 = -\frac{3}{4}$. So, the common ratio is

$$r = \frac{-\frac{3}{4}}{1} = -\frac{3}{4}.$$

The sum of the series is

$$S = \frac{a_1}{1 - r}$$ Formula for sum of an infinite geometric series

$$= \frac{1}{1 - \left(-\frac{3}{4}\right)}$$ Substitute 1 for a_1 and $-\frac{3}{4}$ for r.

$$= \frac{4}{7}.$$ Simplify.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

1. Consider the infinite geometric series

$$\frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \frac{16}{1625} + \frac{32}{3125} + \cdots.$$

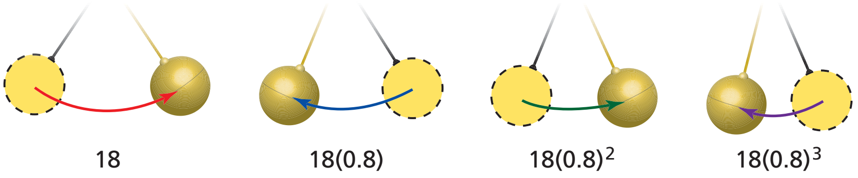
Find and graph the partial sums S_n for $n = 1, 2, 3, 4$, and 5. Then describe what happens to S_n as n increases.

Find the sum of the infinite geometric series, if it exists.

2. $\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{n-1}$ **3.** $\sum_{n=1}^{\infty} 3\left(\frac{5}{4}\right)^{n-1}$ **4.** $3 + \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \cdots$

EXAMPLE 3 Solving a Real-Life Problem

A pendulum that is released to swing freely travels 18 inches on the first swing. On each successive swing, the pendulum travels 80% of the distance of the previous swing. What is the total distance the pendulum swings?



SOLUTION

The total distance traveled by the pendulum is given by the infinite geometric series

$$18 + 18(0.8) + 18(0.8)^2 + 18(0.8)^3 + \cdots.$$

For this series, $a_1 = 18$ and $r = 0.8$. The sum of the series is

$S = \dfrac{a_1}{1 - r}$ Formula for sum of an infinite geometric series

$= \dfrac{18}{1 - 0.8}$ Substitute 18 for a_1 and 0.8 for r.

$= 90.$ Simplify.

▶ The pendulum travels a total distance of 90 inches, or 7.5 feet.

REMEMBER

Because a repeating decimal is a rational number, it can be written as $\frac{a}{b}$, where a and b are integers and $b \neq 0$.

EXAMPLE 4 Writing a Repeating Decimal as a Fraction

Write 0.242424 . . . as a fraction in simplest form.

SOLUTION

Write the repeating decimal as an infinite geometric series.

$$0.242424\ldots = 0.24 + 0.0024 + 0.000024 + 0.00000024 + \cdots$$

For this series, $a_1 = 0.24$ and $r = \dfrac{0.0024}{0.24} = 0.01$. Next, write the sum of the series.

$S = \dfrac{a_1}{1 - r}$ Formula for sum of an infinite geometric series

$= \dfrac{0.24}{1 - 0.01}$ Substitute 0.24 for a_1 and 0.01 for r.

$= \dfrac{0.24}{0.99}$ Simplify.

$= \dfrac{24}{99}$ Write as a quotient of integers.

$= \dfrac{8}{33}$ Simplify.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

5. **WHAT IF?** In Example 3, suppose the pendulum travels 10 inches on its first swing. What is the total distance the pendulum swings?

Write the repeating decimal as a fraction in simplest form.

6. 0.555 . . .
7. 0.727272 . . .
8. 0.131313 . . .

8.4 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The sum S_n of the first n terms of an infinite series is called a(n) ________.

2. **WRITING** Explain how to tell whether the series $\sum_{i=1}^{\infty} a_1 r^{i-1}$ has a sum.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, consider the infinite geometric series. Find and graph the partial sums S_n for $n = 1, 2, 3, 4,$ and 5. Then describe what happens to S_n as n increases. *(See Example 1.)*

3. $\frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \frac{1}{54} + \frac{1}{162} + \cdots$

4. $\frac{2}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \cdots$

5. $4 + \frac{12}{5} + \frac{36}{25} + \frac{108}{125} + \frac{324}{625} + \cdots$

6. $2 + \frac{2}{6} + \frac{2}{36} + \frac{2}{216} + \frac{2}{1296} + \cdots$

In Exercises 7–14, find the sum of the infinite geometric series, if it exists. *(See Example 2.)*

7. $\sum_{n=1}^{\infty} 8\left(\frac{1}{5}\right)^{n-1}$

8. $\sum_{k=1}^{\infty} -6\left(\frac{3}{2}\right)^{k-1}$

9. $\sum_{k=1}^{\infty} \frac{11}{3}\left(\frac{3}{8}\right)^{k-1}$

10. $\sum_{i=1}^{\infty} \frac{2}{5}\left(\frac{5}{3}\right)^{i-1}$

11. $2 + \frac{6}{4} + \frac{18}{16} + \frac{54}{64} + \cdots$

12. $-5 - 2 - \frac{4}{5} - \frac{8}{25} - \cdots$

13. $3 + \frac{5}{2} + \frac{25}{12} + \frac{125}{72} + \cdots$

14. $\frac{1}{2} - \frac{5}{3} + \frac{50}{9} - \frac{500}{27} + \cdots$

ERROR ANALYSIS In Exercises 15 and 16, describe and correct the error in finding the sum of the infinite geometric series.

15. $\sum_{n=1}^{\infty} \left(\frac{7}{2}\right)^{n-1}$

✗ For this series, $a_1 = 1$ and $r = \frac{7}{2}$.

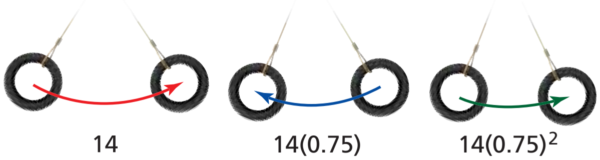
$$S = \frac{a_1}{1-r} = \frac{1}{1-\frac{7}{2}} = \frac{1}{-\frac{5}{2}} = -\frac{2}{5}$$

16. $4 + \frac{8}{3} + \frac{16}{9} + \frac{32}{27} + \cdots$

✗ For this series, $a_1 = 4$ and $r = \frac{4}{\frac{8}{3}} = \frac{3}{2}$.

Because $\left|\frac{3}{2}\right| > 1$, the series has no sum.

17. **MODELING WITH MATHEMATICS** You push your younger cousin on a tire swing one time and then allow your cousin to swing freely. On the first swing, your cousin travels a distance of 14 feet. On each successive swing, your cousin travels 75% of the distance of the previous swing. What is the total distance your cousin swings? *(See Example 3.)*



18. **MODELING WITH MATHEMATICS** A company had a profit of \$350,000 in its first year. Since then, the company's profit has decreased by 12% per year. Assuming this trend continues, what is the total profit the company can make over the course of its lifetime? Justify your answer.

In Exercises 19–24, write the repeating decimal as a fraction in simplest form. *(See Example 4.)*

19. 0.222 . . .

20. 0.444 . . .

21. 0.161616 . . .

22. 0.625625625 . . .

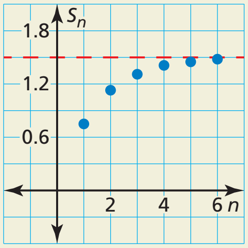
23. 32.323232 . . .

24. 130.130130130 . . .

25. **PROBLEM SOLVING** Find two infinite geometric series whose sums are each 6. Justify your answers.

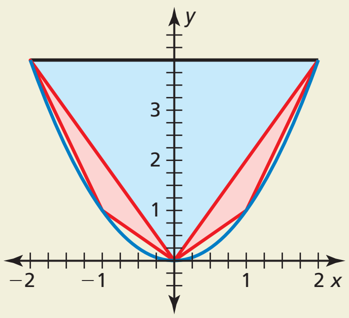
26. HOW DO YOU SEE IT? The graph shows the partial sums of the geometric series $a_1 + a_2 + a_3 + a_4 + \cdots$.

What is the value of $\sum_{n=1}^{\infty} a_n$? Explain.

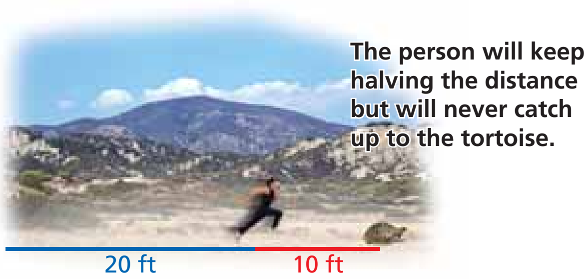


27. MODELING WITH MATHEMATICS A radio station has a daily contest in which a random listener is asked a trivia question. On the first day, the station gives \$500 to the first listener who answers correctly. On each successive day, the winner receives 90% of the winnings from the previous day. What is the total amount of prize money the radio station gives away during the contest?

28. THOUGHT PROVOKING Archimedes used the sum of a geometric series to compute the area enclosed by a parabola and a straight line. In "Quadrature of the Parabola," he proved that the area of the region is $\frac{4}{3}$ the area of the inscribed triangle. The first term of the series for the parabola below is represented by the area of the blue triangle and the second term is represented by the area of the red triangles. Use Archimedes' result to find the area of the region. Then write the area as the sum of an infinite geometric series.



29. DRAWING CONCLUSIONS Can a person running at 20 feet per second ever catch up to a tortoise that runs 10 feet per second when the tortoise has a 20-foot head start? The Greek mathematician Zeno said no. He reasoned as follows:

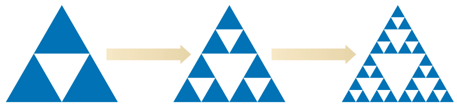


Looking at the race as Zeno did, the distances and the times it takes the person to run those distances both form infinite geometric series. Using the table, show that both series have finite sums. Does the person catch up to the tortoise? Justify your answer.

Distance (ft)	20	10	5	2.5	. . .
Time (sec)	1	0.5	0.25	0.125	. . .

30. MAKING AN ARGUMENT Your friend claims that 0.999 . . . is equal to 1. Is your friend correct? Justify your answer.

31. CRITICAL THINKING The *Sierpinski triangle* is a fractal created using equilateral triangles. The process involves removing smaller triangles from larger triangles by joining the midpoints of the sides of the larger triangles as shown. Assume that the initial triangle has an area of 1 square foot.



a. Let a_n be the total area of all the triangles that are removed at Stage n. Write a rule for a_n.

b. Find $\sum_{n=1}^{\infty} a_n$. Interpret your answer in the context of this situation.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Determine the type of function represented by the table. *(Section 6.7)*

32.

x	−3	−2	−1	0	1
y	0.5	1.5	4.5	13.5	40.5

33.

x	0	4	8	12	16
y	−7	−1	2	2	−1

Determine whether the sequence is *arithmetic*, *geometric*, or *neither*. *(Sections 8.2 and 8.3)*

34. −7, −1, 5, 11, 17, . . .

35. 0, −1, −3, −7, −15, . . .

36. 13.5, 40.5, 121.5, 364.5, . . .

8.5 Using Recursive Rules with Sequences

Essential Question

How can you define a sequence recursively?

A **recursive rule** gives the beginning term(s) of a sequence and a *recursive equation* that tells how a_n is related to one or more preceding terms.

EXPLORATION 1 Evaluating a Recursive Rule

Work with a partner. Use each recursive rule and a spreadsheet to write the first six terms of the sequence. Classify the sequence as arithmetic, geometric, or neither. Explain your reasoning. (The figure shows a partially completed spreadsheet for part (a).)

a. $a_1 = 7,\ a_n = a_{n-1} + 3$

b. $a_1 = 5,\ a_n = a_{n-1} - 2$

c. $a_1 = 1,\ a_n = 2a_{n-1}$

d. $a_1 = 1,\ a_n = \frac{1}{2}(a_{n-1})^2$

e. $a_1 = 3,\ a_n = a_{n-1} + 1$

f. $a_1 = 4,\ a_n = \frac{1}{2}a_{n-1} - 1$

g. $a_1 = 4,\ a_n = \frac{1}{2}a_{n-1}$

h. $a_1 = 4,\ a_2 = 5,\ a_n = a_{n-1} + a_{n-2}$

	A	B
1	n	nth Term
2	1	7
3	2	10
4	3	
5	4	
6	5	
7	6	

B2+3

ATTENDING TO PRECISION

To be proficient in math, you need to communicate precisely to others.

EXPLORATION 2 Writing a Recursive Rule

Work with a partner. Write a recursive rule for the sequence. Explain your reasoning.

a. 3, 6, 9, 12, 15, 18, . . .

b. 18, 14, 10, 6, 2, −2, . . .

c. 3, 6, 12, 24, 48, 96, . . .

d. 128, 64, 32, 16, 8, 4, . . .

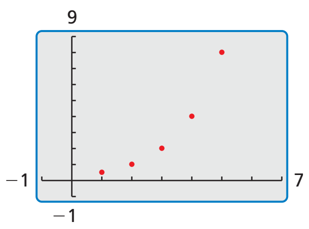
e. 5, 5, 5, 5, 5, 5, . . .

f. 1, 1, 2, 3, 5, 8, . . .

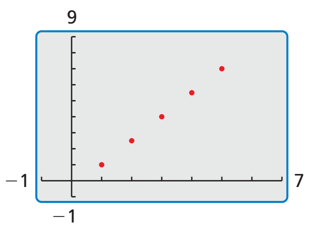
EXPLORATION 3 Writing a Recursive Rule

Work with a partner. Write a recursive rule for the sequence whose graph is shown.

a.



b.



Communicate Your Answer

4. How can you define a sequence recursively?
5. Write a recursive rule that is different from those in Explorations 1–3. Write the first six terms of the sequence. Then graph the sequence and classify it as arithmetic, geometric, or neither.

8.5 Lesson

Core Vocabulary

explicit rule, *p. 442*
recursive rule, *p. 442*

What You Will Learn

- Evaluate recursive rules for sequences.
- Write recursive rules for sequences.
- Translate between recursive and explicit rules for sequences.
- Use recursive rules to solve real-life problems.

Evaluating Recursive Rules

So far in this chapter, you have worked with explicit rules for the nth term of a sequence, such as $a_n = 3n - 2$ and $a_n = 7(0.5)^n$. An **explicit rule** gives a_n as a function of the term's position number n in the sequence.

In this section, you will learn another way to define a sequence—by a *recursive rule*. A **recursive rule** gives the beginning term(s) of a sequence and a *recursive equation* that tells how a_n is related to one or more preceding terms.

EXAMPLE 1 Evaluating Recursive Rules

Write the first six terms of each sequence.

a. $a_0 = 1, a_n = a_{n-1} + 4$

b. $f(1) = 1, f(n) = 3 \cdot f(n-1)$

SOLUTION

a. $a_0 = 1$	1st term		**b.** $f(1) = 1$
$a_1 = a_0 + 4 = 1 + 4 = 5$	2nd term		$f(2) = 3 \cdot f(1) = 3(1) = 3$
$a_2 = a_1 + 4 = 5 + 4 = 9$	3rd term		$f(3) = 3 \cdot f(2) = 3(3) = 9$
$a_3 = a_2 + 4 = 9 + 4 = 13$	4th term		$f(4) = 3 \cdot f(3) = 3(9) = 27$
$a_4 = a_3 + 4 = 13 + 4 = 17$	5th term		$f(5) = 3 \cdot f(4) = 3(27) = 81$
$a_5 = a_4 + 4 = 17 + 4 = 21$	6th term		$f(6) = 3 \cdot f(5) = 3(81) = 243$

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Write the first six terms of the sequence.

1. $a_1 = 3, a_n = a_{n-1} - 7$

2. $a_0 = 162, a_n = 0.5a_{n-1}$

3. $f(0) = 1, f(n) = f(n-1) + n$

4. $a_1 = 4, a_n = 2a_{n-1} - 1$

Writing Recursive Rules

In part (a) of Example 1, the *differences* of consecutive terms of the sequence are constant, so the sequence is arithmetic. In part (b), the *ratios* of consecutive terms are constant, so the sequence is geometric. In general, rules for arithmetic and geometric sequences can be written recursively as follows.

Core Concept

Recursive Equations for Arithmetic and Geometric Sequences

Arithmetic Sequence

$a_n = a_{n-1} + d$, where d is the common difference

Geometric Sequence

$a_n = r \cdot a_{n-1}$, where r is the common ratio

EXAMPLE 2 Writing Recursive Rules

Write a recursive rule for (a) 3, 13, 23, 33, 43, . . . and (b) 16, 40, 100, 250, 625, ...

SOLUTION

Use a table to organize the terms and find the pattern.

COMMON ERROR

A recursive *equation* for a sequence does not include the initial term. To write a recursive *rule* for a sequence, the initial term(s) must be included.

a.

n	1	2	3	4	5
a_n	3	13	23	33	43

$+10 \quad +10 \quad +10 \quad +10$

The sequence is arithmetic with first term $a_1 = 3$ and common difference $d = 10$.

$a_n = a_{n-1} + d$ Recursive equation for arithmetic sequence

$= a_{n-1} + 10$ Substitute 10 for *d*.

▶ A recursive rule for the sequence is $a_1 = 3$, $a_n = a_{n-1} + 10$.

b.

n	1	2	3	4	5
a_n	16	40	100	250	625

$\times\frac{5}{2} \quad \times\frac{5}{2} \quad \times\frac{5}{2} \quad \times\frac{5}{2}$

The sequence is geometric with first term $a_1 = 16$ and common ratio $r = \frac{5}{2}$.

$a_n = r \cdot a_{n-1}$ Recursive equation for geometric sequence

$= \frac{5}{2}a_{n-1}$ Substitute $\frac{5}{2}$ for *r*.

▶ A recursive rule for the sequence is $a_1 = 16$, $a_n = \frac{5}{2}a_{n-1}$.

STUDY TIP

The sequence in part (a) of Example 3 is called the *Fibonacci sequence*. The sequence in part (b) lists *factorial numbers*. You will learn more about *factorials* in Chapter 10.

EXAMPLE 3 Writing Recursive Rules

Write a recursive rule for each sequence.

a. 1, 1, 2, 3, 5, . . . **b.** 1, 1, 2, 6, 24, . . .

SOLUTION

a. The terms have neither a common difference nor a common ratio. Beginning with the third term in the sequence, each term is the sum of the two previous terms.

▶ A recursive rule for the sequence is $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-2} + a_{n-1}$.

b. The terms have neither a common difference nor a common ratio. Denote the first term by $a_0 = 1$. Note that $a_1 = 1 = 1 \cdot a_0$, $a_2 = 2 = 2 \cdot a_1$, $a_3 = 6 = 3 \cdot a_2$, and so on.

▶ A recursive rule for the sequence is $a_0 = 1$, $a_n = n \cdot a_{n-1}$.

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Write a recursive rule for the sequence.

5. 2, 14, 98, 686, 4802, . . .

6. 19, 13, 7, 1, −5, . . .

7. 11, 22, 33, 44, 55, . . .

8. 1, 2, 2, 4, 8, 32, . . .

Translating Between Recursive and Explicit Rules

EXAMPLE 4 Translating from Explicit Rules to Recursive Rules

Write a recursive rule for (a) $a_n = -6 + 8n$ and (b) $a_n = -3\left(\frac{1}{2}\right)^{n-1}$.

SOLUTION

a. The explicit rule represents an arithmetic sequence with first term $a_1 = -6 + 8(1) = 2$ and common difference $d = 8$.

$a_n = a_{n-1} + d$ Recursive equation for arithmetic sequence

$a_n = a_{n-1} + 8$ Substitute 8 for *d*.

▶ A recursive rule for the sequence is $a_1 = 2$, $a_n = a_{n-1} + 8$.

b. The explicit rule represents a geometric sequence with first term $a_1 = -3\left(\frac{1}{2}\right)^0 = -3$ and common ratio $r = \frac{1}{2}$.

$a_n = r \cdot a_{n-1}$ Recursive equation for geometric sequence

$a_n = \frac{1}{2}a_{n-1}$ Substitute $\frac{1}{2}$ for *r*.

▶ A recursive rule for the sequence is $a_1 = -3$, $a_n = \frac{1}{2}a_{n-1}$.

EXAMPLE 5 Translating from Recursive Rules to Explicit Rules

Write an explicit rule for each sequence.

a. $a_1 = -5$, $a_n = a_{n-1} - 2$

b. $a_1 = 10$, $a_n = 2a_{n-1}$

SOLUTION

a. The recursive rule represents an arithmetic sequence with first term $a_1 = -5$ and common difference $d = -2$.

$a_n = a_1 + (n - 1)d$ Explicit rule for arithmetic sequence

$a_n = -5 + (n - 1)(-2)$ Substitute -5 for a_1 and -2 for *d*.

$a_n = -3 - 2n$ Simplify.

▶ An explicit rule for the sequence is $a_n = -3 - 2n$.

b. The recursive rule represents a geometric sequence with first term $a_1 = 10$ and common ratio $r = 2$.

$a_n = a_1r^{n-1}$ Explicit rule for geometric sequence

$a_n = 10(2)^{n-1}$ Substitute 10 for a_1 and 2 for *r*.

▶ An explicit rule for the sequence is $a_n = 10(2)^{n-1}$.

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Write a recursive rule for the sequence.

9. $a_n = 17 - 4n$

10. $a_n = 16(3)^{n-1}$

Write an explicit rule for the sequence.

11. $a_1 = -12$, $a_n = a_{n-1} + 16$

12. $a_1 = 2$, $a_n = -6a_{n-1}$

Solving Real-Life Problems

EXAMPLE 6 Solving a Real-Life Problem

A lake initially contains 5200 fish. Each year, the population declines 30% due to fishing and other causes, so the lake is restocked with 400 fish.

a. Write a recursive rule for the number a_n of fish at the start of the nth year.

b. Find the number of fish at the start of the fifth year.

c. Describe what happens to the population of fish over time.

SOLUTION

a. Write a recursive rule. The initial value is 5200. Because the population declines 30% each year, 70% of the fish remain in the lake from one year to the next. Also, 400 fish are added each year. Here is a verbal model for the recursive equation.

Fish at start of year n	$= 0.7 \cdot$	Fish at start of year $n - 1$	$+$	New fish added
a_n	$= 0.7 \cdot$	a_{n-1}	$+$	400

▶ A recursive rule is $a_1 = 5200$, $a_n = (0.7)a_{n-1} + 400$.

b. Find the number of fish at the start of the fifth year. Enter 5200 (the value of a_1) in a graphing calculator. Then enter the rule

$.7 \times \text{Ans} + 400$

to find a_2. Press the enter button three more times to find $a_5 \approx 2262$.

```
5200
                5200
.7*Ans+400
                4040
                3228
              2659.6
             2261.72
```

▶ There are about 2262 fish in the lake at the start of the fifth year.

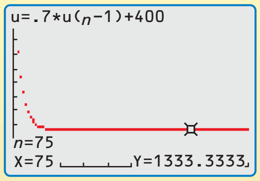
c. Describe what happens to the population of fish over time. Continue pressing enter on the calculator. The screen at the right shows the fish populations for years 44 to 50. Observe that the population of fish approaches 1333.

```
1333.334178
1333.333924
1333.333747
1333.333623
1333.333536
1333.333475
1333.333433
```

▶ Over time, the population of fish in the lake stabilizes at about 1333 fish.

Check

Set a graphing calculator to *sequence* and *dot* modes. Graph the sequence and use the *trace* feature. From the graph, it appears the sequence approaches 1333.



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13. WHAT IF? In Example 6, suppose 75% of the fish remain each year. What happens to the population of fish over time?

EXAMPLE 7 Modeling with Mathematics

REMEMBER
In Section 8.3, you used a formula involving a geometric series to calculate the monthly payment for a similar loan.

You borrow $150,000 at 6% annual interest compounded monthly for 30 years. The monthly payment is $899.33.

- Find the balance after the third payment.
- Due to rounding in the calculations, the last payment is often different from the original payment. Find the amount of the last payment.

SOLUTION

1. **Understand the Problem** You are given the conditions of a loan. You are asked to find the balance after the third payment and the amount of the last payment.

2. **Make a Plan** Because the balance after each payment depends on the balance after the previous payment, write a recursive rule that gives the balance after each payment. Then use a spreadsheet to find the balance after each payment, rounded to the nearest cent.

3. **Solve the Problem** Because the monthly interest rate is $\frac{0.06}{12} = 0.005$, the balance increases by a factor of 1.005 each month, and then the payment of $899.33 is subtracted.

Balance after payment = 1.005 • Balance before payment − Payment

$$a_n = 1.005 \cdot a_{n-1} - 899.33$$

Use a spreadsheet and the recursive rule to find the balance after the third payment and after the 359th payment.

	A	B
1	**Payment number**	**Balance after payment**
2	1	149850.67
3	2	149700.59
4	3	149549.76
358	357	2667.38
359	358	1781.39
360	359	890.97

B2 =Round(1.005*150000−899.33, 2)
B3 =Round(1.005*B2−899.33, 2)
⋮
B360 =Round(1.005*B359−899.33, 2)

▶ The balance after the third payment is $149,549.76. The balance after the 359th payment is $890.97, so the final payment is 1.005(890.97) = $895.42.

4. **Look Back** By continuing the spreadsheet for the 360th payment using the original monthly payment of $899.33, the balance is −3.91.

361	360	−3.91

B361 =Round(1.005*B360−899.33, 2)

This shows an overpayment of $3.91. So, it is reasonable that the last payment is $899.33 − $3.91 = $895.42.

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14. WHAT IF? How do the answers in Example 7 change when the annual interest rate is 7.5% and the monthly payment is $1048.82?

8.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** A recursive __________ tells how the nth term of a sequence is related to one or more preceding terms.

2. **WRITING** Explain the difference between an explicit rule and a recursive rule for a sequence.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, write the first six terms of the sequence. *(See Example 1.)*

3. $a_1 = 1$
$a_n = a_{n-1} + 3$

4. $a_1 = 1$
$a_n = a_{n-1} - 5$

5. $f(0) = 4$
$f(n) = 2f(n-1)$

6. $f(0) = 10$
$f(n) = \frac{1}{2}f(n-1)$

7. $a_1 = 2$
$a_n = (a_{n-1})^2 + 1$

8. $a_1 = 1$
$a_n = (a_{n-1})^2 - 10$

9. $f(0) = 2, f(1) = 4$
$f(n) = f(n-1) - f(n-2)$

10. $f(1) = 2, f(2) = 3$
$f(n) = f(n-1) \bullet f(n-2)$

In Exercises 11–22, write a recursive rule for the sequence. *(See Examples 2 and 3.)*

11. 21, 14, 7, 0, −7, . . .

12. 54, 43, 32, 21, 10, . . .

13. 3, 12, 48, 192, 768, . . .

14. 4, −12, 36, −108, . . .

15. 44, 11, $\frac{11}{4}$, $\frac{11}{16}$, $\frac{11}{64}$, . . .

16. 1, 8, 15, 22, 29, . . .

17. 2, 5, 10, 50, 500, . . .

18. 3, 5, 15, 75, 1125, . . .

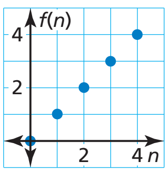
19. 1, 4, 5, 9, 14, . . .

20. 16, 9, 7, 2, 5, . . .

21. 6, 12, 36, 144, 720, . . .

22. −3, −1, 2, 6, 11, . . .

In Exercises 23–26, write a recursive rule for the sequence shown in the graph.

23.


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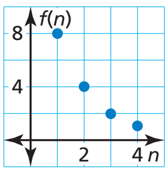

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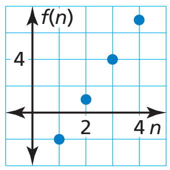
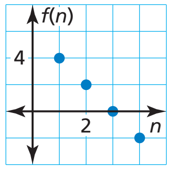


26.


ERROR ANALYSIS **In Exercises 27 and 28, describe and correct the error in writing a recursive rule for the sequence 5, 2, 3, −1, 4, ...**

27.
Beginning with the third term in the sequence, each term a_n equals $a_{n-2} - a_{n-1}$. So, a recursive rule is given by

$$a_n = a_{n-2} - a_{n-1}.$$

28.
Beginning with the second term in the sequence, each term a_n equals $a_{n-1} - 3$. So, a recursive rule is given by

$$a_1 = 5, a_n = a_{n-1} - 3.$$

In Exercises 29–38, write a recursive rule for the sequence. *(See Example 4.)*

29. $a_n = 3 + 4n$

30. $a_n = -2 - 8n$

31. $a_n = 12 - 10n$

32. $a_n = 9 - 5n$

33. $a_n = 12(11)^{n-1}$

34. $a_n = -7(6)^{n-1}$

35. $a_n = 2.5 - 0.6n$

36. $a_n = -1.4 + 0.5n$

37. $a_n = -\frac{1}{2}\left(\frac{1}{4}\right)^{n-1}$

38. $a_n = \frac{1}{4}(5)^{n-1}$

39. REWRITING A FORMULA You have saved \$82 to buy a bicycle. You save an additional \$30 each month. The explicit rule $a_n = 30n + 82$ gives the amount saved after n months. Write a recursive rule for the amount you have saved n months from now.

40. REWRITING A FORMULA Your salary is given by the explicit rule $a_n = 35{,}000(1.04)^{n-1}$, where n is the number of years you have worked. Write a recursive rule for your salary.

In Exercises 41–48, write an explicit rule for the sequence. *(See Example 5.)*

41. $a_1 = 3,\ a_n = a_{n-1} - 6$

42. $a_1 = 16,\ a_n = a_{n-1} + 7$

43. $a_1 = -2,\ a_n = 3a_{n-1}$

44. $a_1 = 13,\ a_n = 4a_{n-1}$

45. $a_1 = -12,\ a_n = a_{n-1} + 9.1$

46. $a_1 = -4,\ a_n = 0.65a_{n-1}$

47. $a_1 = 5,\ a_n = a_{n-1} - \frac{1}{3}$

48. $a_1 = -5,\ a_n = \frac{1}{4}a_{n-1}$

49. REWRITING A FORMULA A grocery store arranges cans in a pyramid-shaped display with 20 cans in the bottom row and two fewer cans in each subsequent row going up. The number of cans in each row is represented by the recursive rule $a_1 = 20$, $a_n = a_{n-1} - 2$. Write an explicit rule for the number of cans in row n.

50. REWRITING A FORMULA The value of a car is given by the recursive rule $a_1 = 25{,}600$, $a_n = 0.86a_{n-1}$, where n is the number of years since the car was new. Write an explicit rule for the value of the car after n years.

51. USING STRUCTURE What is the 1000th term of the sequence whose first term is $a_1 = 4$ and whose nth term is $a_n = a_{n-1} + 6$? Justify your answer.

Ⓐ 4006　　Ⓑ 5998

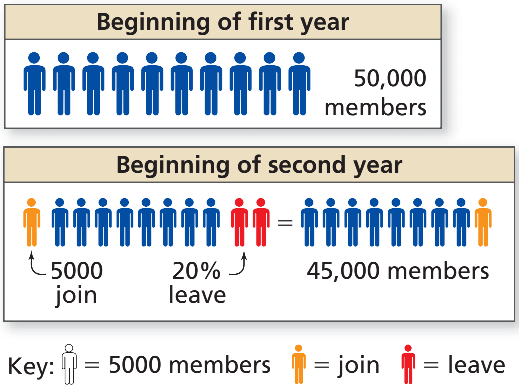
Ⓒ 1010　　Ⓓ 10,000

52. USING STRUCTURE What is the 873rd term of the sequence whose first term is $a_1 = 0.01$ and whose nth term is $a_n = 1.01a_{n-1}$? Justify your answer.

Ⓐ 58.65　　Ⓑ 8.73

Ⓒ 1.08　　Ⓓ 586,459.38

53. PROBLEM SOLVING An online music service initially has 50,000 members. Each year, the company loses 20% of its current members and gains 5000 new members. *(See Example 6.)*

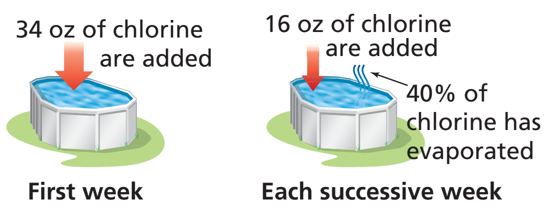


a. Write a recursive rule for the number a_n of members at the start of the nth year.

b. Find the number of members at the start of the fifth year.

c. Describe what happens to the number of members over time.

54. PROBLEM SOLVING You add chlorine to a swimming pool. You add 34 ounces of chlorine the first week and 16 ounces every week thereafter. Each week, 40% of the chlorine in the pool evaporates.



a. Write a recursive rule for the amount of chlorine in the pool at the start of the nth week.

b. Find the amount of chlorine in the pool at the start of the third week.

c. Describe what happens to the amount of chlorine in the pool over time.

55. OPEN-ENDED Give an example of a real-life situation which you can represent with a recursive rule that does not approach a limit. Write a recursive rule that represents the situation.

56. OPEN-ENDED Give an example of a sequence in which each term after the third term is a function of the three terms preceding it. Write a recursive rule for the sequence and find its first eight terms.

57. MODELING WITH MATHEMATICS You borrow \$2000 at 9% annual interest compounded monthly for 2 years. The monthly payment is \$91.37. *(See Example 7.)*

a. Find the balance after the fifth payment.

b. Find the amount of the last payment.

58. MODELING WITH MATHEMATICS You borrow \$10,000 to build an extra bedroom onto your house. The loan is secured for 7 years at an annual interest rate of 11.5%. The monthly payment is \$173.86.

a. Find the balance after the fourth payment.

b. Find the amount of the last payment.

59. COMPARING METHODS In 1202, the mathematician Leonardo Fibonacci wrote *Liber Abaci,* in which he proposed the following rabbit problem:

Begin with a pair of newborn rabbits. When a pair of rabbits is two months old, the rabbits begin producing a new pair of rabbits each month. Assume none of the rabbits die.

Month	1	2	3	4	5	6
Pairs at start of month	1	1	2	3	5	8

This problem produces a sequence called the Fibonacci sequence, which has both a recursive formula and an explicit formula as follows.

Recursive: $a_1 = 1,\ a_2 = 1,\ a_n = a_{n-2} + a_{n-1}$

Explicit: $f_n = \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \dfrac{1}{\sqrt{5}}\left(\dfrac{1-\sqrt{5}}{2}\right)^n,\ n \ge 1$

Use each formula to determine how many rabbits there will be after one year. Justify your answers.

60. USING TOOLS A town library initially has 54,000 books in its collection. Each year, 2% of the books are lost or discarded. The library can afford to purchase 1150 new books each year.

a. Write a recursive rule for the number a_n of books in the library at the beginning of the nth year.

b. Use the *sequence* mode and the *dot* mode of a graphing calculator to graph the sequence. What happens to the number of books in the library over time? Explain.

61. DRAWING CONCLUSIONS A tree farm initially has 9000 trees. Each year, 10% of the trees are harvested and 800 seedlings are planted.

a. Write a recursive rule for the number of trees on the tree farm at the beginning of the nth year.

b. What happens to the number of trees after an extended period of time?

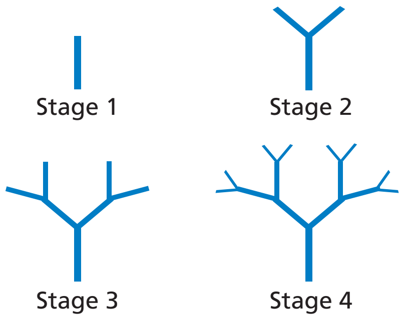
62. DRAWING CONCLUSIONS You sprain your ankle and your doctor prescribes 325 milligrams of an anti-inflammatory drug every 8 hours for 10 days. Sixty percent of the drug is removed from the bloodstream every 8 hours.

a. Write a recursive rule for the amount of the drug in the bloodstream after n doses.

b. The value that a drug level approaches after an extended period of time is called the *maintenance level*. What is the maintenance level of this drug given the prescribed dosage?

c. How does doubling the dosage affect the maintenance level of the drug? Justify your answer.

63. FINDING A PATTERN A fractal tree starts with a single branch (the trunk). At each stage, each new branch from the previous stage grows two more branches, as shown.



a. List the number of new branches in each of the first seven stages. What type of sequence do these numbers form?

b. Write an explicit rule and a recursive rule for the sequence in part (a).

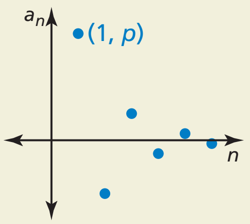
64. THOUGHT PROVOKING Let $a_1 = 34$. Then write the terms of the sequence until you discover a pattern.

$$a_{n+1} = \begin{cases} \frac{1}{2}a_n, & \text{if } a_n \text{ is even} \\ 3a_n + 1, & \text{if } a_n \text{ is odd} \end{cases}$$

Do the same for $a_1 = 25$. What can you conclude?

65. MODELING WITH MATHEMATICS You make a \$500 down payment on a \$3500 diamond ring. You borrow the remaining balance at 10% annual interest compounded monthly. The monthly payment is \$213.59. How long does it take to pay back the loan? What is the amount of the last payment? Justify your answers.

66. HOW DO YOU SEE IT? The graph shows the first six terms of the sequence $a_1 = p$, $a_n = ra_{n-1}$.



a. Describe what happens to the values in the sequence as n increases.

b. Describe the set of possible values for r. Explain your reasoning.

67. REASONING The rule for a recursive sequence is as follows.

$$f(1) = 3, f(2) = 10$$
$$f(n) = 4 + 2f(n-1) - f(n-2)$$

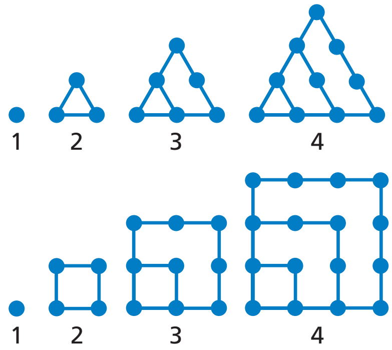
a. Write the first five terms of the sequence.

b. Use finite differences to find a pattern. What type of relationship do the terms of the sequence show?

c. Write an explicit rule for the sequence.

68. MAKING AN ARGUMENT Your friend says it is impossible to write a recursive rule for a sequence that is neither arithmetic nor geometric. Is your friend correct? Justify your answer.

69. CRITICAL THINKING The first four triangular numbers T_n and the first four square numbers S_n are represented by the points in each diagram.



a. Write an explicit rule for each sequence.

b. Write a recursive rule for each sequence.

c. Write a rule for the square numbers in terms of the triangular numbers. Draw diagrams to explain why this rule is true.

70. CRITICAL THINKING You are saving money for retirement. You plan to withdraw \$30,000 at the beginning of each year for 20 years after you retire. Based on the type of investment you are making, you can expect to earn an annual return of 8% on your savings after you retire.

a. Let a_n be your balance n years after retiring. Write a recursive equation that shows how a_n is related to a_{n-1}.

b. Solve the equation from part (a) for a_{n-1}. Find a_0, the minimum amount of money you should have in your account when you retire. (*Hint:* Let $a_{20} = 0$.)

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. Check your solution. *(Section 5.4)*

71. $\sqrt{x} + 2 = 7$

72. $2\sqrt{x} - 5 = 15$

73. $\sqrt[3]{x} + 16 = 19$

74. $2\sqrt[3]{x} - 13 = -5$

The variables x and y vary inversely. Use the given values to write an equation relating x and y. Then find y when $x = 4$. *(Section 7.1)*

75. $x = 2, y = 9$

76. $x = -4, y = 3$

77. $x = 10, y = 32$

8.4–8.5 What Did You Learn?

Core Vocabulary

partial sum, *p. 436*
explicit rule, *p. 442*
recursive rule, *p. 442*

Core Concepts

Section 8.4

Partial Sums of Infinite Geometric Series, *p. 436*
The Sum of an Infinite Geometric Series, *p. 437*

Section 8.5

Evaluating Recursive Rules, *p. 442*
Recursive Equations for Arithmetic and Geometric Sequences, *p. 442*
Translating Between Recursive and Explicit Rules, *p. 444*

Mathematical Practices

1. Describe how labeling the axes in Exercises 3–6 on page 439 clarifies the relationship between the quantities in the problems.
2. What logical progression of arguments can you use to determine whether the statement in Exercise 30 on page 440 is true?
3. Describe how the structure of the equation presented in Exercise 40 on page 448 allows you to determine the starting salary and the raise you receive each year.
4. Does the recursive rule in Exercise 61 on page 449 make sense when $n = 5$? Explain your reasoning.

Performance Task

Integrated Circuits and Moore's Law

In April of 1965, an engineer named Gordon Moore noticed how quickly the size of electronics was shrinking. He predicted how the number of transistors that could fit on a 1-inch diameter circuit would increase over time. In 1965, only 50 transistors fit on the circuit. A decade later, about 65,000 transistors could fit on the circuit. Moore's prediction was accurate and is now known as Moore's Law. What was his prediction? How many transistors will be able to fit on a 1-inch circuit when you graduate from high school?

To explore the answers to this question and more, go to ***BigIdeasMath.com***.

8 Chapter Review

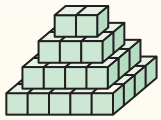
Dynamic Solutions available at *BigIdeasMath.com*

8.1 Defining and Using Sequences and Series *(pp. 409–416)*

Find the sum $\sum_{i=1}^{4}(i^2 - 3)$.

$$\sum_{i=1}^{4}(i^2 - 3) = (1^2 - 3) + (2^2 - 3) + (3^2 - 3) + (4^2 - 3)$$
$$= -2 + 1 + 6 + 13$$
$$= 18$$

1. Describe the pattern shown in the figure. Then write a rule for the nth layer of the figure, where $n = 1$ represents the top layer.



Write the series using summation notation.

2. $7 + 10 + 13 + \cdots + 40$

3. $0 + 2 + 6 + 12 + \cdots$

Find the sum.

4. $\sum_{i=2}^{7}(9 - i^3)$

5. $\sum_{i=1}^{46} i$

6. $\sum_{i=1}^{12} i^2$

7. $\sum_{i=1}^{5} \frac{3 + i}{2}$

8.2 Analyzing Arithmetic Sequences and Series *(pp. 417–424)*

Write a rule for the nth term of the sequence 9, 14, 19, 24, ... Then find a_{14}.

The sequence is arithmetic with first term $a_1 = 9$ and common difference $d = 14 - 9 = 5$. So, a rule for the nth term is

$a_n = a_1 + (n - 1)d$ — Write general rule.

$= 9 + (n - 1)5$ — Substitute 9 for a_1 and 5 for d.

$= 5n + 4.$ — Simplify.

▶ A rule is $a_n = 5n + 4$, and the 14th term is $a_{14} = 5(14) + 4 = 74$.

8. Tell whether the sequence 12, 4, −4, −12, −20, . . . is arithmetic. Explain your reasoning.

Write a rule for the nth term of the arithmetic sequence. Then graph the first six terms of the sequence.

9. 2, 8, 14, 20, . . .

10. $a_{14} = 42, d = 3$

11. $a_6 = -12, a_{12} = -36$

12. Find the sum $\sum_{i=1}^{36}(2 + 3i)$.

13. You take a job with a starting salary of \$37,000. Your employer offers you an annual raise of \$1500 for the next 6 years. Write a rule for your salary in the nth year. What are your total earnings in 6 years?

8.3 Analyzing Geometric Sequences and Series (pp. 425–432)

Find the sum $\sum_{i=1}^{8} 6(3)^{i-1}$.

Step 1 Find the first term and the common ratio.

$a_1 = 6(3)^{1-1} = 6$ Identify first term.

$r = 3$ Identify common ratio.

Step 2 Find the sum.

$S_8 = a_1\left(\frac{1 - r^8}{1 - r}\right)$ Write rule for S_8.

$= 6\left(\frac{1 - 3^8}{1 - 3}\right)$ Substitute 6 for a_1 and 3 for r.

$= 19{,}680$ Simplify.

14. Tell whether the sequence 7, 14, 28, 56, 112, . . . is geometric. Explain your reasoning.

Write a rule for the *n*th term of the geometric sequence. Then graph the first six terms of the sequence.

15. 25, 10, 4, $\frac{8}{5}$, . . .

16. $a_5 = 162$, $r = -3$

17. $a_3 = 16$, $a_5 = 256$

18. Find the sum $\sum_{i=1}^{9} 5(-2)^{i-1}$.

8.4 Finding Sums of Infinite Geometric Series (pp. 435–440)

Find the sum of the series $\sum_{i=1}^{\infty} \left(\frac{4}{5}\right)^{i-1}$, if it exists.

For this series, $a_1 = 1$ and $r = \frac{4}{5}$. Because $\left|\frac{4}{5}\right| < 1$, the sum of the series exists.

The sum of the series is

$S = \frac{a_1}{1 - r}$ Formula for the sum of an infinite geometric series

$= \frac{1}{1 - \frac{4}{5}}$ Substitute 1 for a_1 and $\frac{4}{5}$ for r.

$= 5.$ Simplify.

19. Consider the infinite geometric series $1, -\frac{1}{4}, \frac{1}{16}, -\frac{1}{64}, \frac{1}{256}, \ldots$. Find and graph the partial sums S_n for $n = 1, 2, 3, 4$, and 5. Then describe what happens to S_n as n increases.

20. Find the sum of the infinite geometric series $-2 + \frac{1}{2} - \frac{1}{8} + \frac{1}{32} + \cdots$, if it exists.

21. Write the repeating decimal 0.1212 . . . as a fraction in simplest form.

8.5 Using Recursive Rules with Sequences (pp. 441–450)

a. Write the first six terms of the sequence $a_0 = 46$, $a_n = a_{n-1} - 8$.

$a_0 = 46$ — 1st term

$a_1 = a_0 - 8 = 46 - 8 = 38$ — 2nd term

$a_2 = a_1 - 8 = 38 - 8 = 30$ — 3rd term

$a_3 = a_2 - 8 = 30 - 8 = 22$ — 4th term

$a_4 = a_3 - 8 = 22 - 8 = 14$ — 5th term

$a_5 = a_4 - 8 = 14 - 8 = 6$ — 6th term

b. Write a recursive rule for the sequence 6, 10, 14, 18, 22, ...

Use a table to organize the terms and find the pattern.

n	1	2	3	4	5
a_n	6	10	14	18	22

+4 +4 +4 +4

The sequence is arithmetic with the first term $a_1 = 6$ and common difference $d = 4$.

$a_n = a_{n-1} + d$ — Recursive equation for arithmetic sequence

$= a_{n-1} + 4$ — Substitute 4 for d.

▶ A recursive rule for the sequence is $a_1 = 6$, $a_n = a_{n-1} + 4$.

Write the first six terms of the sequence.

22. $a_1 = 7$, $a_n = a_{n-1} + 11$

23. $a_1 = 6$, $a_n = 4a_{n-1}$

24. $f(0) = 4$, $f(n) = f(n-1) + 2n$

Write a recursive rule for the sequence.

25. $9, 6, 4, \frac{8}{3}, \frac{16}{9}, \ldots$

26. 2, 2, 4, 12, 48, . . .

27. 7, 3, 4, −1, 5, . . .

28. Write a recursive rule for $a_n = 105\left(\frac{3}{5}\right)^{n-1}$.

Write an explicit rule for the sequence.

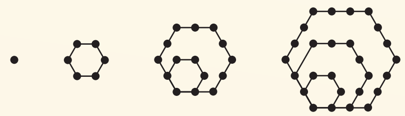
29. $a_1 = -4$, $a_n = a_{n-1} + 26$

30. $a_1 = 8$, $a_n = -5a_{n-1}$

31. $a_1 = 26$, $a_n = \frac{2}{5}a_{n-1}$

32. A town's population increases at a rate of about 4% per year. In 2010, the town had a population of 11,120. Write a recursive rule for the population P_n of the town in year n. Let $n = 1$ represent 2010.

33. The numbers 1, 6, 15, 28, . . . are called hexagonal numbers because they represent the number of dots used to make hexagons, as shown. Write a recursive rule for the nth hexagonal number.



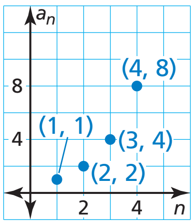
8 Chapter Test

Find the sum.

1. $\sum_{i=1}^{24}(6i - 13)$ **2.** $\sum_{n=1}^{16} n^2$ **3.** $\sum_{k=1}^{\infty} 2(0.8)^{k-1}$ **4.** $\sum_{i=1}^{6} 4(-3)^{i-1}$

Determine whether the graph represents an arithmetic sequence, geometric sequence, or neither. Explain your reasoning. Then write a rule for the *n*th term.

5.



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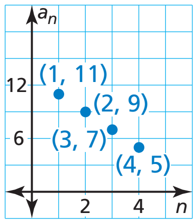

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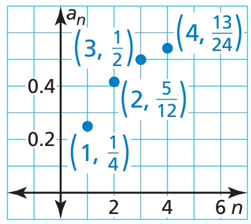


Write a recursive rule for the sequence. Then find a_9.

8. $a_1 = 32, r = \frac{1}{2}$ **9.** $a_n = 2 + 7n$ **10.** $2, 0, -3, -7, -12, \ldots$

11. Write a recursive rule for the sequence $5, -20, 80, -320, 1280, \ldots$. Then write an explicit rule for the sequence using your recursive rule.

12. The numbers a, b, and c are the first three terms of an arithmetic sequence. Is b half of the sum of a and c? Explain your reasoning.

13. Use the pattern of checkerboard quilts shown.

$n = 1, a_n = 1$ $n = 2, a_n = 2$ $n = 3, a_n = 5$ $n = 4, a_n = 8$

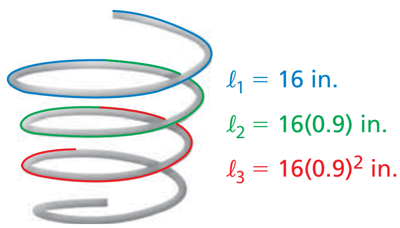
a. What does n represent for each quilt? What does a_n represent?

b. Make a table that shows n and a_n for $n = 1, 2, 3, 4, 5, 6, 7$, and 8.

c. Use the rule $a_n = \frac{n^2}{2} + \frac{1}{4}[1 - (-1)^n]$ to find a_n for $n = 1, 2, 3, 4, 5, 6, 7$, and 8. Compare these values to those in your table in part (b). What can you conclude? Explain.

14. During a baseball season, a company pledges to donate \$5000 to a charity plus \$100 for each home run hit by the local team. Does this situation represent a sequence or a series? Explain your reasoning.

15. The length ℓ_1 of the first loop of a spring is 16 inches. The length ℓ_2 of the second loop is 0.9 times the length of the first loop. The length ℓ_3 of the third loop is 0.9 times the length of the second loop, and so on. Suppose the spring has infinitely many loops, would its length be finite or infinite? Explain. Find the length of the spring, if possible.



8 Cumulative Assessment

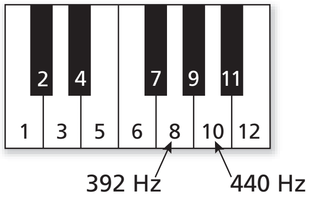
1. The frequencies (in hertz) of the notes on a piano form a geometric sequence. The frequencies of G (labeled 8) and A (labeled 10) are shown in the diagram. What is the approximate frequency of E flat (labeled 4)?

Ⓐ 247 Hz

Ⓑ 311 Hz

Ⓒ 330 Hz

Ⓓ 554 Hz



2. You take out a loan for \$16,000 with an interest rate of 0.75% per month. At the end of each month, you make a payment of \$300.

a. Write a recursive rule for the balance a_n of the loan at the beginning of the nth month.

b. How much do you owe at the beginning of the 18th month?

c. How long will it take to pay off the loan?

d. If you pay \$350 instead of \$300 each month, how long will it take to pay off the loan? How much money will you save? Explain.

3. The table shows that the force F (in pounds) needed to loosen a certain bolt with a wrench depends on the length ℓ (in inches) of the wrench's handle. Write an equation that relates ℓ and F. Describe the relationship.

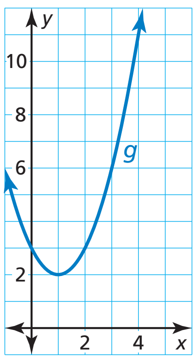
Length, ℓ	4	6	10	12
Force, F	375	250	150	125

4. Order the functions from the least average rate of change to the greatest average rate of change on the interval $1 \le x \le 4$. Justify your answers.

A. $f(x) = 4\sqrt{x} + 2$

B. x and y vary inversely, and $y = 2$ when $x = 5$.

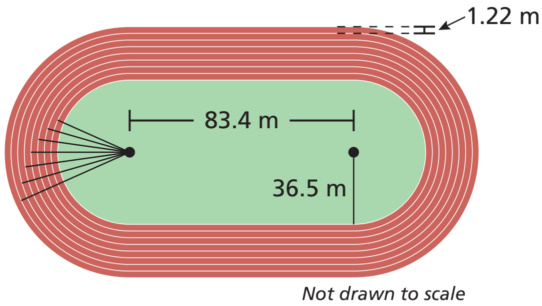
C.



D.

x	y
1	−4
2	−1
3	2
4	5

5. A running track is shaped like a rectangle with two semicircular ends, as shown. The track has 8 lanes that are each 1.22 meters wide. The lanes are numbered from 1 to 8 starting from the inside lane. The distance from the center of a semicircle to the inside of a lane is called the curve radius of that lane. The curve radius of lane 1 is 36.5 meters, as shown in the figure.

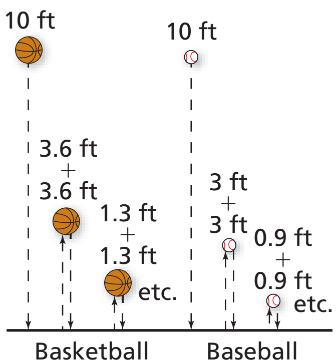


a. Is the sequence formed by the curve radii arithmetic, geometric, or neither? Explain.

b. Write a rule for the sequence formed by the curve radii.

c. World records must be set on tracks that have a curve radius of at most 50 meters in the outside lane. Does the track shown meet the requirement? Explain.

6. The diagram shows the bounce heights of a basketball and a baseball dropped from a height of 10 feet. On each bounce, the basketball bounces to 36% of its previous height, and the baseball bounces to 30% of its previous height. About how much greater is the total distance traveled by the basketball than the total distance traveled by the baseball?



(A) 1.34 feet

(B) 2.00 feet

(C) 2.68 feet

(D) 5.63 feet

7. Classify the solution(s) of each equation as real numbers, imaginary numbers, or pure imaginary numbers. Justify your answers.

a. $x + \sqrt{-16} = 0$

b. $(11 - 2i) - (-3i + 6) = 8 + x$

c. $3x^2 - 14 = -20$

d. $x^2 + 2x = -3$

e. $x^2 = 16$

f. $x^2 - 5x - 8 = 0$