6.1–6.4 Quiz

Tell whether the function represents *exponential growth* or *exponential decay*. Explain your reasoning. *(Sections 6.1 and 6.2)*

1. $f(x) = (4.25)^x$ **2.** $y = \left(\frac{3}{8}\right)^x$ **3.** $y = e^{0.6x}$ **4.** $f(x) = 5e^{-2x}$

Simplify the expression. *(Sections 6.2 and 6.3)*

5. $e^8 \cdot e^4$ **6.** $\frac{15e^3}{3e}$ **7.** $(5e^{4x})^3$

8. $e^{\ln 9}$ **9.** $\log_7 49^x$ **10.** $\log_3 81^{-2x}$

Rewrite the expression in exponential or logarithmic form. *(Section 6.3)*

11. $\log_4 1024 = 5$ **12.** $\log_{1/3} 27 = -3$ **13.** $7^4 = 2401$ **14.** $4^{-2} = 0.0625$

Evaluate the logarithm. If necessary, use a calculator and round your answer to three decimal places. *(Section 6.3)*

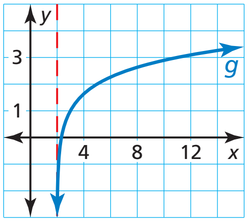
15. $\log 45$ **16.** $\ln 1.4$ **17.** $\log_2 32$

Graph the function and its inverse. *(Section 6.3)*

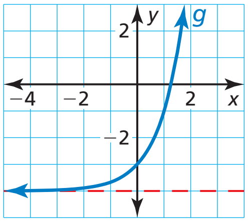
18. $f(x) = \left(\frac{1}{9}\right)^x$ **19.** $y = \ln(x - 7)$ **20.** $f(x) = \log_5(x + 1)$

The graph of *g* is a transformation of the graph of *f*. Write a rule for *g*. *(Section 6.4)*

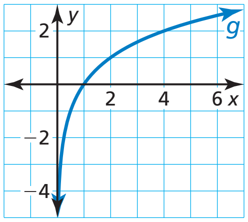
21. $f(x) = \log_3 x$



22. $f(x) = 3^x$



23. $f(x) = \log_{1/2} x$



24. You purchase an antique lamp for \$150. The value of the lamp increases by 2.15% each year. Write an exponential model that gives the value y (in dollars) of the lamp t years after you purchased it. *(Section 6.1)*

25. A local bank advertises two certificate of deposit (CD) accounts that you can use to save money and earn interest. The interest is compounded monthly for both accounts. *(Section 6.1)*

a. You deposit the minimum required amounts in each CD account. How much money is in each account at the end of its term? How much interest does each account earn? Justify your answers.

b. Describe the benefits and drawbacks of each account.

26. The Richter scale is used for measuring the magnitude of an earthquake. The Richter magnitude R is given by $R = 0.67 \ln E + 1.17$, where E is the energy (in kilowatt-hours) released by the earthquake. Graph the model. What is the Richter magnitude for an earthquake that releases 23,000 kilowatt-hours of energy? *(Section 6.4)*

6.5 Properties of Logarithms

Essential Question How can you use properties of exponents to derive properties of logarithms?

Let

$$x = \log_b m \qquad \text{and} \qquad y = \log_b n.$$

The corresponding exponential forms of these two equations are

$$b^x = m \qquad \text{and} \qquad b^y = n.$$

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to understand and use stated assumptions, definitions, and previously established results.

EXPLORATION 1 Product Property of Logarithms

Work with a partner. To derive the Product Property, multiply m and n to obtain

$$mn = b^x b^y = b^{x+y}.$$

The corresponding logarithmic form of $mn = b^{x+y}$ is $\log_b mn = x + y$. So,

$\log_b mn = \underline{\qquad\qquad\qquad}$. Product Property of Logarithms

EXPLORATION 2 Quotient Property of Logarithms

Work with a partner. To derive the Quotient Property, divide m by n to obtain

$$\frac{m}{n} = \frac{b^x}{b^y} = b^{x-y}.$$

The corresponding logarithmic form of $\frac{m}{n} = b^{x-y}$ is $\log_b \frac{m}{n} = x - y$. So,

$\log_b \frac{m}{n} = \underline{\qquad\qquad\qquad}$. Quotient Property of Logarithms

EXPLORATION 3 Power Property of Logarithms

Work with a partner. To derive the Power Property, substitute b^x for m in the expression $\log_b m^n$, as follows.

$\log_b m^n = \log_b(b^x)^n$ Substitute b^x for m.

$= \log_b b^{nx}$ Power of a Power Property of Exponents

$= nx$ Inverse Property of Logarithms

So, substituting $\log_b m$ for x, you have

$\log_b m^n = \underline{\qquad\qquad\qquad}$. Power Property of Logarithms

Communicate Your Answer

4. How can you use properties of exponents to derive properties of logarithms?

5. Use the properties of logarithms that you derived in Explorations 1–3 to evaluate each logarithmic expression.

a. $\log_4 16^3$ **b.** $\log_3 81^{-3}$

c. $\ln e^2 + \ln e^5$ **d.** $2 \ln e^6 - \ln e^5$

e. $\log_5 75 - \log_5 3$ **f.** $\log_4 2 + \log_4 32$

6.5 Lesson

Core Vocabulary

Previous
base
properties of exponents

What You Will Learn

- Use the properties of logarithms to evaluate logarithms.
- Use the properties of logarithms to expand or condense logarithmic expressions.
- Use the change-of-base formula to evaluate logarithms.

Properties of Logarithms

You know that the logarithmic function with base b is the inverse function of the exponential function with base b. Because of this relationship, it makes sense that logarithms have properties similar to properties of exponents.

Core Concept

Properties of Logarithms

Let b, m, and n be positive real numbers with $b \neq 1$.

Product Property $\log_b mn = \log_b m + \log_b n$

Quotient Property $\log_b \frac{m}{n} = \log_b m - \log_b n$

Power Property $\log_b m^n = n \log_b m$

STUDY TIP

These three properties of logarithms correspond to these three properties of exponents.

$a^m a^n = a^{m+n}$

$\frac{a^m}{a^n} = a^{m-n}$

$(a^m)^n = a^{mn}$

EXAMPLE 1 Using Properties of Logarithms

Use $\log_2 3 \approx 1.585$ and $\log_2 7 \approx 2.807$ to evaluate each logarithm.

a. $\log_2 \frac{3}{7}$ **b.** $\log_2 21$ **c.** $\log_2 49$

SOLUTION

a. $\log_2 \frac{3}{7} = \log_2 3 - \log_2 7$ Quotient Property

$\approx 1.585 - 2.807$ Use the given values of $\log_2 3$ and $\log_2 7$.

$= -1.222$ Subtract.

b. $\log_2 21 = \log_2(3 \cdot 7)$ Write 21 as $3 \cdot 7$.

$= \log_2 3 + \log_2 7$ Product Property

$\approx 1.585 + 2.807$ Use the given values of $\log_2 3$ and $\log_2 7$.

$= 4.392$ Add.

c. $\log_2 49 = \log_2 7^2$ Write 49 as 7^2.

$= 2 \log_2 7$ Power Property

$\approx 2(2.807)$ Use the given value $\log_2 7$.

$= 5.614$ Multiply.

COMMON ERROR

Note that in general $\log_b \frac{m}{n} \neq \frac{\log_b m}{\log_b n}$ and $\log_b mn \neq (\log_b m)(\log_b n)$.

Monitoring Progress

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Use $\log_6 5 \approx 0.898$ and $\log_6 8 \approx 1.161$ to evaluate the logarithm.

1. $\log_6 \frac{5}{8}$ **2.** $\log_6 40$ **3.** $\log_6 64$ **4.** $\log_6 125$

Rewriting Logarithmic Expressions

You can use the properties of logarithms to expand and condense logarithmic expressions.

STUDY TIP

When you are expanding or condensing an expression involving logarithms, you can assume that any variables are positive.

EXAMPLE 2 Expanding a Logarithmic Expression

Expand $\ln \frac{5x^7}{y}$.

SOLUTION

$\ln \frac{5x^7}{y} = \ln 5x^7 - \ln y$ Quotient Property

$= \ln 5 + \ln x^7 - \ln y$ Product Property

$= \ln 5 + 7 \ln x - \ln y$ Power Property

EXAMPLE 3 Condensing a Logarithmic Expression

Condense $\log 9 + 3 \log 2 - \log 3$.

SOLUTION

$\log 9 + 3 \log 2 - \log 3 = \log 9 + \log 2^3 - \log 3$ Power Property

$= \log(9 \cdot 2^3) - \log 3$ Product Property

$= \log \frac{9 \cdot 2^3}{3}$ Quotient Property

$= \log 24$ Simplify.

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Expand the logarithmic expression.

5. $\log_6 3x^4$

6. $\ln \frac{5}{12x}$

Condense the logarithmic expression.

7. $\log x - \log 9$

8. $\ln 4 + 3 \ln 3 - \ln 12$

Change-of-Base Formula

Logarithms with any base other than 10 or e can be written in terms of common or natural logarithms using the *change-of-base formula*. This allows you to evaluate any logarithm using a calculator.

Core Concept

Change-of-Base Formula

If a, b, and c are positive real numbers with $b \neq 1$ and $c \neq 1$, then

$$\log_c a = \frac{\log_b a}{\log_b c}.$$

In particular, $\log_c a = \frac{\log a}{\log c}$ and $\log_c a = \frac{\ln a}{\ln c}$.

EXAMPLE 4 Changing a Base Using Common Logarithms

Evaluate $\log_3 8$ using common logarithms.

SOLUTION

$$\log_3 8 = \frac{\log 8}{\log 3} \qquad \log_c a = \frac{\log a}{\log c}$$

$$\approx \frac{0.9031}{0.4771} \approx 1.893 \qquad \text{Use a calculator. Then divide.}$$

ANOTHER WAY

In Example 4, $\log_3 8$ can be evaluated using natural logarithms.

$$\log_3 8 = \frac{\ln 8}{\ln 3} \approx 1.893$$

Notice that you get the same answer whether you use natural logarithms or common logarithms in the change-of-base formula.

EXAMPLE 5 Changing a Base Using Natural Logarithms

Evaluate $\log_6 24$ using natural logarithms.

SOLUTION

$$\log_6 24 = \frac{\ln 24}{\ln 6} \qquad \log_c a = \frac{\ln a}{\ln c}$$

$$\approx \frac{3.1781}{1.7918} \approx 1.774 \qquad \text{Use a calculator. Then divide.}$$

EXAMPLE 6 Solving a Real-Life Problem

For a sound with intensity I (in watts per square meter), the loudness $L(I)$ of the sound (in decibels) is given by the function

$$L(I) = 10 \log \frac{I}{I_0}$$

where I_0 is the intensity of a barely audible sound (about 10^{-12} watts per square meter). An artist in a recording studio turns up the volume of a track so that the intensity of the sound doubles. By how many decibels does the loudness increase?

SOLUTION

Let I be the original intensity, so that $2I$ is the doubled intensity.

$$\text{increase in loudness} = L(2I) - L(I) \qquad \text{Write an expression.}$$

$$= 10 \log \frac{2I}{I_0} - 10 \log \frac{I}{I_0} \qquad \text{Substitute.}$$

$$= 10 \left(\log \frac{2I}{I_0} - \log \frac{I}{I_0} \right) \qquad \text{Distributive Property}$$

$$= 10 \left(\log 2 + \log \frac{I}{I_0} - \log \frac{I}{I_0} \right) \qquad \text{Product Property}$$

$$= 10 \log 2 \qquad \text{Simplify.}$$

▶ The loudness increases by 10 log 2 decibels, or about 3 decibels.

Monitoring Progress

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Use the change-of-base formula to evaluate the logarithm.

9. $\log_5 8$ **10.** $\log_8 14$ **11.** $\log_{26} 9$ **12.** $\log_{12} 30$

13. WHAT IF? In Example 6, the artist turns up the volume so that the intensity of the sound triples. By how many decibels does the loudness increase?

6.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** To condense the expression $\log_3 2x + \log_3 y$, you need to use the __________ Property of Logarithms.

2. **WRITING** Describe two ways to evaluate $\log_7 12$ using a calculator.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, use $\log_7 4 \approx 0.712$ and $\log_7 12 \approx 1.277$ to evaluate the logarithm. *(See Example 1.)*

3. $\log_7 3$
4. $\log_7 48$
5. $\log_7 16$
6. $\log_7 64$
7. $\log_7 \frac{1}{4}$
8. $\log_7 \frac{1}{3}$

In Exercises 9–12, match the expression with the logarithm that has the same value. Justify your answer.

9. $\log_3 6 - \log_3 2$
10. $2 \log_3 6$
11. $6 \log_3 2$
12. $\log_3 6 + \log_3 2$

A. $\log_3 64$
B. $\log_3 3$
C. $\log_3 12$
D. $\log_3 36$

In Exercises 13–20, expand the logarithmic expression. *(See Example 2.)*

13. $\log_3 4x$
14. $\log_8 3x$
15. $\log 10x^5$
16. $\ln 3x^4$
17. $\ln \frac{x}{3y}$
18. $\ln \frac{6x^2}{y^4}$
19. $\log_7 5\sqrt{x}$
20. $\log_5 \sqrt[3]{x^2 y}$

ERROR ANALYSIS In Exercises 21 and 22, describe and correct the error in expanding the logarithmic expression.

21. ✗ $\log_2 5x = (\log_2 5)(\log_2 x)$

22. ✗ $\ln 8x^3 = 3 \ln 8 + \ln x$

In Exercises 23–30, condense the logarithmic expression. *(See Example 3.)*

23. $\log_4 7 - \log_4 10$
24. $\ln 12 - \ln 4$
25. $6 \ln x + 4 \ln y$
26. $2 \log x + \log 11$
27. $\log_5 4 + \frac{1}{3} \log_5 x$
28. $6 \ln 2 - 4 \ln y$
29. $5 \ln 2 + 7 \ln x + 4 \ln y$
30. $\log_3 4 + 2 \log_3 \frac{1}{2} + \log_3 x$

31. **REASONING** Which of the following is *not* equivalent to $\log_5 \frac{y^4}{3x}$? Justify your answer.

(A) $4 \log_5 y - \log_5 3x$

(B) $4 \log_5 y - \log_5 3 + \log_5 x$

(C) $4 \log_5 y - \log_5 3 - \log_5 x$

(D) $\log_5 y^4 - \log_5 3 - \log_5 x$

32. **REASONING** Which of the following equations is correct? Justify your answer.

(A) $\log_7 x + 2 \log_7 y = \log_7(x + y^2)$

(B) $9 \log x - 2 \log y = \log \frac{x^9}{y^2}$

(C) $5 \log_4 x + 7 \log_2 y = \log_6 x^5 y^7$

(D) $\log_9 x - 5 \log_9 y = \log_9 \frac{x}{5y}$

In Exercises 33–40, use the change-of-base formula to evaluate the logarithm. *(See Examples 4 and 5.)*

33. $\log_4 7$

34. $\log_5 13$

35. $\log_9 15$

36. $\log_8 22$

37. $\log_6 17$

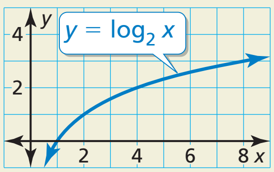
38. $\log_2 28$

39. $\log_7 \frac{3}{16}$

40. $\log_3 \frac{9}{40}$

41. MAKING AN ARGUMENT Your friend claims you can use the change-of-base formula to graph $y = \log_3 x$ using a graphing calculator. Is your friend correct? Explain your reasoning.

42. HOW DO YOU SEE IT? Use the graph to determine the value of $\dfrac{\log 8}{\log 2}$.

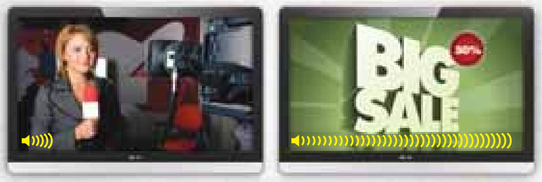


MODELING WITH MATHEMATICS **In Exercises 43 and 44, use the function $L(I)$ given in Example 6.**

43. The blue whale can produce sound with an intensity that is 1 million times greater than the intensity of the loudest sound a human can make. Find the difference in the decibel levels of the sounds made by a blue whale and a human. *(See Example 6.)*

44. The intensity of the sound of a certain television advertisement is 10 times greater than the intensity of the television program. By how many decibels does the loudness increase?

Intensity of Television Sound



During show: Intensity $= I$

During ad: Intensity $= 10I$

45. REWRITING A FORMULA Under certain conditions, the wind speed s (in knots) at an altitude of h meters above a grassy plain can be modeled by the function

$$s(h) = 2 \ln 100h.$$

a. By what amount does the wind speed increase when the altitude doubles?

b. Show that the given function can be written in terms of common logarithms as

$$s(h) = \frac{2}{\log e}(\log h + 2).$$

46. THOUGHT PROVOKING Determine whether the formula

$$\log_b(M + N) = \log_b M + \log_b N$$

is true for all positive, real values of M, N, and b (with $b \neq 1$). Justify your answer.

47. USING STRUCTURE Use the properties of exponents to prove the change-of-base formula. (*Hint:* Let $x = \log_b a$, $y = \log_b c$, and $z = \log_c a$.)

48. CRITICAL THINKING Describe *three* ways to transform the graph of $f(x) = \log x$ to obtain the graph of $g(x) = \log 100x - 1$. Justify your answers.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the inequality by graphing. *(Section 3.6)*

49. $x^2 - 4 > 0$

50. $2(x - 6)^2 - 5 \geq 37$

51. $x^2 + 13x + 42 < 0$

52. $-x^2 - 4x + 6 \leq -6$

Solve the equation by graphing the related system of equations. *(Section 3.5)*

53. $4x^2 - 3x - 6 = -x^2 + 5x + 3$

54. $-(x + 3)(x - 2) = x^2 - 6x$

55. $2x^2 - 4x - 5 = -(x + 3)^2 + 10$

56. $-(x + 7)^2 + 5 = (x + 10)^2 - 3$

6.6 Solving Exponential and Logarithmic Equations

Essential Question

How can you solve exponential and logarithmic equations?

EXPLORATION 1 Solving Exponential and Logarithmic Equations

Work with a partner. Match each equation with the graph of its related system of equations. Explain your reasoning. Then use the graph to solve the equation.

a. $e^x = 2$

b. $\ln x = -1$

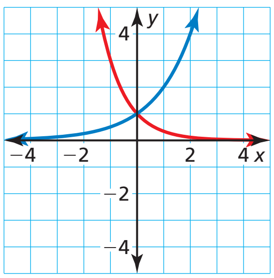
c. $2^x = 3^{-x}$

d. $\log_4 x = 1$

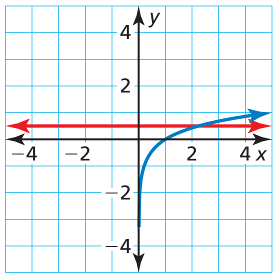
e. $\log_5 x = \frac{1}{2}$

f. $4^x = 2$

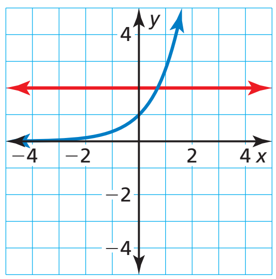
A.



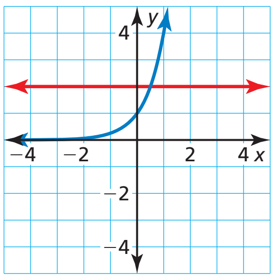
B.



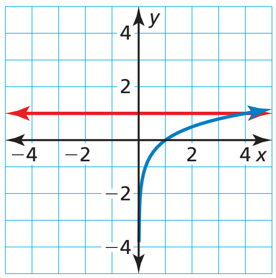
C.



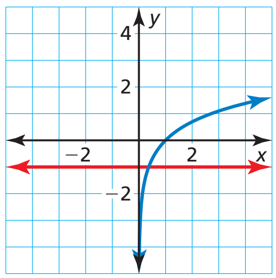
D.



E.



F.



MAKING SENSE OF PROBLEMS

To be proficient in math, you need to plan a solution pathway rather than simply jumping into a solution attempt.

EXPLORATION 2 Solving Exponential and Logarithmic Equations

Work with a partner. Look back at the equations in Explorations 1(a) and 1(b). Suppose you want a more accurate way to solve the equations than using a graphical approach.

a. Show how you could use a *numerical approach* by creating a table. For instance, you might use a spreadsheet to solve the equations.

b. Show how you could use an *analytical approach*. For instance, you might try solving the equations by using the inverse properties of exponents and logarithms.

Communicate Your Answer

3. How can you solve exponential and logarithmic equations?

4. Solve each equation using any method. Explain your choice of method.

a. $16^x = 2$

b. $2^x = 4^{2x+1}$

c. $2^x = 3^{x+1}$

d. $\log x = \frac{1}{2}$

e. $\ln x = 2$

f. $\log_3 x = \frac{3}{2}$

6.6 Lesson

What You Will Learn

- Solve exponential equations.
- Solve logarithmic equations.
- Solve exponential and logarithmic inequalities.

Core Vocabulary

exponential equations, *p. 334*
logarithmic equations, *p. 335*

Previous
extraneous solution
inequality

Solving Exponential Equations

Exponential equations are equations in which variable expressions occur as exponents. The result below is useful for solving certain exponential equations.

Core Concept

Property of Equality for Exponential Equations

Algebra If b is a positive real number other than 1, then $b^x = b^y$ if and only if $x = y$.

Example If $3^x = 3^5$, then $x = 5$. If $x = 5$, then $3^x = 3^5$.

The preceding property is useful for solving an exponential equation when each side of the equation uses the same base (or can be rewritten to use the same base). When it is not convenient to write each side of an exponential equation using the same base, you can try to solve the equation by taking a logarithm of each side.

EXAMPLE 1 Solving Exponential Equations

Solve each equation.

a. $100^x = \left(\frac{1}{10}\right)^{x-3}$ **b.** $2^x = 7$

SOLUTION

a.

$100^x = \left(\frac{1}{10}\right)^{x-3}$ Write original equation.

$(10^2)^x = (10^{-1})^{x-3}$ Rewrite 100 and $\frac{1}{10}$ as powers with base 10.

$10^{2x} = 10^{-x+3}$ Power of a Power Property

$2x = -x + 3$ Property of Equality for Exponential Equations

$x = 1$ Solve for x.

Check

$$100^1 \stackrel{?}{=} \left(\frac{1}{10}\right)^{1-3}$$

$$100 \stackrel{?}{=} \left(\frac{1}{10}\right)^{-2}$$

$$100 = 100 \checkmark$$

b.

$2^x = 7$ Write original equation.

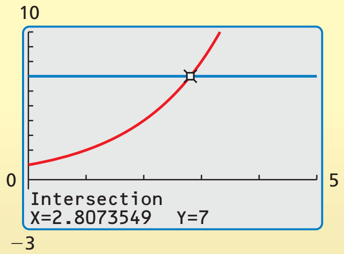
$\log_2 2^x = \log_2 7$ Take $\log_2$ of each side.

$x = \log_2 7$ $\log_b b^x = x$

$x \approx 2.807$ Use a calculator.

Check

Enter $y = 2^x$ and $y = 7$ in a graphing calculator. Use the *intersect* feature to find the intersection point of the graphs. The graphs intersect at about (2.807, 7). So, the solution of $2^x = 7$ is about 2.807. ✓



An important application of exponential equations is *Newton's Law of Cooling*. This law states that for a cooling substance with initial temperature T_0, the temperature T after t minutes can be modeled by

$$T = (T_0 - T_R)e^{-rt} + T_R$$

where T_R is the surrounding temperature and r is the cooling rate of the substance.

LOOKING FOR STRUCTURE

Notice that Newton's Law of Cooling models the temperature of a cooling body by adding a constant function, T_R, to a decaying exponential function, $(T_0 - T_R)e^{-rt}$.

EXAMPLE 2 Solving a Real-Life Problem

You are cooking *aleecha*, an Ethiopian stew. When you take it off the stove, its temperature is 212°F. The room temperature is 70°F, and the cooling rate of the stew is $r = 0.046$. How long will it take to cool the stew to a serving temperature of 100°F?

SOLUTION

Use Newton's Law of Cooling with $T = 100$, $T_0 = 212$, $T_R = 70$, and $r = 0.046$.

$T = (T_0 - T_R)e^{-rt} + T_R$	Newton's Law of Cooling
$100 = (212 - 70)e^{-0.046t} + 70$	Substitute for T, T_0, T_R, and r.
$30 = 142e^{-0.046t}$	Subtract 70 from each side.
$0.211 \approx e^{-0.046t}$	Divide each side by 142.
$\ln 0.211 \approx \ln e^{-0.046t}$	Take natural log of each side.
$-1.556 \approx -0.046t$	$\ln e^x = \log_e e^x = x$
$33.8 \approx t$	Divide each side by -0.046.

▶ You should wait about 34 minutes before serving the stew.

Monitoring Progress

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Solve the equation.

1. $2^x = 5$

2. $7^{9x} = 15$

3. $4e^{-0.3x} - 7 = 13$

4. WHAT IF? In Example 2, how long will it take to cool the stew to 100°F when the room temperature is 75°F?

Solving Logarithmic Equations

Logarithmic equations are equations that involve logarithms of variable expressions. You can use the next property to solve some types of logarithmic equations.

Core Concept

Property of Equality for Logarithmic Equations

Algebra If b, x, and y are positive real numbers with $b \neq 1$, then $\log_b x = \log_b y$ if and only if $x = y$.

Example If $\log_2 x = \log_2 7$, then $x = 7$. If $x = 7$, then $\log_2 x = \log_2 7$.

The preceding property implies that if you are given an equation $x = y$, then you can exponentiate each side to obtain an equation of the form $b^x = b^y$. This technique is useful for solving some logarithmic equations.

EXAMPLE 3 Solving Logarithmic Equations

Solve (a) $\ln(4x - 7) = \ln(x + 5)$ and (b) $\log_2(5x - 17) = 3$.

SOLUTION

a. $\ln(4x - 7) = \ln(x + 5)$ Write original equation.

$4x - 7 = x + 5$ Property of Equality for Logarithmic Equations

$3x - 7 = 5$ Subtract x from each side.

$3x = 12$ Add 7 to each side.

$x = 4$ Divide each side by 3.

Check

$\ln(4 \cdot 4 - 7) \stackrel{?}{=} \ln(4 + 5)$

$\ln(16 - 7) \stackrel{?}{=} \ln 9$

$\ln 9 = \ln 9$ ✓

b. $\log_2(5x - 17) = 3$ Write original equation.

$2^{\log_2(5x-17)} = 2^3$ Exponentiate each side using base 2.

$5x - 17 = 8$ $b^{\log_b x} = x$

$5x = 25$ Add 17 to each side.

$x = 5$ Divide each side by 5.

Check

$\log_2(5 \cdot 5 - 17) \stackrel{?}{=} 3$

$\log_2(25 - 17) \stackrel{?}{=} 3$

$\log_2 8 \stackrel{?}{=} 3$

Because $2^3 = 8$, $\log_2 8 = 3$. ✓

Because the domain of a logarithmic function generally does not include all real numbers, be sure to check for extraneous solutions of logarithmic equations. You can do this algebraically or graphically.

EXAMPLE 4 Solving a Logarithmic Equation

Solve $\log 2x + \log(x - 5) = 2$.

SOLUTION

$\log 2x + \log(x - 5) = 2$ Write original equation.

$\log[2x(x - 5)] = 2$ Product Property of Logarithms

$10^{\log[2x(x-5)]} = 10^2$ Exponentiate each side using base 10.

$2x(x - 5) = 100$ $b^{\log_b x} = x$

$2x^2 - 10x = 100$ Distributive Property

$2x^2 - 10x - 100 = 0$ Write in standard form.

$x^2 - 5x - 50 = 0$ Divide each side by 2.

$(x - 10)(x + 5) = 0$ Factor.

$x = 10$ or $x = -5$ Zero-Product Property

▶ The apparent solution $x = -5$ is extraneous. So, the only solution is $x = 10$.

Check

$\log(2 \cdot 10) + \log(10 - 5) \stackrel{?}{=} 2$

$\log 20 + \log 5 \stackrel{?}{=} 2$

$\log 100 \stackrel{?}{=} 2$

$2 = 2$ ✓

$\log[2 \cdot (-5)] + \log(-5 - 5) \stackrel{?}{=} 2$

$\log(-10) + \log(-10) \stackrel{?}{=} 2$

Because $\log(-10)$ is not defined, -5 is not a solution. ✗

Monitoring Progress

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Solve the equation. Check for extraneous solutions.

5. $\ln(7x - 4) = \ln(2x + 11)$

6. $\log_2(x - 6) = 5$

7. $\log 5x + \log(x - 1) = 2$

8. $\log_4(x + 12) + \log_4 x = 3$

Solving Exponential and Logarithmic Inequalities

Exponential inequalities are inequalities in which variable expressions occur as exponents, and *logarithmic inequalities* are inequalities that involve logarithms of variable expressions. To solve exponential and logarithmic inequalities algebraically, use these properties. Note that the properties are true for $\leq$ and $\geq$.

STUDY TIP

Be sure you understand that these properties of inequality are only true for values of $b > 1$.

Exponential Property of Inequality: If b is a positive real number greater than 1, then $b^x > b^y$ if and only if $x > y$, and $b^x < b^y$ if and only if $x < y$.

Logarithmic Property of Inequality: If b, x, and y are positive real numbers with $b > 1$, then $\log_b x > \log_b y$ if and only if $x > y$, and $\log_b x < \log_b y$ if and only if $x < y$.

You can also solve an inequality by taking a logarithm of each side or by exponentiating.

EXAMPLE 5 Solving an Exponential Inequality

Solve $3^x < 20$.

SOLUTION

$3^x < 20$ — Write original inequality.

$\log_3 3^x < \log_3 20$ — Take $\log_3$ of each side.

$x < \log_3 20$ — $\log_b b^x = x$

▶ The solution is $x < \log_3 20$. Because $\log_3 20 \approx 2.727$, the approximate solution is $x < 2.727$.

EXAMPLE 6 Solving a Logarithmic Inequality

Solve $\log x \leq 2$.

SOLUTION

Method 1 Use an algebraic approach.

$\log x \leq 2$ — Write original inequality.

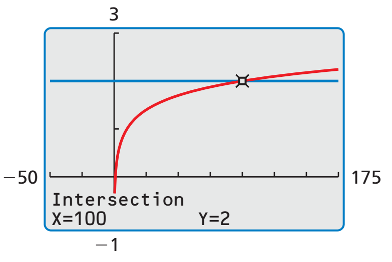
$10^{\log_{10} x} \leq 10^2$ — Exponentiate each side using base 10.

$x \leq 100$ — $b^{\log_b x} = x$

▶ Because $\log x$ is only defined when $x > 0$, the solution is $0 < x \leq 100$.

Method 2 Use a graphical approach.

Graph $y = \log x$ and $y = 2$ in the same viewing window. Use the *intersect* feature to determine that the graphs intersect when $x = 100$. The graph of $y = \log x$ is on or below the graph of $y = 2$ when $0 < x \leq 100$.



▶ The solution is $0 < x \leq 100$.

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Solve the inequality.

9. $e^x < 2$ **10.** $10^{2x-6} > 3$ **11.** $\log x + 9 < 45$ **12.** $2 \ln x - 1 > 4$

6.6 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The equation $3^{x-1} = 34$ is an example of a(n) ___________ equation.

2. **WRITING** Compare the methods for solving exponential and logarithmic equations.

3. **WRITING** When do logarithmic equations have extraneous solutions?

4. **COMPLETE THE SENTENCE** If b is a positive real number other than 1, then $b^x = b^y$ if and only if _________.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–16, solve the equation. *(See Example 1.)*

5. $7^{3x+5} = 7^{1-x}$

6. $e^{2x} = e^{3x-1}$

7. $5^{x-3} = 25^{x-5}$

8. $6^{2x-6} = 36^{3x-5}$

9. $3^x = 7$

10. $5^x = 33$

11. $49^{5x+2} = \left(\frac{1}{7}\right)^{11-x}$

12. $512^{5x-1} = \left(\frac{1}{8}\right)^{-4-x}$

13. $7^{5x} = 12$

14. $11^{6x} = 38$

15. $3e^{4x} + 9 = 15$

16. $2e^{2x} - 7 = 5$

17. **MODELING WITH MATHEMATICS** The length ℓ (in centimeters) of a scalloped hammerhead shark can be modeled by the function

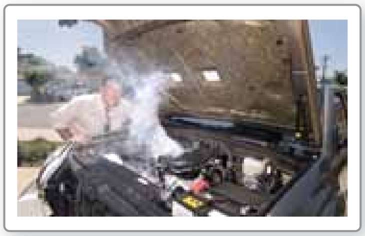
$$\ell = 266 - 219e^{-0.05t}$$

where t is the age (in years) of the shark. How old is a shark that is 175 centimeters long?

18. **MODELING WITH MATHEMATICS** One hundred grams of radium are stored in a container. The amount R (in grams) of radium present after t years can be modeled by $R = 100e^{-0.00043t}$. After how many years will only 5 grams of radium be present?

In Exercises 19 and 20, use Newton's Law of Cooling to solve the problem. *(See Example 2.)*

19. You are driving on a hot day when your car overheats and stops running. The car overheats at 280°F and can be driven again at 230°F. When it is 80°F outside, the cooling rate of the car is $r = 0.0058$. How long do you have to wait until you can continue driving?



20. You cook a turkey until the internal temperature reaches 180°F. The turkey is placed on the table until the internal temperature reaches 100°F and it can be carved. When the room temperature is 72°F, the cooling rate of the turkey is $r = 0.067$. How long do you have to wait until you can carve the turkey?

In Exercises 21–32, solve the equation. *(See Example 3.)*

21. $\ln(4x - 7) = \ln(x + 11)$

22. $\ln(2x - 4) = \ln(x + 6)$

23. $\log_2(3x - 4) = \log_2 5$

24. $\log(7x + 3) = \log 38$

25. $\log_2(4x + 8) = 5$

26. $\log_3(2x + 1) = 2$

27. $\log_7(4x + 9) = 2$

28. $\log_5(5x + 10) = 4$

29. $\log(12x - 9) = \log 3x$

30. $\log_6(5x + 9) = \log_6 6x$

31. $\log_2(x^2 - x - 6) = 2$

32. $\log_3(x^2 + 9x + 27) = 2$

In Exercises 33–40, solve the equation. Check for extraneous solutions. *(See Example 4.)*

33. $\log_2 x + \log_2(x - 2) = 3$

34. $\log_6 3x + \log_6(x - 1) = 3$

35. $\ln x + \ln(x + 3) = 4$

36. $\ln x + \ln(x - 2) = 5$

37. $\log_3 3x^2 + \log_3 3 = 2$

38. $\log_4(-x) + \log_4(x + 10) = 2$

39. $\log_3(x - 9) + \log_3(x - 3) = 2$

40. $\log_5(x + 4) + \log_5(x + 1) = 2$

ERROR ANALYSIS In Exercises 41 and 42, describe and correct the error in solving the equation.

41.

$$\log_3(5x - 1) = 4$$
$$3^{\log_3(5x - 1)} = 4^3$$
$$5x - 1 = 64$$
$$5x = 65$$
$$x = 13$$

42.

$$\log_4(x + 12) + \log_4 x = 3$$
$$\log_4[(x + 12)(x)] = 3$$
$$4^{\log_4[(x + 12)(x)]} = 4^3$$
$$(x + 12)(x) = 64$$
$$x^2 + 12x - 64 = 0$$
$$(x + 16)(x - 4) = 0$$
$$x = -16 \quad \text{or} \quad x = 4$$

43. PROBLEM SOLVING You deposit \$100 in an account that pays 6% annual interest. How long will it take for the balance to reach \$1000 for each frequency of compounding?

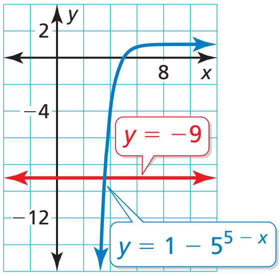
a. annual **b.** quarterly

c. daily **d.** continuously

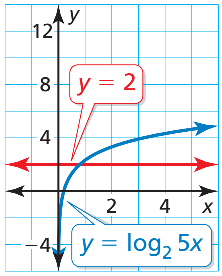
44. MODELING WITH MATHEMATICS The *apparent magnitude* of a star is a measure of the brightness of the star as it appears to observers on Earth. The apparent magnitude M of the dimmest star that can be seen with a telescope is $M = 5 \log D + 2$, where D is the diameter (in millimeters) of the telescope's objective lens. What is the diameter of the objective lens of a telescope that can reveal stars with a magnitude of 12?

45. ANALYZING RELATIONSHIPS Approximate the solution of each equation using the graph.

a. $1 - 5^{5-x} = -9$



b. $\log_2 5x = 2$



46. MAKING AN ARGUMENT Your friend states that a logarithmic equation cannot have a negative solution because logarithmic functions are not defined for negative numbers. Is your friend correct? Justify your answer.

In Exercises 47–54, solve the inequality. *(See Examples 5 and 6.)*

47. $9^x > 54$

48. $4^x \le 36$

49. $\ln x \ge 3$

50. $\log_4 x < 4$

51. $3^{4x-5} < 8$

52. $e^{3x+4} > 11$

53. $-3 \log_5 x + 6 \le 9$

54. $-4 \log_5 x - 5 \ge 3$

55. COMPARING METHODS Solve $\log_5 x < 2$ algebraically and graphically. Which method do you prefer? Explain your reasoning.

56. PROBLEM SOLVING You deposit \$1000 in an account that pays 3.5% annual interest compounded monthly. When is your balance at least \$1200? \$3500?

57. PROBLEM SOLVING An investment that earns a rate of return r doubles in value in t years, where $t = \dfrac{\ln 2}{\ln(1 + r)}$ and r is expressed as a decimal. What rates of return will double the value of an investment in less than 10 years?

58. PROBLEM SOLVING Your family purchases a new car for \$20,000. Its value decreases by 15% each year. During what interval does the car's value exceed \$10,000?

USING TOOLS In Exercises 59–62, use a graphing calculator to solve the equation.

59. $\ln 2x = 3^{-x+2}$

60. $\log x = 7^{-x}$

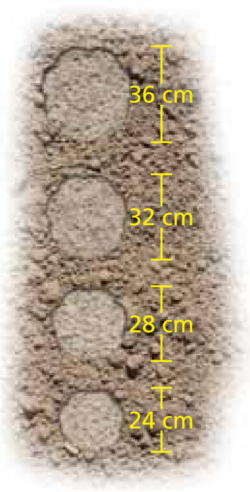
61. $\log x = 3^{x-3}$

62. $\ln 2x = e^{x-3}$

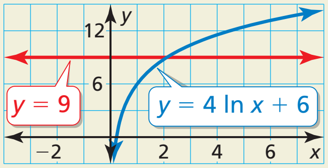
63. **REWRITING A FORMULA** A biologist can estimate the age of an African elephant by measuring the length of its footprint and using the equation $\ell = 45 - 25.7e^{-0.09a}$, where ℓ is the length (in centimeters) of the footprint and a is the age (in years).

a. Rewrite the equation, solving for a in terms of ℓ.

b. Use the equation in part (a) to find the ages of the elephants whose footprints are shown.



64. **HOW DO YOU SEE IT?** Use the graph to solve the inequality $4 \ln x + 6 > 9$. Explain your reasoning.



65. **OPEN-ENDED** Write an exponential equation that has a solution of $x = 4$. Then write a logarithmic equation that has a solution of $x = -3$.

66. **THOUGHT PROVOKING** Give examples of logarithmic or exponential equations that have one solution, two solutions, and no solutions.

CRITICAL THINKING In Exercises 67–72, solve the equation.

67. $2^{x+3} = 5^{3x-1}$

68. $10^{3x-8} = 2^{5-x}$

69. $\log_3(x - 6) = \log_9 2x$

70. $\log_4 x = \log_8 4x$

71. $2^{2x} - 12 \cdot 2^x + 32 = 0$

72. $5^{2x} + 20 \cdot 5^x - 125 = 0$

73. **WRITING** In Exercises 67–70, you solved exponential and logarithmic equations with different bases. Describe general methods for solving such equations.

74. **PROBLEM SOLVING** When X-rays of a fixed wavelength strike a material x centimeters thick, the intensity $I(x)$ of the X-rays transmitted through the material is given by $I(x) = I_0e^{-\mu x}$, where I_0 is the initial intensity and μ is a value that depends on the type of material and the wavelength of the X-rays. The table shows the values of μ for various materials and X-rays of medium wavelength.

Material	Aluminum	Copper	Lead
Value of μ	0.43	3.2	43

a. Find the thickness of aluminum shielding that reduces the intensity of X-rays to 30% of their initial intensity. (*Hint*: Find the value of x for which $I(x) = 0.3I_0$.)

b. Repeat part (a) for the copper shielding.

c. Repeat part (a) for the lead shielding.

d. Your dentist puts a lead apron on you before taking X-rays of your teeth to protect you from harmful radiation. Based on your results from parts (a)–(c), explain why lead is a better material to use than aluminum or copper.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write an equation in point-slope form of the line that passes through the given point and has the given slope. *(Skills Review Handbook)*

75. $(1, -2)$; $m = 4$

76. $(3, 2)$; $m = -2$

77. $(3, -8)$; $m = -\frac{1}{3}$

78. $(2, 5)$; $m = 2$

Use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function. *(Section 4.9)*

79. $(-3, -50), (-2, -13), (-1, 0), (0, 1), (1, 2), (2, 15), (3, 52), (4, 125)$

80. $(-3, 139), (-2, 32), (-1, 1), (0, -2), (1, -1), (2, 4), (3, 37), (4, 146)$

81. $(-3, -327), (-2, -84), (-1, -17), (0, -6), (1, -3), (2, -32), (3, -189), (4, -642)$

6.7 Modeling with Exponential and Logarithmic Functions

Essential Question How can you recognize polynomial, exponential, and logarithmic models?

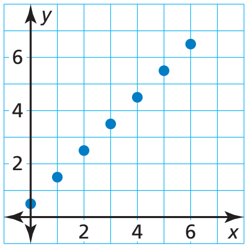
EXPLORATION 1 Recognizing Different Types of Models

Work with a partner. Match each type of model with the appropriate scatter plot. Use a regression program to find a model that fits the scatter plot.

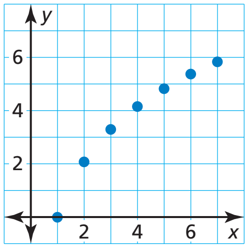
a. linear (positive slope) **b.** linear (negative slope) **c.** quadratic

d. cubic **e.** exponential **f.** logarithmic

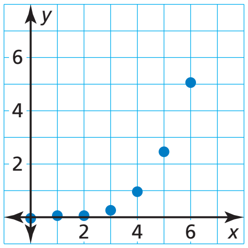
A.



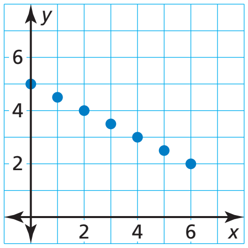
B.



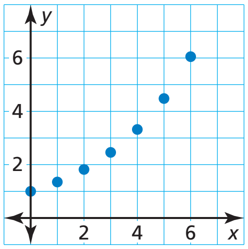
C.



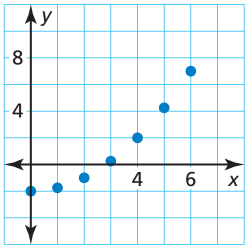
D.



E.



F.



USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technological tools to explore and deepen your understanding of concepts.

EXPLORATION 2 Exploring Gaussian and Logistic Models

Work with a partner. Two common types of functions that are related to exponential functions are given. Use a graphing calculator to graph each function. Then determine the domain, range, intercept, and asymptote(s) of the function.

a. Gaussian Function: $f(x) = e^{-x^2}$ **b.** Logistic Function: $f(x) = \dfrac{1}{1 + e^{-x}}$

Communicate Your Answer

3. How can you recognize polynomial, exponential, and logarithmic models?

4. Use the Internet or some other reference to find real-life data that can be modeled using one of the types given in Exploration 1. Create a table and a scatter plot of the data. Then use a regression program to find a model that fits the data.

6.7 Lesson

Core Vocabulary

Previous
finite differences
common ratio
point-slope form

What You Will Learn

- Classify data sets.
- Write exponential functions.
- Use technology to find exponential and logarithmic models.

Classifying Data

You have analyzed *finite differences* of data with equally-spaced inputs to determine what type of polynomial function can be used to model the data. For exponential data with equally-spaced inputs, the outputs are multiplied by a constant factor. So, consecutive outputs form a constant ratio.

EXAMPLE 1 Classifying Data Sets

Determine the type of function represented by each table.

a.

x	-2	-1	0	1	2	3	4
y	0.5	1	2	4	8	16	32

b.

x	-2	0	2	4	6	8	10
y	2	0	2	8	18	32	50

SOLUTION

a. The inputs are equally spaced. Look for a pattern in the outputs.

x	-2	-1	0	1	2	3	4
y	0.5	1	2	4	8	16	32

$\times 2$ $\quad \times 2$ $\quad \times 2$ $\quad \times 2$ $\quad \times 2$ $\quad \times 2$

As x increases by 1, y is multiplied by 2. So, the common ratio is 2, and the data in the table represent an exponential function.

b. The inputs are equally spaced. The outputs do not have a common ratio. So, analyze the finite differences.

x	-2	0	2	4	6	8	10
y	2	0	2	8	18	32	50

$-2 \quad 2 \quad 6 \quad 10 \quad 14 \quad 18$ first differences

$4 \quad 4 \quad 4 \quad 4 \quad 4$ second differences

The second differences are constant. So, the data in the table represent a quadratic function.

REMEMBER

First differences of linear functions are constant, second differences of quadratic functions are constant, and so on.

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Determine the type of function represented by the table. Explain your reasoning.

1.

x	0	10	20	30
y	15	12	9	6

2.

x	0	2	4	6
y	27	9	3	1

Writing Exponential Functions

You know that two points determine a line. Similarly, two points determine an exponential curve.

EXAMPLE 2 Writing an Exponential Function Using Two Points

Write an exponential function $y = ab^x$ whose graph passes through (1, 6) and (3, 54).

SOLUTION

Step 1 Substitute the coordinates of the two given points into $y = ab^x$.

$$6 = ab^1 \qquad \text{Equation 1: Substitute 6 for } y \text{ and 1 for } x.$$

$$54 = ab^3 \qquad \text{Equation 2: Substitute 54 for } y \text{ and 3 for } x.$$

Step 2 Solve for a in Equation 1 to obtain $a = \frac{6}{b}$ and substitute this expression for a in Equation 2.

$$54 = \left(\frac{6}{b}\right)b^3 \qquad \text{Substitute } \frac{6}{b} \text{ for } a \text{ in Equation 2.}$$

$$54 = 6b^2 \qquad \text{Simplify.}$$

$$9 = b^2 \qquad \text{Divide each side by 6.}$$

$$3 = b \qquad \text{Take the positive square root because } b > 0.$$

REMEMBER

You know that b must be positive by the definition of an exponential function.

Step 3 Determine that $a = \frac{6}{b} = \frac{6}{3} = 2$.

▶ So, the exponential function is $y = 2(3^x)$.

Data do not always show an *exact* exponential relationship. When the data in a scatter plot show an *approximately* exponential relationship, you can model the data with an exponential function.

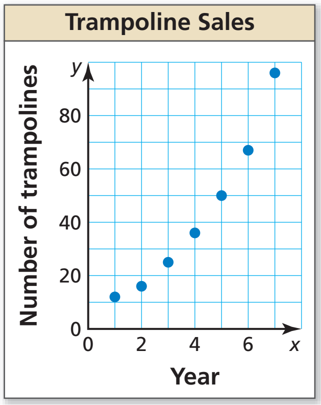
EXAMPLE 3 Finding an Exponential Model

A store sells trampolines. The table shows the numbers y of trampolines sold during the xth year that the store has been open. Write a function that models the data.

Year, x	Number of trampolines, y
1	12
2	16
3	25
4	36
5	50
6	67
7	96

SOLUTION

Step 1 Make a scatter plot of the data. The data appear exponential.



Step 2 Choose any two points to write a model, such as (1, 12) and (4, 36). Substitute the coordinates of these two points into $y = ab^x$.

$$12 = ab^1$$

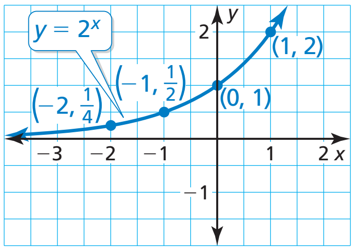
$$36 = ab^4$$

Solve for a in the first equation to obtain $a = \frac{12}{b}$. Substitute to obtain $b = \sqrt[3]{3} \approx 1.44$ and $a = \frac{12}{\sqrt[3]{3}} \approx 8.32$.

▶ So, an exponential function that models the data is $y = 8.32(1.44)^x$.

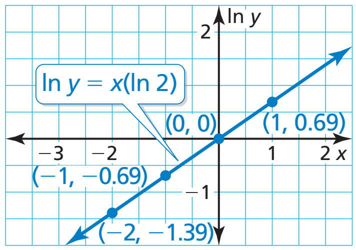
A set of more than two points (x, y) fits an exponential pattern if and only if the set of transformed points $(x, \ln y)$ fits a linear pattern.

Graph of points (x, y)



The graph is an exponential curve.

Graph of points $(x, \ln y)$



The graph is a line.

EXAMPLE 4 Writing a Model Using Transformed Points

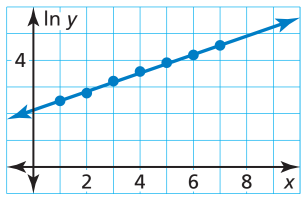
Use the data from Example 3. Create a scatter plot of the data pairs $(x, \ln y)$ to show that an exponential model should be a good fit for the original data pairs (x, y). Then write an exponential model for the original data.

SOLUTION

Step 1 Create a table of data pairs $(x, \ln y)$.

x	1	2	3	4	5	6	7
$\ln y$	2.48	2.77	3.22	3.58	3.91	4.20	4.56

Step 2 Plot the transformed points as shown. The points lie close to a line, so an exponential model should be a good fit for the original data.



Step 3 Find an exponential model $y = ab^x$ by choosing any two points on the line, such as (1, 2.48) and (7, 4.56). Use these points to write an equation of the line. Then solve for y.

$$\ln y - 2.48 = 0.35(x - 1) \qquad \text{Equation of line}$$

$$\ln y = 0.35x + 2.13 \qquad \text{Simplify.}$$

$$y = e^{0.35x + 2.13} \qquad \text{Exponentiate each side using base e.}$$

$$y = e^{0.35x}(e^{2.13}) \qquad \text{Use properties of exponents.}$$

$$y = 8.41(1.42)^x \qquad \text{Simplify.}$$

So, an exponential function that models the data is $y = 8.41(1.42)^x$.

LOOKING FOR STRUCTURE

Because the axes are x and $\ln y$, the point-slope form is rewritten as $\ln y - \ln y_1 = m(x - x_1)$. The slope of the line through (1, 2.48) and (7, 4.56) is

$$\frac{4.56 - 2.48}{7 - 1} \approx 0.35.$$

Monitoring Progress

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Write an exponential function $y = ab^x$ whose graph passes through the given points.

3. (2, 12), (3, 24)

4. (1, 2), (3, 32)

5. (2, 16), (5, 2)

6. **WHAT IF?** Repeat Examples 3 and 4 using the sales data from another store.

Year, x	1	2	3	4	5	6	7
Number of trampolines, y	15	23	40	52	80	105	140

Using Technology

You can use technology to find best-fit models for exponential and logarithmic data.

EXAMPLE 5 Finding an Exponential Model

Use a graphing calculator to find an exponential model for the data in Example 3. Then use this model and the models in Examples 3 and 4 to predict the number of trampolines sold in the eighth year. Compare the predictions.

SOLUTION

Enter the data into a graphing calculator and perform an exponential regression. The model is $y = 8.46(1.42)^x$.

```
ExpReg
 y=a*b^x
 a=8.457377971
 b=1.418848603
 r²=.9972445053
 r=.9986213023
```

Substitute $x = 8$ into each model to predict the number of trampolines sold in the eighth year.

Example 3: $y = 8.32(1.44)^8 \approx 154$

Example 4: $y = 8.41(1.42)^8 \approx 139$

Regression model: $y = 8.46(1.42)^8 \approx 140$

▶ The predictions are close for the regression model and the model in Example 4 that used transformed points. These predictions are less than the prediction for the model in Example 3.

EXAMPLE 6 Finding a Logarithmic Model

The atmospheric pressure decreases with increasing altitude. At sea level, the average air pressure is 1 atmosphere (1.033227 kilograms per square centimeter). The table shows the pressures p (in atmospheres) at selected altitudes h (in kilometers). Use a graphing calculator to find a logarithmic model of the form $h = a + b \ln p$ that represents the data. Estimate the altitude when the pressure is 0.75 atmosphere.

Air pressure, *p*	1	0.55	0.25	0.12	0.06	0.02
Altitude, *h*	0	5	10	15	20	25

Weather balloons carry instruments that send back information such as wind speed, temperature, and air pressure.

SOLUTION

Enter the data into a graphing calculator and perform a logarithmic regression. The model is $h = 0.86 - 6.45 \ln p$.

```
LnReg
 y=a+blnx
 a=.8626578705
 b=-6.447382985
 r²=.9925582287
 r=-.996272166
```

Substitute $p = 0.75$ into the model to obtain

$h = 0.86 - 6.45 \ln 0.75 \approx 2.7.$

▶ So, when the air pressure is 0.75 atmosphere, the altitude is about 2.7 kilometers.

Monitoring Progress

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7. Use a graphing calculator to find an exponential model for the data in Monitoring Progress Question 6.

8. Use a graphing calculator to find a logarithmic model of the form $p = a + b \ln h$ for the data in Example 6. Explain why the result is an error message.

6.7 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** Given a set of more than two data pairs (x, y), you can decide whether a(n) __________ function fits the data well by making a scatter plot of the points $(x, \ln y)$.

2. **WRITING** Given a table of values, explain how you can determine whether an exponential function is a good model for a set of data pairs (x, y).

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, determine the type of function represented by the table. Explain your reasoning. *(See Example 1.)*

3.

x	0	3	6	9	12	15
y	0.25	1	4	16	64	256

4.

x	-4	-3	-2	-1	0	1	2
y	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

5.

x	5	10	15	20	25	30
y	4	3	7	16	30	49

6.

x	-3	1	5	9	13
y	8	-3	-14	-25	-36

In Exercises 7–16, write an exponential function $y = ab^x$ whose graph passes through the given points. *(See Example 2.)*

7. (1, 3), (2, 12)

8. (2, 24), (3, 144)

9. (3, 1), (5, 4)

10. (3, 27), (5, 243)

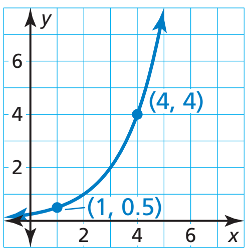
11. (1, 2), (3, 50)

12. (1, 40), (3, 640)

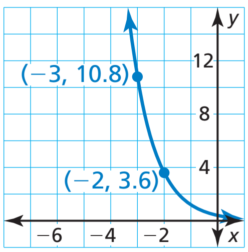
13. $(-1, 10)$, (4, 0.31)

14. (2, 6.4), (5, 409.6)

15.



16.



ERROR ANALYSIS **In Exercises 17 and 18, describe and correct the error in determining the type of function represented by the data.**

17.

x	0	1	2	3	4
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

$\times 3$ $\quad \times 3$ $\quad \times 3$ $\quad \times 3$

The outputs have a common ratio of 3, so the data represent a linear function.

18.

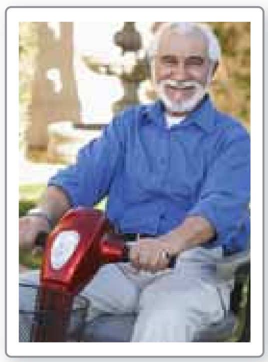
x	-2	-1	1	2	4
y	3	6	12	24	48

$\times 2$ $\quad \times 2$ $\quad \times 2$ $\quad \times 2$

The outputs have a common ratio of 2, so the data represent an exponential function.

19. **MODELING WITH MATHEMATICS** A store sells motorized scooters. The table shows the numbers y of scooters sold during the xth year that the store has been open. Write a function that models the data. *(See Example 3.)*

x	y
1	9
2	14
3	19
4	25
5	37
6	53
7	71



20. **MODELING WITH MATHEMATICS** The table shows the numbers y of visits to a website during the xth month. Write a function that models the data. Then use your model to predict the number of visits after 1 year.

x	1	2	3	4	5	6	7
y	22	39	70	126	227	408	735

In Exercises 21–24, determine whether the data show an exponential relationship. Then write a function that models the data.

21.

x	1	6	11	16	21
y	12	28	76	190	450

22.

x	-3	-1	1	3	5
y	2	7	24	68	194

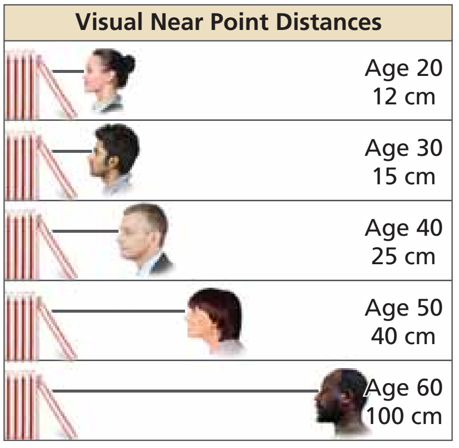
23.

x	0	10	20	30	40	50	60
y	66	58	48	42	31	26	21

24.

x	-20	-13	-6	1	8	15
y	25	19	14	11	8	6

25. **MODELING WITH MATHEMATICS** Your visual near point is the closest point at which your eyes can see an object distinctly. The diagram shows the near point y (in centimeters) at age x (in years). Create a scatter plot of the data pairs $(x, \ln y)$ to show that an exponential model should be a good fit for the original data pairs (x, y). Then write an exponential model for the original data. *(See Example 4.)*



26. **MODELING WITH MATHEMATICS** Use the data from Exercise 19. Create a scatter plot of the data pairs $(x, \ln y)$ to show that an exponential model should be a good fit for the original data pairs (x, y). Then write an exponential model for the original data.

In Exercises 27–30, create a scatter plot of the points $(x, \ln y)$ to determine whether an exponential model fits the data. If so, find an exponential model for the data.

27.

x	1	2	3	4	5
y	18	36	72	144	288

28.

x	1	4	7	10	13
y	3.3	10.1	30.6	92.7	280.9

29.

x	-13	-6	1	8	15
y	9.8	12.2	15.2	19	23.8

30.

x	-8	-5	-2	1	4
y	1.4	1.67	5.32	6.41	7.97

31. **USING TOOLS** Use a graphing calculator to find an exponential model for the data in Exercise 19. Then use the model to predict the number of motorized scooters sold in the tenth year. *(See Example 5.)*

32. **USING TOOLS** A doctor measures an astronaut's pulse rate y (in beats per minute) at various times x (in minutes) after the astronaut has finished exercising. The results are shown in the table. Use a graphing calculator to find an exponential model for the data. Then use the model to predict the astronaut's pulse rate after 16 minutes.

x	y
0	172
2	132
4	110
6	92
8	84
10	78
12	75

33. **USING TOOLS** An object at a temperature of 160°C is removed from a furnace and placed in a room at 20°C. The table shows the temperatures d (in degrees Celsius) at selected times t (in hours) after the object was removed from the furnace. Use a graphing calculator to find a logarithmic model of the form $t = a + b \ln d$ that represents the data. Estimate how long it takes for the object to cool to 50°C.
(See Example 6.)

d	160	90	56	38	29	24
t	0	1	2	3	4	5

34. **USING TOOLS** The f-stops on a camera control the amount of light that enters the camera. Let s be a measure of the amount of light that strikes the film and let f be the f-stop. The table shows several f-stops on a 35-millimeter camera. Use a graphing calculator to find a logarithmic model of the form $s = a + b \ln f$ that represents the data. Estimate the amount of light that strikes the film when $f = 5.657$.

f	**s**
1.414	1
2.000	2
2.828	3
4.000	4
11.314	7

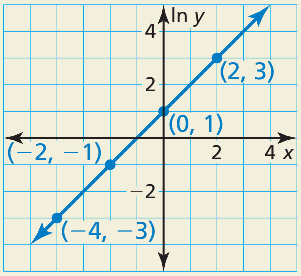
35. **DRAWING CONCLUSIONS** The table shows the average weight (in kilograms) of an Atlantic cod that is x years old from the Gulf of Maine.

Age, x	1	2	3	4	5
Weight, y	0.751	1.079	1.702	2.198	3.438

a. Show that an exponential model fits the data. Then find an exponential model for the data.

b. By what percent does the weight of an Atlantic cod increase each year in this period of time? Explain.

36. **HOW DO YOU SEE IT?** The graph shows a set of data points $(x, \ln y)$. Do the data pairs (x, y) fit an exponential pattern? Explain your reasoning.



37. **MAKING AN ARGUMENT** Your friend says it is possible to find a logarithmic model of the form $d = a + b \ln t$ for the data in Exercise 33. Is your friend correct? Explain.

38. **THOUGHT PROVOKING** Is it possible to write y as an exponential function of x? Explain your reasoning. (Assume p is positive.)

x	**y**
1	p
2	$2p$
3	$4p$
4	$8p$
5	$16p$

39. **CRITICAL THINKING** You plant a sunflower seedling in your garden. The height h (in centimeters) of the seedling after t weeks can be modeled by the *logistic function*

$$h(t) = \frac{256}{1 + 13e^{-0.65t}}.$$

a. Find the time it takes the sunflower seedling to reach a height of 200 centimeters.

b. Use a graphing calculator to graph the function. Interpret the meaning of the asymptote in the context of this situation.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Tell whether x and y are in a proportional relationship. Explain your reasoning.
(Skills Review Handbook)

40. $y = \frac{x}{2}$ **41.** $y = 3x - 12$ **42.** $y = \frac{5}{x}$ **43.** $y = -2x$

Identify the focus, directrix, and axis of symmetry of the parabola. Then graph the equation.
(Section 2.3)

44. $x = \frac{1}{8}y^2$ **45.** $y = 4x^2$ **46.** $x^2 = 3y$ **47.** $y^2 = \frac{2}{5}x$

6.5–6.7 What Did You Learn?

Core Vocabulary

exponential equations, *p. 334*
logarithmic equations, *p. 335*

Core Concepts

Section 6.5

Properties of Logarithms, *p. 328*
Change-of-Base Formula, *p. 329*

Section 6.6

Property of Equality for Exponential Equations, *p. 334*
Property of Equality for Logarithmic Equations, *p. 335*

Section 6.7

Classifying Data, *p. 342*
Writing Exponential Functions, *p. 343*
Using Exponential and Logarithmic Regression, *p. 345*

Mathematical Practices

1. Explain how you used properties of logarithms to rewrite the function in part (b) of Exercise 45 on page 332.
2. How can you use cases to analyze the argument given in Exercise 46 on page 339?

Performance Task

Measuring Natural Disasters

In 2005, an earthquake measuring 4.1 on the Richter scale barely shook the city of Ocotillo, California, leaving virtually no damage. But in 1906, an earthquake with an estimated 8.2 on the same scale devastated the city of San Francisco. Does twice the measurement on the Richter scale mean twice the intensity of the earthquake?

To explore the answers to these questions and more, go to ***BigIdeasMath.com***.

Dynamic Solutions available at *BigIdeasMath.com*

6.1 Exponential Growth and Decay Functions (pp. 295–302)

Tell whether the function $y = 3^x$ represents *exponential growth* or *exponential decay*. Then graph the function.

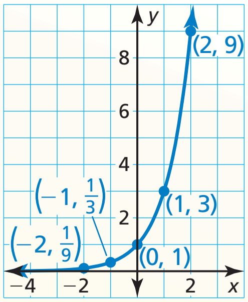
Step 1 Identify the value of the base. The base, 3, is greater than 1, so the function represents exponential growth.

Step 2 Make a table of values.

Step 3 Plot the points from the table.

Step 4 Draw, from *left to right*, a smooth curve that begins just above the x-axis, passes through the plotted points, and moves up to the right.

x	-2	-1	0	1	2
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9



Tell whether the function represents exponential growth or exponential decay. Identify the percent increase or decrease. Then graph the function.

1. $f(x) = \left(\frac{1}{3}\right)^x$ **2.** $y = 5^x$ **3.** $f(x) = (0.2)^x$

4. You deposit \$1500 in an account that pays 7% annual interest. Find the balance after 2 years when the interest is compounded daily.

6.2 The Natural Base *e* (pp. 303–308)

Simplify each expression.

a. $\dfrac{18e^{13}}{2e^7} = 9e^{13-7} = 9e^6$ **b.** $(2e^{3x})^3 = 2^3(e^{3x})^3 = 8e^{9x}$

Simplify the expression.

5. $e^4 \cdot e^{11}$ **6.** $\dfrac{20e^3}{10e^6}$ **7.** $(-3e^{-5x})^2$

Tell whether the function represents *exponential growth* or *exponential decay*. Then graph the function.

8. $f(x) = \frac{1}{3}e^x$ **9.** $y = 6e^{-x}$ **10.** $y = 3e^{-0.75x}$

6.3 Logarithms and Logarithmic Functions (pp. 309–316)

Find the inverse of the function $y = \ln(x - 2)$.

$y = \ln(x - 2)$ Write original function.

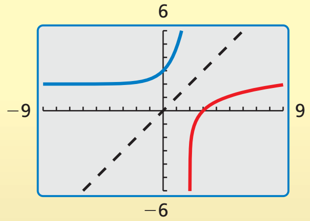
$x = \ln(y - 2)$ Switch x and y.

$e^x = y - 2$ Write in exponential form.

$e^x + 2 = y$ Add 2 to each side.

▶ The inverse of $y = \ln(x - 2)$ is $y = e^x + 2$.

Check



The graphs appear to be reflections of each other in the line $y = x$. ✓

Evaluate the logarithm.

11. $\log_2 8$

12. $\log_6 \frac{1}{36}$

13. $\log_5 1$

Find the inverse of the function.

14. $f(x) = 8^x$

15. $y = \ln(x - 4)$

16. $y = \log(x + 9)$

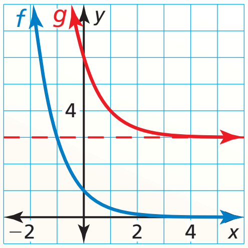
17. Graph $y = \log_{1/5} x$.

6.4 Transformations of Exponential and Logarithmic Functions (pp. 317–324)

Describe the transformation of $f(x) = \left(\frac{1}{3}\right)^x$ represented by $g(x) = \left(\frac{1}{3}\right)^{x-1} + 3$. Then graph each function.

Notice that the function is of the form $g(x) = \left(\frac{1}{3}\right)^{x-h} + k$, where $h = 1$ and $k = 3$.

► So, the graph of g is a translation 1 unit right and 3 units up of the graph of f.



Describe the transformation of f represented by g. Then graph each function.

18. $f(x) = e^{-x}$, $g(x) = e^{-5x} - 8$

19. $f(x) = \log_4 x$, $g(x) = \frac{1}{2}\log_4(x + 5)$

Write a rule for g.

20. Let the graph of g be a vertical stretch by a factor of 3, followed by a translation 6 units left and 3 units up of the graph of $f(x) = e^x$.

21. Let the graph of g be a translation 2 units down, followed by a reflection in the y-axis of the graph of $f(x) = \log x$.

6.5 Properties of Logarithms (pp. 327–332)

Expand $\ln \frac{12x^5}{y}$.

$$\ln \frac{12x^5}{y} = \ln 12x^5 - \ln y \quad \text{Quotient Property}$$

$$= \ln 12 + \ln x^5 - \ln y \quad \text{Product Property}$$

$$= \ln 12 + 5 \ln x - \ln y \quad \text{Power Property}$$

Expand or condense the logarithmic expression.

22. $\log_8 3xy$

23. $\log 10x^3y$

24. $\ln \frac{3y}{x^5}$

25. $3 \log_7 4 + \log_7 6$

26. $\log_2 12 - 2 \log_2 x$

27. $2 \ln x + 5 \ln 2 - \ln 8$

Use the change-of-base formula to evaluate the logarithm.

28. $\log_2 10$

29. $\log_7 9$

30. $\log_{23} 42$

6.6 Solving Exponential and Logarithmic Equations (pp. 333–340)

Solve $\ln(3x - 9) = \ln(2x + 6)$.

$\ln(3x - 9) = \ln(2x + 6)$	Write original equation.
$3x - 9 = 2x + 6$	Property of Equality for Logarithmic Equations
$x - 9 = 6$	Subtract $2x$ from each side.
$x = 15$	Add 9 to each side.

Check

$\ln(3 \cdot 15 - 9) \stackrel{?}{=} \ln(2 \cdot 15 + 6)$

$\ln(45 - 9) \stackrel{?}{=} \ln(30 + 6)$

$\ln 36 = \ln 36$ ✓

Solve the equation. Check for extraneous solutions.

31. $5^x = 8$

32. $\log_3(2x - 5) = 2$

33. $\ln x + \ln(x + 2) = 3$

Solve the inequality.

34. $6^x > 12$

35. $\ln x \leq 9$

36. $e^{4x-2} \geq 16$

6.7 Modeling with Exponential and Logarithmic Functions (pp. 341–348)

Write an exponential function whose graph passes through (1, 3) and (4, 24).

Step 1 Substitute the coordinates of the two given points into $y = ab^x$.

$3 = ab^1$	Equation 1: Substitute 3 for y and 1 for x.
$24 = ab^4$	Equation 2: Substitute 24 for y and 4 for x.

Step 2 Solve for a in Equation 1 to obtain $a = \frac{3}{b}$ and substitute this expression for a in Equation 2.

$24 = \left(\frac{3}{b}\right)b^4$	Substitute $\frac{3}{b}$ for a in Equation 2.
$24 = 3b^3$	Simplify.
$8 = b^3$	Divide each side by 3.
$2 = b$	Take cube root of each side.

Step 3 Determine that $a = \frac{3}{b} = \frac{3}{2}$.

▶ So, the exponential function is $y = \frac{3}{2}(2^x)$.

Write an exponential model for the data pairs (x, y).

37. (3, 8), (5, 2)

38.

x	1	2	3	4
$\ln y$	1.64	2.00	2.36	2.72

39. A shoe store sells a new type of basketball shoe. The table shows the pairs sold s over time t (in weeks). Use a graphing calculator to find a logarithmic model of the form $s = a + b \ln t$ that represents the data. Estimate how many pairs of shoes are sold after 6 weeks.

Week, t	1	3	5	7	9
Pairs sold, s	5	32	48	58	65

6 Chapter Test

Graph the equation. State the domain, range, and asymptote.

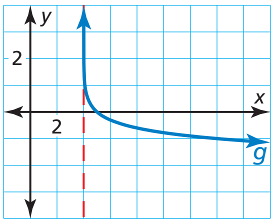
1. $y = \left(\frac{1}{2}\right)^x$

2. $y = \log_{1/5} x$

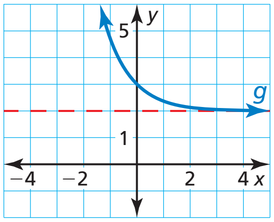
3. $y = 4e^{-2x}$

Describe the transformation of *f* represented by *g*. Then write a rule for *g*.

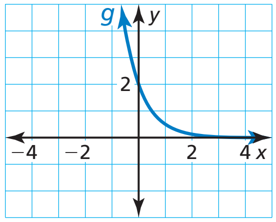
4. $f(x) = \log x$



5. $f(x) = e^x$



6. $f(x) = \left(\frac{1}{4}\right)^x$



Evaluate the logarithm. Use $\log_3 4 \approx 1.262$ and $\log_3 13 \approx 2.335$, if necessary.

7. $\log_3 52$

8. $\log_3 \frac{13}{9}$

9. $\log_3 16$

10. $\log_3 8 + \log_3 \frac{1}{2}$

11. Describe the similarities and differences in solving the equations $4^{5x-2} = 16$ and $\log_4(10x + 6) = 1$. Then solve each equation.

12. Without calculating, determine whether $\log_5 11$, $\frac{\log 11}{\log 5}$, and $\frac{\ln 11}{\ln 5}$ are equivalent expressions. Explain your reasoning.

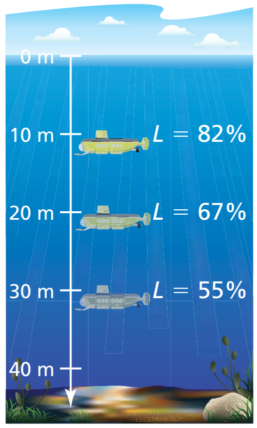
13. The amount y of oil collected by a petroleum company drilling on the U.S. continental shelf can be modeled by $y = 12.263 \ln x - 45.381$, where y is measured in billions of barrels and x is the number of wells drilled. About how many barrels of oil would you expect to collect after drilling 1000 wells? Find the inverse function and describe the information you obtain from finding the inverse.

14. The percent L of surface light that filters down through bodies of water can be modeled by the exponential function $L(x) = 100e^{kx}$, where k is a measure of the murkiness of the water and x is the depth (in meters) below the surface.

a. A recreational submersible is traveling in clear water with a k-value of about -0.02. Write a function that gives the percent of surface light that filters down through clear water as a function of depth.

b. Tell whether your function in part (a) represents exponential growth or exponential decay. Explain your reasoning.

c. Estimate the percent of surface light available at a depth of 40 meters.

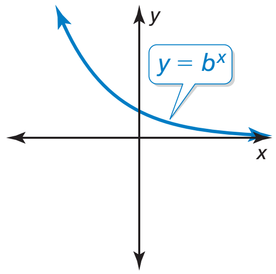


15. The table shows the values y (in dollars) of a new snowmobile after x years of ownership. Describe three different ways to find an exponential model that represents the data. Then write and use a model to find the year when the snowmobile is worth $2500.

Year, *x*	0	1	2	3	4
Value, *y*	4200	3780	3402	3061.80	2755.60

6 Cumulative Assessment

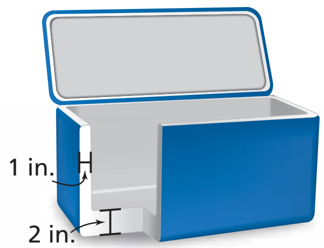
1. Select every value of b for the equation $y = b^x$ that could result in the graph shown.



1.08	0.94	e^2
2.04	$e^{-1/2}$	$\frac{5}{4}$

2. Your friend claims more interest is earned when an account pays interest compounded continuously than when it pays interest compounded daily. Do you agree with your friend? Justify your answer.

3. You are designing a rectangular picnic cooler with a length four times its width and height twice its width. The cooler has insulation that is 1 inch thick on each of the four sides and 2 inches thick on the top and bottom.



a. Let x represent the width of the cooler. Write a polynomial function T that gives the volume of the rectangular prism formed by the outer surfaces of the cooler.

b. Write a polynomial function C for the volume of the inside of the cooler.

c. Let I be a polynomial function that represents the volume of the insulation. How is I related to T and C?

d. Write I in standard form. What is the volume of the insulation when the width of the cooler is 8 inches?

4. What is the solution to the logarithmic inequality $-4 \log_2 x \geq -20$?

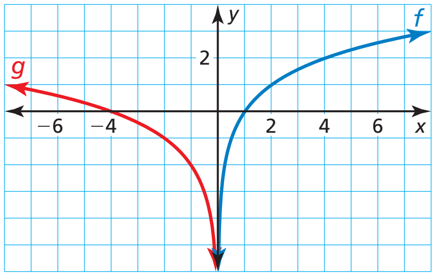
(A) $x \leq 32$

(B) $0 \leq x \leq 32$

(C) $0 < x \leq 32$

(D) $x \geq 32$

5. Describe the transformation of $f(x) = \log_2 x$ represented by the graph of g.



6. Let $f(x) = 2x^3 - 4x^2 + 8x - 1$, $g(x) = 2x - 3x^4 - 6x^3 + 5$, and $h(x) = -7 + x^2 + x$. Order the following functions from least degree to greatest degree.

A. $(f + g)(x)$

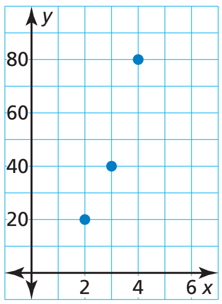
B. $(hg)(x)$

C. $(h - f)(x)$

D. $(fh)(x)$

7. Write an exponential model that represents each data set. Compare the two models.

a.



b.

x	2	3	4	5	6
y	4.5	13.5	40.5	121.5	364.5

8. Choose a method to solve each quadratic equation. Explain your choice of method.

a. $x^2 + 4x = 10$

b. $x^2 = -12$

c. $4(x - 1)^2 = 6x + 2$

d. $x^2 - 3x - 18 = 0$

9. At the annual pumpkin-tossing contest, contestants compete to see whose catapult will send pumpkins the longest distance. The table shows the horizontal distances y (in feet) a pumpkin travels when launched at different angles x (in degrees). Create a scatter plot of the data. Do the data show a linear, quadratic, or exponential relationship? Use technology to find a model for the data. Find the angle(s) at which a launched pumpkin travels 500 feet.

Angle (degrees), x	20	30	40	50	60	70
Distance (feet), y	372	462	509	501	437	323