6 Exponential and Logarithmic Functions

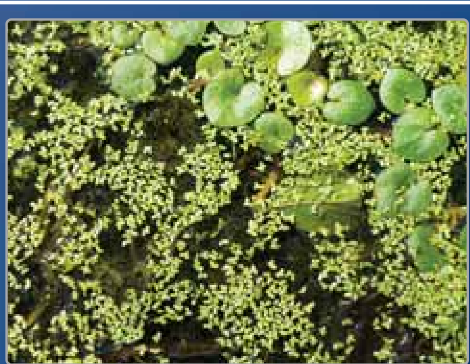
6.1 Exponential Growth and Decay Functions
6.2 The Natural Base *e*
6.3 Logarithms and Logarithmic Functions
6.4 Transformations of Exponential and Logarithmic Functions
6.5 Properties of Logarithms
6.6 Solving Exponential and Logarithmic Equations
6.7 Modeling with Exponential and Logarithmic Functions

Astronaut Health *(p. 347)*

Cooking *(p. 335)*

Recording Studio *(p. 330)*

Tornado Wind Speed *(p. 315)*



Duckweed Growth *(p. 301)*

Maintaining Mathematical Proficiency

Using Exponents

Example 1 Evaluate $\left(-\frac{1}{3}\right)^4$.

$$\left(-\frac{1}{3}\right)^4 = \left(-\frac{1}{3}\right) \cdot \left(-\frac{1}{3}\right) \cdot \left(-\frac{1}{3}\right) \cdot \left(-\frac{1}{3}\right)$$ Rewrite $\left(-\frac{1}{3}\right)^4$ as repeated multiplication.

$$= \left(\frac{1}{9}\right) \cdot \left(-\frac{1}{3}\right) \cdot \left(-\frac{1}{3}\right)$$ Multiply.

$$= \left(-\frac{1}{27}\right) \cdot \left(-\frac{1}{3}\right)$$ Multiply.

$$= \frac{1}{81}$$ Multiply.

Evaluate the expression.

1. $3 \cdot 2^4$

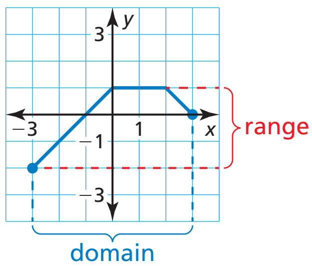
2. $(-2)^5$

3. $-\left(\frac{5}{6}\right)^2$

4. $\left(\frac{3}{4}\right)^3$

Finding the Domain and Range of a Function

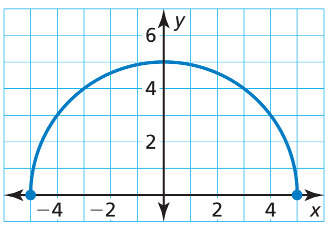
Example 2 Find the domain and range of the function represented by the graph.



The domain is $-3 \leq x \leq 3$.
The range is $-2 \leq y \leq 1$.

Find the domain and range of the function represented by the graph.

5.



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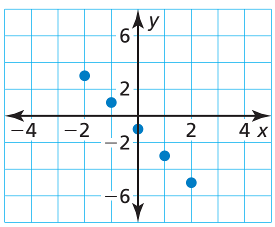

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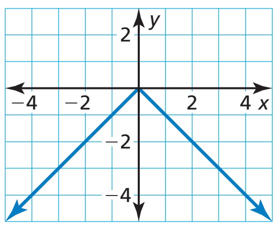


8. ABSTRACT REASONING Consider the expressions -4^n and $(-4)^n$, where n is an integer. For what values of n is each expression negative? positive? Explain your reasoning.

Mathematical Practices

Mathematically proficient students know when it is appropriate to use general methods and shortcuts.

Exponential Models

Core Concept

Consecutive Ratio Test for Exponential Models

Consider a table of values of the given form.

x	0	1	2	3	4	5	6	7	8	9
y	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9

If the consecutive ratios of the y-values are all equal to a common value r, then y can be modeled by an exponential function. When $r > 1$, the model represents exponential *growth*.

$$r = \frac{a_{n+1}}{a_n} \qquad \text{Common ratio}$$

$$y = a_0 r^x \qquad \text{Exponential model}$$

EXAMPLE 1 Modeling Real-Life Data

The table shows the amount A (in dollars) in a savings account over time. Write a model for the amount in the account as a function of time t (in years). Then use the model to find the amount after 10 years.

Year, t	0	1	2	3	4	5
Amount, A	\$1000	\$1040	\$1081.60	\$1124.86	\$1169.86	\$1216.65

SOLUTION

Begin by determining whether the ratios of consecutive amounts are equal.

$$\frac{1040}{1000} = 1.04, \quad \frac{1081.60}{1040} = 1.04, \quad \frac{1124.86}{1081.60} \approx 1.04, \quad \frac{1169.86}{1124.86} \approx 1.04, \quad \frac{1216.65}{1169.86} \approx 1.04$$

The ratios of consecutive amounts are equal, so the amount A after t years can be modeled by

$$A = 1000(1.04)^t.$$

Using this model, the amount when $t = 10$ is $A = 1000(1.04)^{10} = \$1480.24$.

Monitoring Progress

Determine whether the data can be modeled by an exponential or linear function. Explain your reasoning. Then write the appropriate model and find y when $x = 10$.

1.

x	0	1	2	3	4
y	1	2	4	8	16

2.

x	0	1	2	3	4
y	0	4	8	12	16

3.

x	0	1	2	3	4
y	1	4	7	10	13

4.

x	0	1	2	3	4
y	1	3	9	27	81

6.1 Exponential Growth and Decay Functions

Essential Question

Essential Question What are some of the characteristics of the graph of an exponential function?

You can use a graphing calculator to evaluate an exponential function. For example, consider the exponential function $f(x) = 2^x$.

Function Value	Graphing Calculator Keystrokes	Display
$f(-3.1) = 2^{-3.1}$	2 [^] [(–)] 3.1 [ENTER]	0.1166291
$f\left(\frac{2}{3}\right) = 2^{2/3}$	2 [^] [(] 2 [÷] 3 [)] [ENTER]	1.5874011

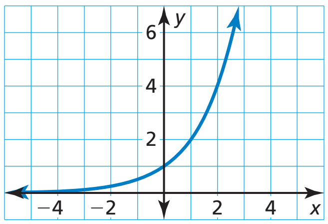
EXPLORATION 1 Identifying Graphs of Exponential Functions

Work with a partner. Match each exponential function with its graph. Use a table of values to sketch the graph of the function, if necessary.

a. $f(x) = 2^x$ **b.** $f(x) = 3^x$ **c.** $f(x) = 4^x$

d. $f(x) = \left(\frac{1}{2}\right)^x$ **e.** $f(x) = \left(\frac{1}{3}\right)^x$ **f.** $f(x) = \left(\frac{1}{4}\right)^x$

A.



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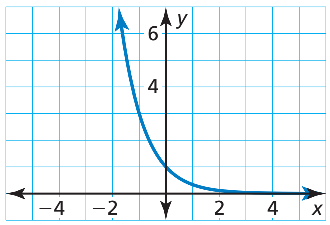

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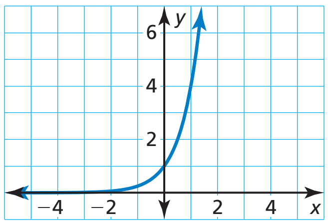

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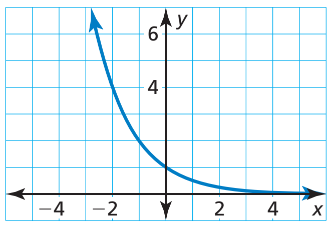

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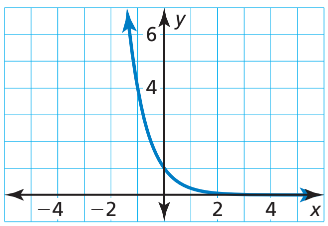

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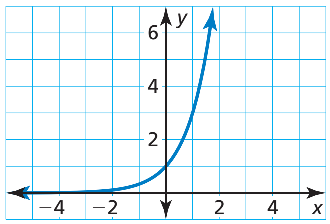


CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to justify your conclusions and communicate them to others.

EXPLORATION 2 Characteristics of Graphs of Exponential Functions

Work with a partner. Use the graphs in Exploration 1 to determine the domain, range, and y-intercept of the graph of $f(x) = b^x$, where b is a positive real number other than 1. Explain your reasoning.

Communicate Your Answer

3. What are some of the characteristics of the graph of an exponential function?

4. In Exploration 2, is it possible for the graph of $f(x) = b^x$ to have an x-intercept? Explain your reasoning.

6.1 Lesson

Core Vocabulary

exponential function, *p. 296*
exponential growth function, *p. 296*
growth factor, *p. 296*
asymptote, *p. 296*
exponential decay function, *p. 296*
decay factor, *p. 296*

Previous
properties of exponents

What You Will Learn

- Graph exponential growth and decay functions.
- Use exponential models to solve real-life problems.

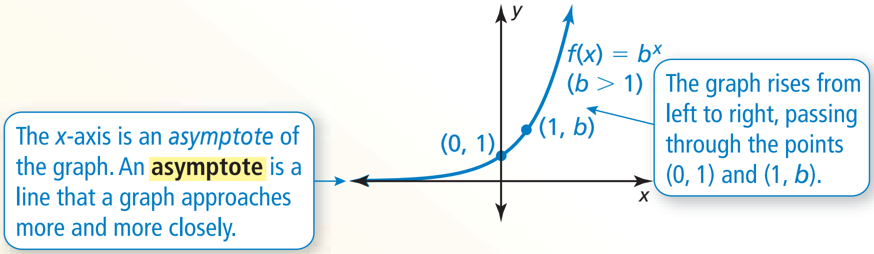
Exponential Growth and Decay Functions

An **exponential function** has the form $y = ab^x$, where $a \neq 0$ and the base b is a positive real number other than 1. If $a > 0$ and $b > 1$, then $y = ab^x$ is an **exponential growth function**, and b is called the **growth factor**. The simplest type of exponential growth function has the form $y = b^x$.

Core Concept

Parent Function for Exponential Growth Functions

The function $f(x) = b^x$, where $b > 1$, is the parent function for the family of exponential growth functions with base b. The graph shows the general shape of an exponential growth function.



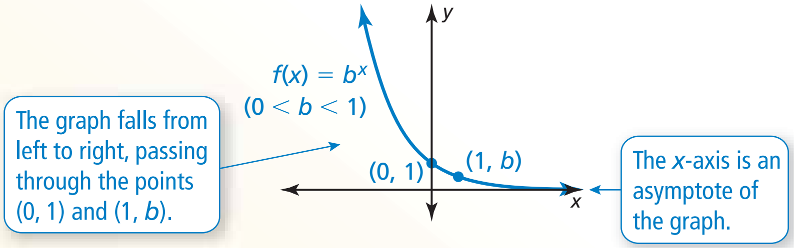
The domain of $f(x) = b^x$ is all real numbers. The range is $y > 0$.

If $a > 0$ and $0 < b < 1$, then $y = ab^x$ is an **exponential decay function**, and b is called the **decay factor**.

Core Concept

Parent Function for Exponential Decay Functions

The function $f(x) = b^x$, where $0 < b < 1$, is the parent function for the family of exponential decay functions with base b. The graph shows the general shape of an exponential decay function.



The domain of $f(x) = b^x$ is all real numbers. The range is $y > 0$.

EXAMPLE 1 Graphing Exponential Growth and Decay Functions

Tell whether each function represents *exponential growth* or *exponential decay*. Then graph the function.

a. $y = 2^x$

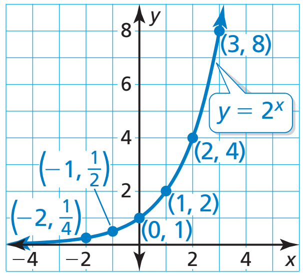
b. $y = \left(\frac{1}{2}\right)^x$

SOLUTION

a. Step 1 Identify the value of the base. The base, 2, is greater than 1, so the function represents exponential growth.

Step 2 Make a table of values.

x	-2	-1	0	1	2	3
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



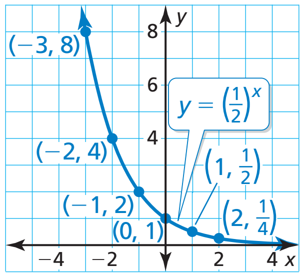
Step 3 Plot the points from the table.

Step 4 Draw, from *left to right*, a smooth curve that begins just above the x-axis, passes through the plotted points, and moves up to the right.

b. Step 1 Identify the value of the base. The base, $\frac{1}{2}$, is greater than 0 and less than 1, so the function represents exponential decay.

Step 2 Make a table of values.

x	-3	-2	-1	0	1	2
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$



Step 3 Plot the points from the table.

Step 4 Draw, from *right to left*, a smooth curve that begins just above the x-axis, passes through the plotted points, and moves up to the left.

Monitoring Progress

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Tell whether the function represents *exponential growth* or *exponential decay*. Then graph the function.

1. $y = 4^x$

2. $y = \left(\frac{2}{3}\right)^x$

3. $f(x) = (0.25)^x$

4. $f(x) = (1.5)^x$

Exponential Models

Some real-life quantities increase or decrease by a fixed percent each year (or some other time period). The amount y of such a quantity after t years can be modeled by one of these equations.

Exponential Growth Model

$$y = a(1 + r)^t$$

Exponential Decay Model

$$y = a(1 - r)^t$$

Note that a is the initial amount and r is the percent increase or decrease written as a decimal. The quantity $1 + r$ is the growth factor, and $1 - r$ is the decay factor.

EXAMPLE 2 Solving a Real-Life Problem

The value of a car y (in thousands of dollars) can be approximated by the model $y = 25(0.85)^t$, where t is the number of years since the car was new.

a. Tell whether the model represents exponential growth or exponential decay.

b. Identify the annual percent increase or decrease in the value of the car.

c. Estimate when the value of the car will be \$8000.

REASONING QUANTITATIVELY

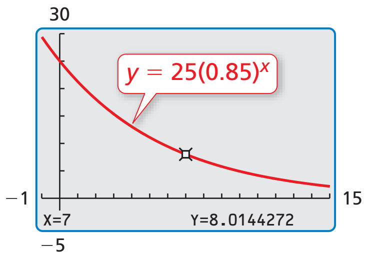
The percent decrease, 15%, tells you how much value the car *loses* each year. The decay factor, 0.85, tells you what fraction of the car's value *remains* each year.

SOLUTION

a. The base, 0.85, is greater than 0 and less than 1, so the model represents exponential decay.

b. Because t is given in years and the decay factor $0.85 = 1 - 0.15$, the annual percent decrease is 0.15, or 15%.

c. Use the *trace* feature of a graphing calculator to determine that $y \approx 8$ when $t = 7$. After 7 years, the value of the car will be about \$8000.



EXAMPLE 3 Writing an Exponential Model

In 2000, the world population was about 6.09 billion. During the next 13 years, the world population increased by about 1.18% each year.

a. Write an exponential growth model giving the population y (in billions) t years after 2000. Estimate the world population in 2005.

b. Estimate the year when the world population was 7 billion.

SOLUTION

a. The initial amount is $a = 6.09$, and the percent increase is $r = 0.0118$. So, the exponential growth model is

$$y = a(1 + r)^t \qquad \text{Write exponential growth model.}$$

$$= 6.09(1 + 0.0118)^t \qquad \text{Substitute 6.09 for } a \text{ and 0.0118 for } r.$$

$$= 6.09(1.0118)^t. \qquad \text{Simplify.}$$

Using this model, you can estimate the world population in 2005 ($t = 5$) to be $y = 6.09(1.0118)^5 \approx 6.46$ billion.

X	Y1
6	6.5341
7	6.6112
8	6.6892
9	6.7681
10	6.848
11	6.9288
12	7.0106

X=12

b. Use the *table* feature of a graphing calculator to determine that $y \approx 7$ when $t = 12$. So, the world population was about 7 billion in 2012.

Monitoring Progress

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5. WHAT IF? In Example 2, the value of the car can be approximated by the model $y = 25(0.9)^t$. Identify the annual percent decrease in the value of the car. Estimate when the value of the car will be \$8000.

6. WHAT IF? In Example 3, assume the world population increased by 1.5% each year. Write an equation to model this situation. Estimate the year when the world population was 7 billion.

EXAMPLE 4 Rewriting an Exponential Function

The amount y (in grams) of the radioactive isotope chromium-51 remaining after t days is $y = a(0.5)^{t/28}$, where a is the initial amount (in grams). What percent of the chromium-51 decays each day?

SOLUTION

$y = a(0.5)^{t/28}$	Write original function.
$= a[(0.5)^{1/28}]^t$	Power of a Power Property
$\approx a(0.9755)^t$	Evaluate power.
$= a(1 - 0.0245)^t$	Rewrite in form $y = a(1 - r)^t$.

▶ The daily decay rate is about 0.0245, or 2.45%.

Compound interest is interest paid on an initial investment, called the *principal*, and on previously earned interest. Interest earned is often expressed as an *annual* percent, but the interest is usually compounded more than once per year. So, the exponential growth model $y = a(1 + r)^t$ must be modified for compound interest problems.

Core Concept

Compound Interest

Consider an initial principal P deposited in an account that pays interest at an annual rate r (expressed as a decimal), compounded n times per year. The amount A in the account after t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

EXAMPLE 5 Finding the Balance in an Account

You deposit \$9000 in an account that pays 1.46% annual interest. Find the balance after 3 years when the interest is compounded quarterly.

SOLUTION

With interest compounded quarterly (4 times per year), the balance after 3 years is

$A = P\left(1 + \frac{r}{n}\right)^{nt}$	Write compound interest formula.
$= 9000\left(1 + \frac{0.0146}{4}\right)^{4 \cdot 3}$	$P = 9000$, $r = 0.0146$, $n = 4$, $t = 3$
$\approx 9402.21.$	Use a calculator.

▶ The balance at the end of 3 years is \$9402.21.

Monitoring Progress

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7. The amount y (in grams) of the radioactive isotope iodine-123 remaining after t hours is $y = a(0.5)^{t/13}$, where a is the initial amount (in grams). What percent of the iodine-123 decays each hour?

8. **WHAT IF?** In Example 5, find the balance after 3 years when the interest is compounded daily.

6.1 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** In the exponential growth model $y = 2.4(1.5)^x$, identify the initial amount, the growth factor, and the percent increase.

2. **WHICH ONE DOESN'T BELONG?** Which characteristic of an exponential decay function does *not* belong with the other three? Explain your reasoning.

base of 0.8	decay factor of 0.8
decay rate of 20%	80% decrease

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, evaluate the expression for (a) $x = -2$ and (b) $x = 3$.

3. 2^x

4. 4^x

5. $8 \cdot 3^x$

6. $6 \cdot 2^x$

7. $5 + 3^x$

8. $2^x - 2$

In Exercises 9–18, tell whether the function represents *exponential growth* or *exponential decay*. Then graph the function. *(See Example 1.)*

9. $y = 6^x$

10. $y = 7^x$

11. $y = \left(\frac{1}{6}\right)^x$

12. $y = \left(\frac{1}{8}\right)^x$

13. $y = \left(\frac{4}{3}\right)^x$

14. $y = \left(\frac{2}{5}\right)^x$

15. $y = (1.2)^x$

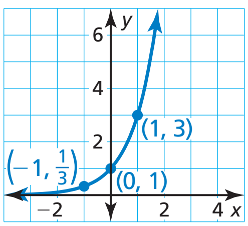
16. $y = (0.75)^x$

17. $y = (0.6)^x$

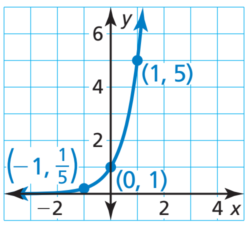
18. $y = (1.8)^x$

ANALYZING RELATIONSHIPS **In Exercises 19 and 20, use the graph of $f(x) = b^x$ to identify the value of the base b.**

19.



20.



21. **MODELING WITH MATHEMATICS** The value of a mountain bike y (in dollars) can be approximated by the model $y = 200(0.75)^t$, where t is the number of years since the bike was new. *(See Example 2.)*

    a. Tell whether the model represents exponential growth or exponential decay.

    b. Identify the annual percent increase or decrease in the value of the bike.

    c. Estimate when the value of the bike will be \$50.

22. **MODELING WITH MATHEMATICS** The population P (in thousands) of Austin, Texas, during a recent decade can be approximated by $y = 494.29(1.03)^t$, where t is the number of years since the beginning of the decade.

    a. Tell whether the model represents exponential growth or exponential decay.

    b. Identify the annual percent increase or decrease in population.

    c. Estimate when the population was about 590,000.

23. **MODELING WITH MATHEMATICS** In 2006, there were approximately 233 million cell phone subscribers in the United States. During the next 4 years, the number of cell phone subscribers increased by about 6% each year. *(See Example 3.)*

    a. Write an exponential growth model giving the number of cell phone subscribers y (in millions) t years after 2006. Estimate the number of cell phone subscribers in 2008.

    b. Estimate the year when the number of cell phone subscribers was about 278 million.

24. MODELING WITH MATHEMATICS You take a 325 milligram dosage of ibuprofen. During each subsequent hour, the amount of medication in your bloodstream decreases by about 29% each hour.

a. Write an exponential decay model giving the amount y (in milligrams) of ibuprofen in your bloodstream t hours after the initial dose.

b. Estimate how long it takes for you to have 100 milligrams of ibuprofen in your bloodstream.

JUSTIFYING STEPS **In Exercises 25 and 26, justify each step in rewriting the exponential function.**

25.

$y = a(3)^{t/14}$	Write original function.
$= a[(3)^{1/14}]^t$	
$\approx a(1.0816)^t$	
$= a(1 + 0.0816)^t$	

26.

$y = a(0.1)^{t/3}$	Write original function.
$= a[(0.1)^{1/3}]^t$	
$\approx a(0.4642)^t$	
$= a(1 - 0.5358)^t$	

27. PROBLEM SOLVING When a plant or animal dies, it stops acquiring carbon-14 from the atmosphere. The amount y (in grams) of carbon-14 in the body of an organism after t years is $y = a(0.5)^{t/5730}$, where a is the initial amount (in grams). What percent of the carbon-14 is released each year? *(See Example 4.)*

28. PROBLEM SOLVING The number y of duckweed fronds in a pond after t days is $y = a(1230.25)^{t/16}$, where a is the initial number of fronds. By what percent does the duckweed increase each day?

In Exercises 29–36, rewrite the function in the form $y = a(1 + r)^t$ or $y = a(1 - r)^t$. Then state the growth or decay rate.

29. $y = a(2)^{t/3}$

30. $y = a(4)^{t/6}$

31. $y = a(0.5)^{t/12}$

32. $y = a(0.25)^{t/9}$

33. $y = a\left(\frac{2}{3}\right)^{t/10}$

34. $y = a\left(\frac{5}{4}\right)^{t/22}$

35. $y = a(2)^{8t}$

36. $y = a\left(\frac{1}{3}\right)^{3t}$

37. PROBLEM SOLVING You deposit \$5000 in an account that pays 2.25% annual interest. Find the balance after 5 years when the interest is compounded quarterly. *(See Example 5.)*

38. DRAWING CONCLUSIONS You deposit \$2200 into three separate bank accounts that each pay 3% annual interest. How much interest does each account earn after 6 years?

Account	Compounding	Interest after 6 years
1	quarterly	
2	monthly	
3	daily	

39. ERROR ANALYSIS You invest \$500 in the stock of a company. The value of the stock decreases 2% each year. Describe and correct the error in writing a model for the value of the stock after t years.

$$y = \begin{pmatrix}\text{Initial}\\\text{amount}\end{pmatrix}\begin{pmatrix}\text{Decay}\\\text{factor}\end{pmatrix}^t$$

$$y = 500(0.02)^t$$

40. ERROR ANALYSIS You deposit \$250 in an account that pays 1.25% annual interest. Describe and correct the error in finding the balance after 3 years when the interest is compounded quarterly.

$$A = 250\left(1 + \frac{1.25}{4}\right)^{4\cdot 3}$$

$$A = \$6533.29$$

In Exercises 41–44, use the given information to find the amount A in the account earning compound interest after 6 years when the principal is \$3500.

41. $r = 2.16\%$, compounded quarterly

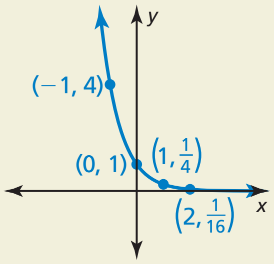
42. $r = 2.29\%$, compounded monthly

43. $r = 1.83\%$, compounded daily

44. $r = 1.26\%$, compounded monthly

45. **USING STRUCTURE** A website recorded the number y of referrals it received from social media websites over a 10-year period. The results can be modeled by $y = 2500(1.50)^t$, where t is the year and $0 \le t \le 9$. Interpret the values of a and b in this situation. What is the annual percent increase? Explain.

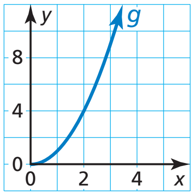
46. **HOW DO YOU SEE IT?** Consider the graph of an exponential function of the form $f(x) = ab^x$.



a. Determine whether the graph of f represents exponential growth or exponential decay.

b. What are the domain and range of the function? Explain.

47. **MAKING AN ARGUMENT** Your friend says the graph of $f(x) = 2^x$ increases at a faster rate than the graph of $g(x) = x^2$ when $x \ge 0$. Is your friend correct? Explain your reasoning.



48. **THOUGHT PROVOKING** The function $f(x) = b^x$ represents an exponential decay function. Write a second exponential decay function in terms of b and x.

49. **PROBLEM SOLVING** The population p of a small town after x years can be modeled by the function $p = 6850(1.03)^x$. What is the average rate of change in the population over the first 6 years? Justify your answer.

50. **REASONING** Consider the exponential function $f(x) = ab^x$.

a. Show that $\dfrac{f(x+1)}{f(x)} = b$.

b. Use the equation in part (a) to explain why there is no exponential function of the form $f(x) = ab^x$ whose graph passes through the points in the table below.

x	0	1	2	3	4
y	4	4	8	24	72

51. **PROBLEM SOLVING** The number E of eggs a Leghorn chicken produces per year can be modeled by the equation $E = 179.2(0.89)^{w/52}$, where w is the age (in weeks) of the chicken and $w \ge 22$.

a. Identify the decay factor and the percent decrease.

b. Graph the model.

c. Estimate the egg production of a chicken that is 2.5 years old.

d. Explain how you can rewrite the given equation so that time is measured in years rather than in weeks.

52. **CRITICAL THINKING** You buy a new stereo for \$1300 and are able to sell it 4 years later for \$275. Assume that the resale value of the stereo decays exponentially with time. Write an equation giving the resale value V (in dollars) of the stereo as a function of the time t (in years) since you bought it.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the expression. *(Skills Review Handbook)*

53. $x^9 \cdot x^2$

54. $\dfrac{x^4}{x^3}$

55. $4x \cdot 6x$

56. $\left(\dfrac{4x^8}{2x^6}\right)^4$

57. $\dfrac{x + 3x}{2}$

58. $\dfrac{6x}{2} + 4x$

59. $\dfrac{12x}{4x} + 5x$

60. $(2x \cdot 3x^5)^3$

6.2 The Natural Base e

Essential Question

What is the natural base e?

So far in your study of mathematics, you have worked with special numbers such as π and i. Another special number is called the *natural base* and is denoted by e. The natural base e is irrational, so you cannot find its exact value.

EXPLORATION 1 Approximating the Natural Base e

Work with a partner. One way to approximate the natural base e is to approximate the sum

$$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots.$$

Use a spreadsheet or a graphing calculator to approximate this sum. Explain the steps you used. How many decimal places did you use in your approximation?

USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technological tools to explore and deepen your understanding of concepts.

EXPLORATION 2 Approximating the Natural Base e

Work with a partner. Another way to approximate the natural base e is to consider the expression

$$\left(1 + \frac{1}{x}\right)^x.$$

As x increases, the value of this expression approaches the value of e. Copy and complete the table. Then use the results in the table to approximate e. Compare this approximation to the one you obtained in Exploration 1.

x	10^1	10^2	10^3	10^4	10^5	10^6
$\left(1 + \frac{1}{x}\right)^x$						

EXPLORATION 3 Graphing a Natural Base Function

Work with a partner. Use your approximate value of e in Exploration 1 or 2 to complete the table. Then sketch the graph of the *natural base exponential function* $y = e^x$. You can use a graphing calculator and the e^x key to check your graph. What are the domain and range of $y = e^x$? Justify your answers.

x	-2	-1	0	1	2
$y = e^x$					

Communicate Your Answer

4. What is the natural base e?
5. Repeat Exploration 3 for the natural base exponential function $y = e^{-x}$. Then compare the graph of $y = e^x$ to the graph of $y = e^{-x}$.
6. The natural base e is used in a wide variety of real-life applications. Use the Internet or some other reference to research some of the real-life applications of e.

6.2 Lesson

Core Vocabulary

natural base *e*, *p. 304*

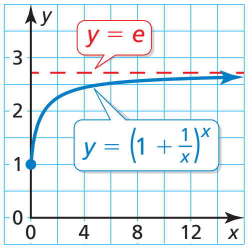
Previous
irrational number
properties of exponents
percent increase
percent decrease
compound interest

What You Will Learn

- Define and use the natural base e.
- Graph natural base functions.
- Solve real-life problems.

The Natural Base e

The history of mathematics is marked by the discovery of special numbers, such as π and i. Another special number is denoted by the letter e. The number is called the **natural base e**. The expression $\left(1 + \frac{1}{x}\right)^x$ approaches e as x increases, as shown in the graph and table.



x	10^1	10^2	10^3	10^4	10^5	10^6
$\left(1 + \frac{1}{x}\right)^x$	2.59374	2.70481	2.71692	2.71815	2.71827	2.71828

Core Concept

The Natural Base e

The natural base e is irrational. It is defined as follows:

As x approaches $+\infty$, $\left(1 + \frac{1}{x}\right)^x$ approaches $e \approx 2.71828182846$.

EXAMPLE 1 Simplifying Natural Base Expressions

Simplify each expression.

a. $e^3 \cdot e^6$ **b.** $\frac{16e^5}{4e^4}$ **c.** $(3e^{-4x})^2$

SOLUTION

a. $e^3 \cdot e^6 = e^{3+6}$
$= e^9$

b. $\frac{16e^5}{4e^4} = 4e^{5-4}$
$= 4e$

c. $(3e^{-4x})^2 = 3^2(e^{-4x})^2$
$= 9e^{-8x}$
$= \frac{9}{e^{8x}}$

Check

You can use a calculator to check the equivalence of numerical expressions involving e.

```
e^(3)*e^(6)
     8103.083928
e^(9)
     8103.083928
```

Monitoring Progress

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Simplify the expression.

1. $e^7 \cdot e^4$ **2.** $\frac{24e^8}{8e^5}$ **3.** $(10e^{-3x})^3$

Graphing Natural Base Functions

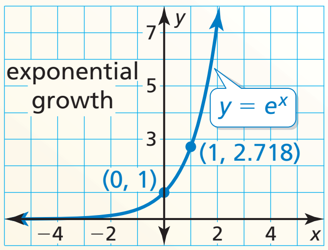
Core Concept

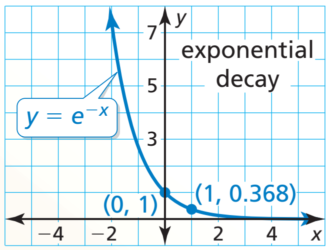
Natural Base Functions

A function of the form $y = ae^{rx}$ is called a *natural base exponential function*.

- When $a > 0$ and $r > 0$, the function is an exponential growth function.
- When $a > 0$ and $r < 0$, the function is an exponential decay function.

The graphs of the basic functions $y = e^x$ and $y = e^{-x}$ are shown.





EXAMPLE 2 Graphing Natural Base Functions

Tell whether each function represents *exponential growth* or *exponential decay*. Then graph the function.

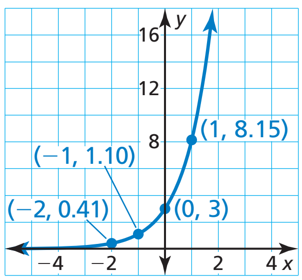
a. $y = 3e^x$

b. $f(x) = e^{-0.5x}$

SOLUTION

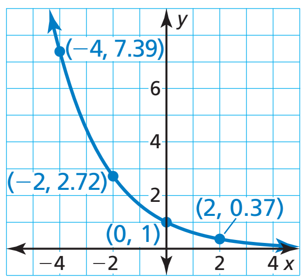
a. Because $a = 3$ is positive and $r = 1$ is positive, the function is an exponential growth function. Use a table to graph the function.

x	-2	-1	0	1
y	0.41	1.10	3	8.15



b. Because $a = 1$ is positive and $r = -0.5$ is negative, the function is an exponential decay function. Use a table to graph the function.

x	-4	-2	0	2
y	7.39	2.72	1	0.37



LOOKING FOR STRUCTURE

You can rewrite natural base exponential functions to find percent rates of change. In Example 2(b),

$$\begin{aligned} f(x) &= e^{-0.5x} \\ &= (e^{-0.5})^x \\ &\approx (0.6065)^x \\ &= (1 - 0.3935)^x. \end{aligned}$$

So, the percent decrease is about 39.35%.

Monitoring Progress

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Tell whether the function represents *exponential growth* or *exponential decay*. Then graph the function.

4. $y = \frac{1}{2}e^x$

5. $y = 4e^{-x}$

6. $f(x) = 2e^{2x}$

Solving Real-Life Problems

You have learned that the balance of an account earning compound interest is given by $A = P\left(1 + \frac{r}{n}\right)^{nt}$. As the frequency n of compounding approaches positive infinity, the compound interest formula approximates the following formula.

Core Concept

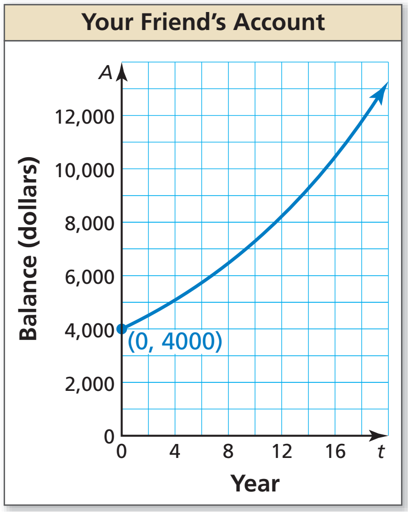
Continuously Compounded Interest

When interest is compounded *continuously*, the amount A in an account after t years is given by the formula

$$A = Pe^{rt}$$

where P is the principal and r is the annual interest rate expressed as a decimal.

EXAMPLE 3 Modeling with Mathematics



You and your friend each have accounts that earn annual interest compounded continuously. The balance A (in dollars) of your account after t years can be modeled by $A = 4500e^{0.04t}$. The graph shows the balance of your friend's account over time. Which account has a greater principal? Which has a greater balance after 10 years?

SOLUTION

1. **Understand the Problem** You are given a graph and an equation that represent account balances. You are asked to identify the account with the greater principal and the account with the greater balance after 10 years.

2. **Make a Plan** Use the equation to find your principal and account balance after 10 years. Then compare these values to the graph of your friend's account.

3. **Solve the Problem** The equation $A = 4500e^{0.04t}$ is of the form $A = Pe^{rt}$, where $P = 4500$. So, your principal is \$4500. Your balance A when $t = 10$ is

$$A = 4500e^{0.04(10)} = \$6713.21.$$

Because the graph passes through (0, 4000), your friend's principal is \$4000. The graph also shows that the balance is about \$7250 when $t = 10$.

▶ So, your account has a greater principal, but your friend's account has a greater balance after 10 years.

4. **Look Back** Because your friend's account has a lesser principal but a greater balance after 10 years, the average rate of change from $t = 0$ to $t = 10$ should be greater for your friend's account than for your account.

Your account: $\dfrac{A(10) - A(0)}{10 - 0} = \dfrac{6713.21 - 4500}{10} = 221.321$

Your friend's account: $\dfrac{A(10) - A(0)}{10 - 0} \approx \dfrac{7250 - 4000}{10} = 325$ ✓

MAKING CONJECTURES

You can also use this reasoning to conclude that your friend's account has a greater annual interest rate than your account.

Monitoring Progress

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7. You deposit \$4250 in an account that earns 5% annual interest compounded continuously. Compare the balance after 10 years with the accounts in Example 3.

6.2 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** What is the natural base e?

2. **WRITING** Tell whether the function $f(x) = \frac{1}{3}e^{4x}$ represents exponential growth or exponential decay. Explain.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–12, simplify the expression. *(See Example 1.)*

3. $e^3 \cdot e^5$
4. $e^{-4} \cdot e^6$
5. $\dfrac{11e^9}{22e^{10}}$
6. $\dfrac{27e^7}{3e^4}$
7. $(5e^{7x})^4$
8. $(4e^{-2x})^3$
9. $\sqrt{9e^{6x}}$
10. $\sqrt[3]{8e^{12x}}$
11. $e^x \cdot e^{-6x} \cdot e^8$
12. $e^x \cdot e^4 \cdot e^{x+3}$

ERROR ANALYSIS **In Exercises 13 and 14, describe and correct the error in simplifying the expression.**

13. ✗
$$(4e^{3x})^2 = 4e^{(3x)(2)} = 4e^{6x}$$

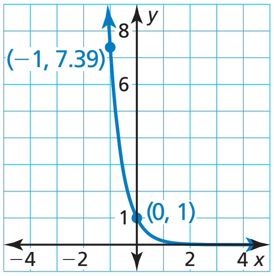
14. ✗
$$\frac{e^{5x}}{e^{-2x}} = e^{5x-2x} = e^{3x}$$

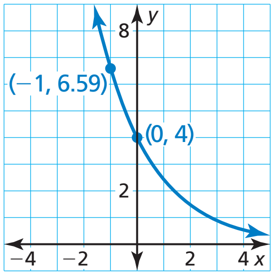
In Exercises 15–22, tell whether the function represents *exponential growth* or *exponential decay*. Then graph the function. *(See Example 2.)*

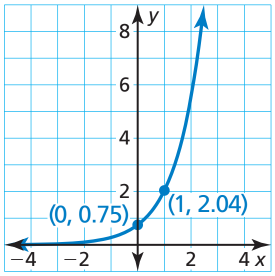
15. $y = e^{3x}$
16. $y = e^{-2x}$
17. $y = 2e^{-x}$
18. $y = 3e^{2x}$
19. $y = 0.5e^x$
20. $y = 0.25e^{-3x}$
21. $y = 0.4e^{-0.25x}$
22. $y = 0.6e^{0.5x}$

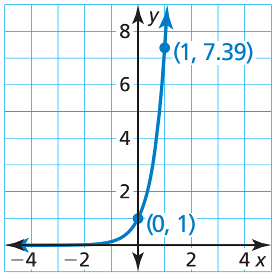
ANALYZING EQUATIONS **In Exercises 23–26, match the function with its graph. Explain your reasoning.**

23. $y = e^{2x}$
24. $y = e^{-2x}$
25. $y = 4e^{-0.5x}$
26. $y = 0.75e^x$

A.


B.


C.


D.


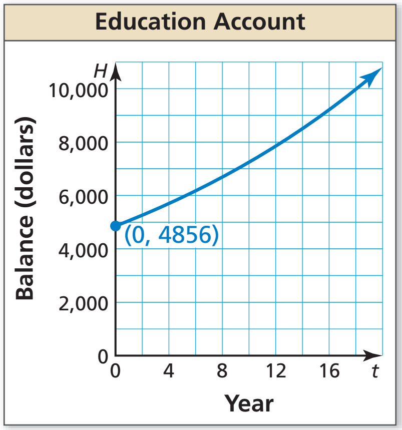
USING STRUCTURE **In Exercises 27–30, use the properties of exponents to rewrite the function in the form $y = a(1 + r)^t$ or $y = a(1 - r)^t$. Then find the percent rate of change.**

27. $y = e^{-0.25t}$
28. $y = e^{-0.75t}$
29. $y = 2e^{0.4t}$
30. $y = 0.5e^{0.8t}$

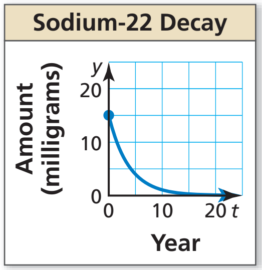
USING TOOLS **In Exercises 31–34, use a table of values or a graphing calculator to graph the function. Then identify the domain and range.**

31. $y = e^{x-2}$
32. $y = e^{x+1}$
33. $y = 2e^x + 1$
34. $y = 3e^x - 5$

35. MODELING WITH MATHEMATICS Investment accounts for a house and education earn annual interest compounded continuously. The balance H (in dollars) of the house fund after t years can be modeled by $H = 3224e^{0.05t}$. The graph shows the balance in the education fund over time. Which account has the greater principal? Which account has a greater balance after 10 years? *(See Example 3.)*



36. MODELING WITH MATHEMATICS Tritium and sodium-22 decay over time. In a sample of tritium, the amount y (in milligrams) remaining after t years is given by $y = 10e^{-0.0562t}$. The graph shows the amount of sodium-22 in a sample over time. Which sample started with a greater amount? Which has a greater amount after 10 years?



37. OPEN-ENDED Find values of a, b, r, and q such that $f(x) = ae^{rx}$ and $g(x) = be^{qx}$ are exponential decay functions, but $\dfrac{f(x)}{g(x)}$ represents exponential growth.

38. THOUGHT PROVOKING Explain why $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$ approximates $A = Pe^{rt}$ as n approaches positive infinity.

39. WRITING Can the natural base e be written as a ratio of two integers? Explain.

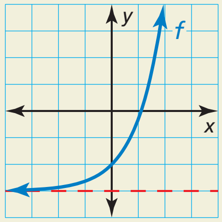
40. MAKING AN ARGUMENT Your friend evaluates $f(x) = e^{-x}$ when $x = 1000$ and concludes that the graph of $y = f(x)$ has an x-intercept at $(1000, 0)$. Is your friend correct? Explain your reasoning.

41. DRAWING CONCLUSIONS You invest \$2500 in an account to save for college. Account 1 pays 6% annual interest compounded quarterly. Account 2 pays 4% annual interest compounded continuously. Which account should you choose to obtain the greater amount in 10 years? Justify your answer.

42. HOW DO YOU SEE IT? Use the graph to complete each statement.

a. $f(x)$ approaches ____ as x approaches $+\infty$.

b. $f(x)$ approaches ____ as x approaches $-\infty$.



43. PROBLEM SOLVING The growth of *Mycobacterium tuberculosis* bacteria can be modeled by the function $N(t) = ae^{0.166t}$, where N is the number of cells after t hours and a is the number of cells when $t = 0$.

a. At 1:00 P.M., there are 30 *M. tuberculosis* bacteria in a sample. Write a function that gives the number of bacteria after 1:00 P.M.

b. Use a graphing calculator to graph the function in part (a).

c. Describe how to find the number of cells in the sample at 3:45 P.M.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write the number in scientific notation. *(Skills Review Handbook)*

44. 0.006 **45.** 5000 **46.** 26,000,000 **47.** 0.000000047

Find the inverse of the function. Then graph the function and its inverse. *(Section 5.6)*

48. $y = 3x + 5$ **49.** $y = x^2 - 1, x \le 0$

50. $y = \sqrt{x + 6}$ **51.** $y = x^3 - 2$

6.3 Logarithms and Logarithmic Functions

Essential Question

What are some of the characteristics of the graph of a logarithmic function?

Every exponential function of the form $f(x) = b^x$, where b is a positive real number other than 1, has an inverse function that you can denote by $g(x) = \log_b x$. This inverse function is called a *logarithmic function with base b*.

EXPLORATION 1 Rewriting Exponential Equations

Work with a partner. Find the value of x in each exponential equation. Explain your reasoning. Then use the value of x to rewrite the exponential equation in its equivalent logarithmic form, $x = \log_b y$.

a. $2^x = 8$ **b.** $3^x = 9$ **c.** $4^x = 2$

d. $5^x = 1$ **e.** $5^x = \frac{1}{5}$ **f.** $8^x = 4$

EXPLORATION 2 Graphing Exponential and Logarithmic Functions

Work with a partner. Complete each table for the given exponential function. Use the results to complete the table for the given logarithmic function. Explain your reasoning. Then sketch the graphs of f and g in the same coordinate plane.

a.

x	-2	-1	0	1	2
$f(x) = 2^x$					

x					
$g(x) = \log_2 x$	-2	-1	0	1	2

b.

x	-2	-1	0	1	2
$f(x) = 10^x$					

x					
$g(x) = \log_{10} x$	-2	-1	0	1	2

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to justify your conclusions and communicate them to others.

EXPLORATION 3 Characteristics of Graphs of Logarithmic Functions

Work with a partner. Use the graphs you sketched in Exploration 2 to determine the domain, range, x-intercept, and asymptote of the graph of $g(x) = \log_b x$, where b is a positive real number other than 1. Explain your reasoning.

Communicate Your Answer

4. What are some of the characteristics of the graph of a logarithmic function?

5. How can you use the graph of an exponential function to obtain the graph of a logarithmic function?

6.3 Lesson

Core Vocabulary

logarithm of y with base b, *p. 310*
common logarithm, *p. 311*
natural logarithm, *p. 311*

Previous
inverse functions

What You Will Learn

- Define and evaluate logarithms.
- Use inverse properties of logarithmic and exponential functions.
- Graph logarithmic functions.

Logarithms

You know that $2^2 = 4$ and $2^3 = 8$. However, for what value of x does $2^x = 6$? Mathematicians define this x-value using a *logarithm* and write $x = \log_2 6$. The definition of a logarithm can be generalized as follows.

Core Concept

Definition of Logarithm with Base *b*

Let b and y be positive real numbers with $b \neq 1$. The **logarithm of y with base b** is denoted by $\log_b y$ and is defined as

$$\log_b y = x \qquad \text{if and only if} \qquad b^x = y.$$

The expression $\log_b y$ is read as "log base b of y."

This definition tells you that the equations $\log_b y = x$ and $b^x = y$ are equivalent. The first is in *logarithmic form*, and the second is in *exponential form*.

EXAMPLE 1 Rewriting Logarithmic Equations

Rewrite each equation in exponential form.

a. $\log_2 16 = 4$ **b.** $\log_4 1 = 0$ **c.** $\log_{12} 12 = 1$ **d.** $\log_{1/4} 4 = -1$

SOLUTION

Logarithmic Form	**Exponential Form**
a. $\log_2 16 = 4$	$2^4 = 16$
b. $\log_4 1 = 0$	$4^0 = 1$
c. $\log_{12} 12 = 1$	$12^1 = 12$
d. $\log_{1/4} 4 = -1$	$\left(\frac{1}{4}\right)^{-1} = 4$

EXAMPLE 2 Rewriting Exponential Equations

Rewrite each equation in logarithmic form.

a. $5^2 = 25$ **b.** $10^{-1} = 0.1$ **c.** $8^{2/3} = 4$ **d.** $6^{-3} = \frac{1}{216}$

SOLUTION

Exponential Form	**Logarithmic Form**
a. $5^2 = 25$	$\log_5 25 = 2$
b. $10^{-1} = 0.1$	$\log_{10} 0.1 = -1$
c. $8^{2/3} = 4$	$\log_8 4 = \frac{2}{3}$
d. $6^{-3} = \frac{1}{216}$	$\log_6 \frac{1}{216} = -3$

Parts (b) and (c) of Example 1 illustrate two special logarithm values that you should learn to recognize. Let b be a positive real number such that $b \neq 1$.

Logarithm of 1

$\log_b 1 = 0$ because $b^0 = 1$.

Logarithm of b with Base b

$\log_b b = 1$ because $b^1 = b$.

EXAMPLE 3 Evaluating Logarithmic Expressions

Evaluate each logarithm.

a. $\log_4 64$ **b.** $\log_5 0.2$ **c.** $\log_{1/5} 125$ **d.** $\log_{36} 6$

SOLUTION

To help you find the value of $\log_b y$, ask yourself what power of b gives you y.

a. What power of 4 gives you 64? $4^3 = 64$, so $\log_4 64 = 3$.

b. What power of 5 gives you 0.2? $5^{-1} = 0.2$, so $\log_5 0.2 = -1$.

c. What power of $\frac{1}{5}$ gives you 125? $\left(\frac{1}{5}\right)^{-3} = 125$, so $\log_{1/5} 125 = -3$.

d. What power of 36 gives you 6? $36^{1/2} = 6$, so $\log_{36} 6 = \frac{1}{2}$.

A **common logarithm** is a logarithm with base 10. It is denoted by $\log_{10}$ or simply by log. A **natural logarithm** is a logarithm with base e. It can be denoted by $\log_e$ but is usually denoted by ln.

Common Logarithm

$\log_{10} x = \log x$

Natural Logarithm

$\log_e x = \ln x$

EXAMPLE 4 Evaluating Common and Natural Logarithms

Evaluate (a) log 8 and (b) ln 0.3 using a calculator. Round your answer to three decimal places.

SOLUTION

Most calculators have keys for evaluating common and natural logarithms.

a. $\log 8 \approx 0.903$

b. $\ln 0.3 \approx -1.204$

Check your answers by rewriting each logarithm in exponential form and evaluating.

```
log(8)
      .903089987
ln(0.3)
    -1.203972804
```

Check

```
10^(0.903)
      7.99834255
e^(-1.204)
     .2999918414
```

Monitoring Progress

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Rewrite the equation in exponential form.

1. $\log_3 81 = 4$ **2.** $\log_7 7 = 1$ **3.** $\log_{14} 1 = 0$ **4.** $\log_{1/2} 32 = -5$

Rewrite the equation in logarithmic form.

5. $7^2 = 49$ **6.** $50^0 = 1$ **7.** $4^{-1} = \frac{1}{4}$ **8.** $256^{1/8} = 2$

Evaluate the logarithm. If necessary, use a calculator and round your answer to three decimal places.

9. $\log_2 32$ **10.** $\log_{27} 3$ **11.** $\log 12$ **12.** $\ln 0.75$

Using Inverse Properties

By the definition of a logarithm, it follows that the logarithmic function $g(x) = \log_b x$ is the inverse of the exponential function $f(x) = b^x$. This means that

$$g(f(x)) = \log_b b^x = x \qquad \text{and} \qquad f(g(x)) = b^{\log_b x} = x.$$

In other words, exponential functions and logarithmic functions "undo" each other.

EXAMPLE 5 Using Inverse Properties

Simplify (a) $10^{\log 4}$ and (b) $\log_5 25^x$.

SOLUTION

a. $10^{\log 4} = 4$ — $b^{\log_b x} = x$

b. $\log_5 25^x = \log_5 (5^2)^x$ — Express 25 as a power with base 5.

$= \log_5 5^{2x}$ — Power of a Power Property

$= 2x$ — $\log_b b^x = x$

EXAMPLE 6 Finding Inverse Functions

Find the inverse of each function.

a. $f(x) = 6^x$

b. $y = \ln(x + 3)$

SOLUTION

a. From the definition of logarithm, the inverse of $f(x) = 6^x$ is $g(x) = \log_6 x$.

b.

$y = \ln(x + 3)$ — Write original function.

$x = \ln(y + 3)$ — Switch x and y.

$e^x = y + 3$ — Write in exponential form.

$e^x - 3 = y$ — Subtract 3 from each side.

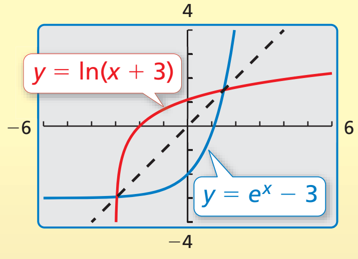
▶ The inverse of $y = \ln(x + 3)$ is $y = e^x - 3$.

Check

a. $f(g(x)) = 6^{\log_6 x} = x$ ✓

$g(f(x)) = \log_6 6^x = x$ ✓

b.



The graphs appear to be reflections of each other in the line $y = x$. ✓

Monitoring Progress

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Simplify the expression.

13. $8^{\log_8 x}$

14. $\log_7 7^{-3x}$

15. $\log_2 64^x$

16. $e^{\ln 20}$

17. Find the inverse of $y = 4^x$.

18. Find the inverse of $y = \ln(x - 5)$.

Graphing Logarithmic Functions

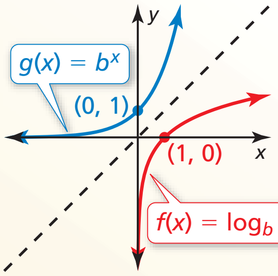
You can use the inverse relationship between exponential and logarithmic functions to graph logarithmic functions.

Core Concept

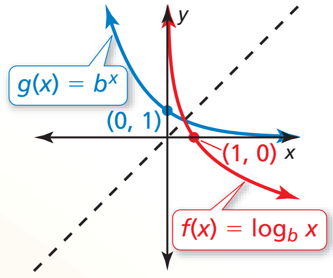
Parent Graphs for Logarithmic Functions

The graph of $f(x) = \log_b x$ is shown below for $b > 1$ and for $0 < b < 1$. Because $f(x) = \log_b x$ and $g(x) = b^x$ are inverse functions, the graph of $f(x) = \log_b x$ is the reflection of the graph of $g(x) = b^x$ in the line $y = x$.

Graph of $f(x) = \log_b x$ for $b > 1$



Graph of $f(x) = \log_b x$ for $0 < b < 1$



Note that the y-axis is a vertical asymptote of the graph of $f(x) = \log_b x$. The domain of $f(x) = \log_b x$ is $x > 0$, and the range is all real numbers.

EXAMPLE 7 Graphing a Logarithmic Function

Graph $f(x) = \log_3 x$.

SOLUTION

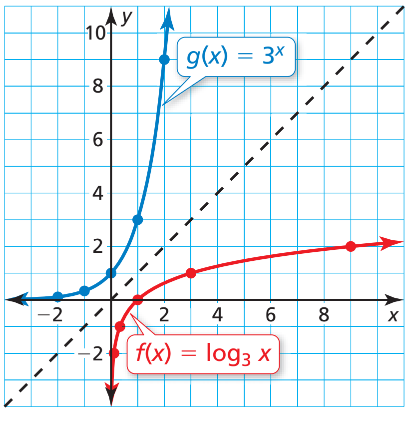
Step 1 Find the inverse of f. From the definition of logarithm, the inverse of $f(x) = \log_3 x$ is $g(x) = 3^x$.

Step 2 Make a table of values for $g(x) = 3^x$.

x	-2	-1	0	1	2
$g(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

Step 3 Plot the points from the table and connect them with a smooth curve.

Step 4 Because $f(x) = \log_3 x$ and $g(x) = 3^x$ are inverse functions, the graph of f is obtained by reflecting the graph of g in the line $y = x$. To do this, reverse the coordinates of the points on g and plot these new points on the graph of f.



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Graph the function.

19. $y = \log_2 x$

20. $f(x) = \log_5 x$

21. $y = \log_{1/2} x$

6.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. COMPLETE THE SENTENCE A logarithm with base 10 is called a(n) __________ logarithm.

2. COMPLETE THE SENTENCE The expression $\log_3 9$ is read as ____________.

3. WRITING Describe the relationship between $y = 7^x$ and $y = \log_7 x$.

4. DIFFERENT WORDS, SAME QUESTION Which is different? Find "both" answers.

What power of 4 gives you 16?	What is log base 4 of 16?
Evaluate 4^2.	Evaluate $\log_4 16$.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, rewrite the equation in exponential form. *(See Example 1.)*

5. $\log_3 9 = 2$

6. $\log_4 4 = 1$

7. $\log_6 1 = 0$

8. $\log_7 343 = 3$

9. $\log_{1/2} 16 = -4$

10. $\log_3 \frac{1}{3} = -1$

In Exercises 11–16, rewrite the equation in logarithmic form. *(See Example 2.)*

11. $6^2 = 36$

12. $12^0 = 1$

13. $16^{-1} = \frac{1}{16}$

14. $5^{-2} = \frac{1}{25}$

15. $125^{2/3} = 25$

16. $49^{1/2} = 7$

In Exercises 17–24, evaluate the logarithm. *(See Example 3.)*

17. $\log_3 81$

18. $\log_7 49$

19. $\log_3 3$

20. $\log_{1/2} 1$

21. $\log_5 \frac{1}{625}$

22. $\log_8 \frac{1}{512}$

23. $\log_4 0.25$

24. $\log_{10} 0.001$

25. NUMBER SENSE Order the logarithms from least value to greatest value.

$\log_5 23$	$\log_6 38$	$\log_7 8$	$\log_2 10$

26. WRITING Explain why the expressions $\log_2(-1)$ and $\log_1 1$ are not defined.

In Exercises 27–32, evaluate the logarithm using a calculator. Round your answer to three decimal places. *(See Example 4.)*

27. $\log 6$

28. $\ln 12$

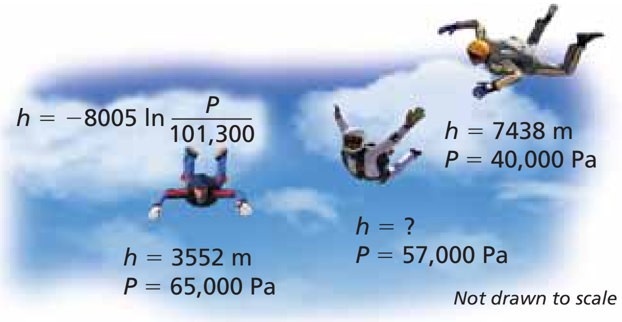
29. $\ln \frac{1}{3}$

30. $\log \frac{2}{7}$

31. $3 \ln 0.5$

32. $\log 0.6 + 1$

33. MODELING WITH MATHEMATICS Skydivers use an instrument called an *altimeter* to track their altitude as they fall. The altimeter determines altitude by measuring air pressure. The altitude h (in meters) above sea level is related to the air pressure P (in pascals) by the function shown in the diagram. What is the altitude above sea level when the air pressure is 57,000 pascals?



34. MODELING WITH MATHEMATICS The pH value for a substance measures how acidic or alkaline the substance is. It is given by the formula $\text{pH} = -\log[\text{H}^+]$, where H^+ is the hydrogen ion concentration (in moles per liter). Find the pH of each substance.

a. baking soda: $[\text{H}^+] = 10^{-8}$ moles per liter

b. vinegar: $[\text{H}^+] = 10^{-3}$ moles per liter

In Exercises 35–40, simplify the expression. *(See Example 5.)*

35. $7^{\log_7 x}$

36. $3^{\log_3 5x}$

37. $e^{\ln 4}$

38. $10^{\log 15}$

39. $\log_3 3^{2x}$

40. $\ln e^{x+1}$

41. ERROR ANALYSIS Describe and correct the error in rewriting $4^{-3} = \frac{1}{64}$ in logarithmic form.

✗ $\log_4(-3) = \frac{1}{64}$

42. ERROR ANALYSIS Describe and correct the error in simplifying the expression $\log_4 64^x$.

✗
$$\begin{aligned} \log_4 64^x &= \log_4(16 \cdot 4^x) \\ &= \log_4(4^2 \cdot 4^x) \\ &= \log_4 4^{2+x} \\ &= 2 + x \end{aligned}$$

In Exercises 43–52, find the inverse of the function. *(See Example 6.)*

43. $y = 0.3^x$

44. $y = 11^x$

45. $y = \log_2 x$

46. $y = \log_{1/5} x$

47. $y = \ln(x - 1)$

48. $y = \ln 2x$

49. $y = e^{3x}$

50. $y = e^{x-4}$

51. $y = 5^x - 9$

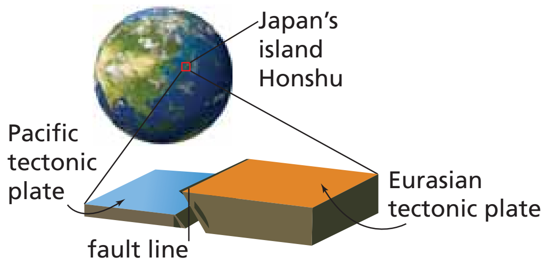
52. $y = 13 + \log x$

53. PROBLEM SOLVING The wind speed s (in miles per hour) near the center of a tornado can be modeled by $s = 93 \log d + 65$, where d is the distance (in miles) that the tornado travels.

a. In 1925, a tornado traveled 220 miles through three states. Estimate the wind speed near the center of the tornado.

b. Find the inverse of the given function. Describe what the inverse represents.

54. MODELING WITH MATHEMATICS The energy magnitude M of an earthquake can be modeled by $M = \frac{2}{3} \log E - 9.9$, where E is the amount of energy released (in ergs).



a. In 2011, a powerful earthquake in Japan, caused by the slippage of two tectonic plates along a fault, released 2.24×10^{28} ergs. What was the energy magnitude of the earthquake?

b. Find the inverse of the given function. Describe what the inverse represents.

In Exercises 55–60, graph the function. *(See Example 7.)*

55. $y = \log_4 x$

56. $y = \log_6 x$

57. $y = \log_{1/3} x$

58. $y = \log_{1/4} x$

59. $y = \log_2 x - 1$

60. $y = \log_3(x + 2)$

USING TOOLS In Exercises 61–64, use a graphing calculator to graph the function. Determine the domain, range, and asymptote of the function.

61. $y = \log(x + 2)$

62. $y = -\ln x$

63. $y = \ln(-x)$

64. $y = 3 - \log x$

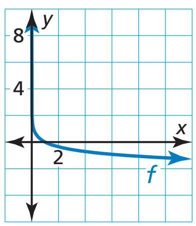
65. MAKING AN ARGUMENT Your friend states that every logarithmic function will pass through the point (1, 0). Is your friend correct? Explain your reasoning.

66. ANALYZING RELATIONSHIPS Rank the functions in order from the least average rate of change to the greatest average rate of change over the interval $1 \le x \le 10$.

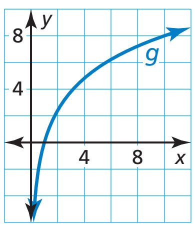
a. $y = \log_6 x$

b. $y = \log_{3/5} x$

c.



d.



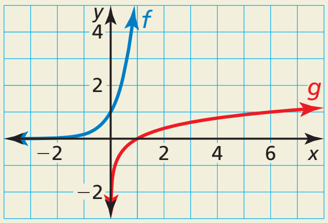
67. **PROBLEM SOLVING** Biologists have found that the length ℓ (in inches) of an alligator and its weight w (in pounds) are related by the function $\ell = 27.1 \ln w - 32.8$.

a. Use a graphing calculator to graph the function.

b. Use your graph to estimate the weight of an alligator that is 10 feet long.

c. Use the *zero* feature to find the x-intercept of the graph of the function. Does this x-value make sense in the context of the situation? Explain.

68. **HOW DO YOU SEE IT?** The figure shows the graphs of the two functions f and g.



a. Compare the end behavior of the logarithmic function g to that of the exponential function f.

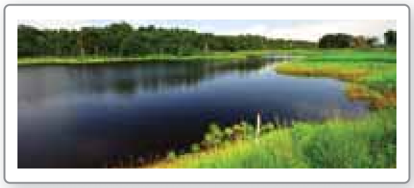
b. Determine whether the functions are inverse functions. Explain.

c. What is the base of each function? Explain.

69. **PROBLEM SOLVING** A study in Florida found that the number s of fish species in a pool or lake can be modeled by the function

$$s = 30.6 - 20.5 \log A + 3.8(\log A)^2$$

where A is the area (in square meters) of the pool or lake.



a. Use a graphing calculator to graph the function on the domain $200 \le A \le 35{,}000$.

b. Use your graph to estimate the number of species in a lake with an area of 30,000 square meters.

c. Use your graph to estimate the area of a lake that contains six species of fish.

d. Describe what happens to the number of fish species as the area of a pool or lake increases. Explain why your answer makes sense.

70. **THOUGHT PROVOKING** Write a logarithmic function that has an output of -4. Then sketch the graph of your function.

71. **CRITICAL THINKING** Evaluate each logarithm. (*Hint*: For each logarithm $\log_b x$, rewrite b and x as powers of the same base.)

a. $\log_{125} 25$ **b.** $\log_8 32$

c. $\log_{27} 81$ **d.** $\log_4 128$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Let $f(x) = \sqrt[3]{x}$. Write a rule for g that represents the indicated transformation of the graph of f. *(Section 5.3)*

72. $g(x) = -f(x)$

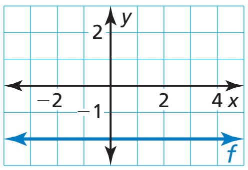
73. $g(x) = f\left(\frac{1}{2}x\right)$

74. $g(x) = f(-x) + 3$

75. $g(x) = f(x + 2)$

Identify the function family to which f belongs. Compare the graph of f to the graph of its parent function. *(Section 1.1)*

76.



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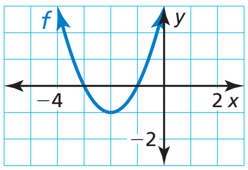

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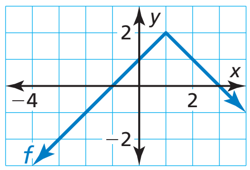


6.4 Transformations of Exponential and Logarithmic Functions

Essential Question

How can you transform the graphs of exponential and logarithmic functions?

EXPLORATION 1 Identifying Transformations

Work with a partner. Each graph shown is a transformation of the parent function

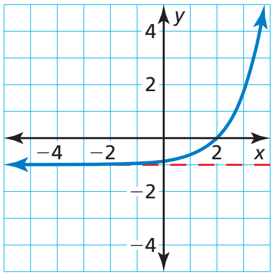
$f(x) = e^x$ or $f(x) = \ln x$.

Match each function with its graph. Explain your reasoning. Then describe the transformation of f represented by g.

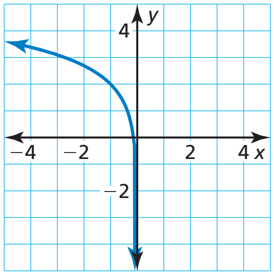
a. $g(x) = e^{x+2} - 3$ **b.** $g(x) = -e^{x+2} + 1$ **c.** $g(x) = e^{x-2} - 1$

d. $g(x) = \ln(x + 2)$ **e.** $g(x) = 2 + \ln x$ **f.** $g(x) = 2 + \ln(-x)$

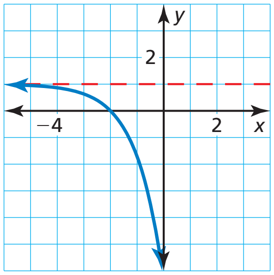
A.



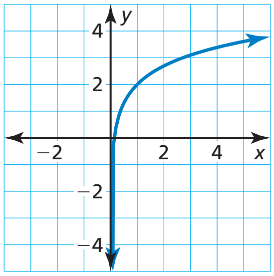
B.



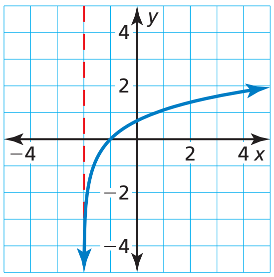
C.



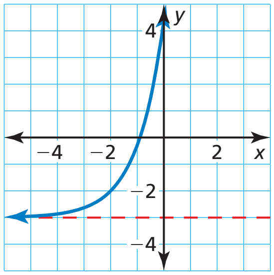
D.



E.



F.



EXPLORATION 2 Characteristics of Graphs

Work with a partner. Determine the domain, range, and asymptote of each function in Exploration 1. Justify your answers.

REASONING QUANTITATIVELY

To be proficient in math, you need to make sense of quantities and their relationships in problem situations.

Communicate Your Answer

3. How can you transform the graphs of exponential and logarithmic functions?

4. Find the inverse of each function in Exploration 1. Then check your answer by using a graphing calculator to graph each function and its inverse in the same viewing window.

6.4 Lesson

Core Vocabulary

Previous
exponential function
logarithmic function
transformations

What You Will Learn

- Transform graphs of exponential functions.
- Transform graphs of logarithmic functions.
- Write transformations of graphs of exponential and logarithmic functions.

Transforming Graphs of Exponential Functions

You can transform graphs of exponential and logarithmic functions in the same way you transformed graphs of functions in previous chapters. Examples of transformations of the graph of $f(x) = 4^x$ are shown below.

Core Concept

Transformation	$f(x)$ Notation	Examples	
Horizontal Translation Graph shifts left or right.	$f(x - h)$	$g(x) = 4^{x-3}$ $g(x) = 4^{x+2}$	3 units right 2 units left
Vertical Translation Graph shifts up or down.	$f(x) + k$	$g(x) = 4^x + 5$ $g(x) = 4^x - 1$	5 units up 1 unit down
Reflection Graph flips over x- or y-axis.	$f(-x)$ $-f(x)$	$g(x) = 4^{-x}$ $g(x) = -4^x$	in the y-axis in the x-axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y-axis.	$f(ax)$	$g(x) = 4^{2x}$ $g(x) = 4^{x/2}$	shrink by a factor of $\frac{1}{2}$ stretch by a factor of 2
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis.	$a \cdot f(x)$	$g(x) = 3(4^x)$ $g(x) = \frac{1}{4}(4^x)$	stretch by a factor of 3 shrink by a factor of $\frac{1}{4}$

EXAMPLE 1 Translating an Exponential Function

Describe the transformation of $f(x) = \left(\frac{1}{2}\right)^x$ represented by $g(x) = \left(\frac{1}{2}\right)^x - 4$. Then graph each function.

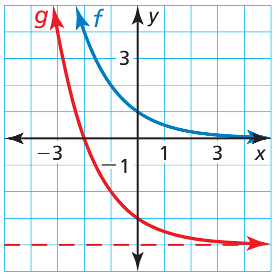
SOLUTION

Notice that the function is of the form $g(x) = \left(\frac{1}{2}\right)^x + k$.

Rewrite the function to identify k.

$$g(x) = \left(\frac{1}{2}\right)^x + \underbrace{(-4)}_{k}$$

- Because $k = -4$, the graph of g is a translation 4 units down of the graph of f.



STUDY TIP

Notice in the graph that the vertical translation also shifted the asymptote 4 units down, so the range of g is $y > -4$.

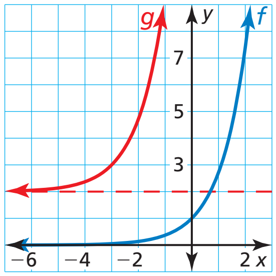
EXAMPLE 2 Translating a Natural Base Exponential Function

Describe the transformation of $f(x) = e^x$ represented by $g(x) = e^{x+3} + 2$. Then graph each function.

SOLUTION

Notice that the function is of the form $g(x) = e^{x-h} + k$. Rewrite the function to identify h and k.

$$g(x) = e^{x-\underset{h}{(-3)}} + \underset{k}{2}$$



▶ Because $h = -3$ and $k = 2$, the graph of g is a translation 3 units left and 2 units up of the graph of f.

STUDY TIP

Notice in the graph that the vertical translation also shifted the asymptote 2 units up, so the range of g is $y > 2$.

EXAMPLE 3 Transforming Exponential Functions

Describe the transformation of f represented by g. Then graph each function.

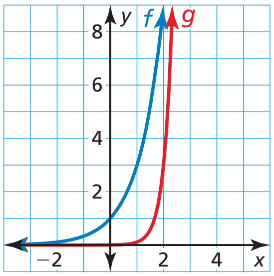
a. $f(x) = 3^x$, $g(x) = 3^{3x-5}$

b. $f(x) = e^{-x}$, $g(x) = -\frac{1}{8}e^{-x}$

SOLUTION

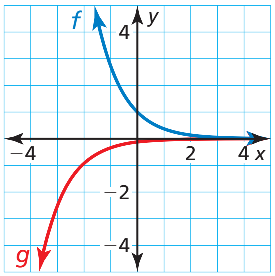
a. Notice that the function is of the form $g(x) = 3^{ax-h}$, where $a = 3$ and $h = 5$.

▶ So, the graph of g is a translation 5 units right, followed by a horizontal shrink by a factor of $\frac{1}{3}$ of the graph of f.



b. Notice that the function is of the form $g(x) = ae^{-x}$, where $a = -\frac{1}{8}$.

▶ So, the graph of g is a reflection in the x-axis and a vertical shrink by a factor of $\frac{1}{8}$ of the graph of f.



LOOKING FOR STRUCTURE

In Example 3(a), the horizontal shrink follows the translation. In the function $h(x) = 3^{3(x-5)}$, the translation 5 units right follows the horizontal shrink by a factor of $\frac{1}{3}$.

Monitoring Progress

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Describe the transformation of f represented by g. Then graph each function.

1. $f(x) = 2^x$, $g(x) = 2^{x-3} + 1$

2. $f(x) = e^{-x}$, $g(x) = e^{-x} - 5$

3. $f(x) = 0.4^x$, $g(x) = 0.4^{-2x}$

4. $f(x) = e^x$, $g(x) = -e^{x+6}$

Transforming Graphs of Logarithmic Functions

Examples of transformations of the graph of $f(x) = \log x$ are shown below.

Core Concept

Transformation	$f(x)$ Notation	Examples	
Horizontal Translation Graph shifts left or right.	$f(x-h)$	$g(x) = \log(x-4)$ $g(x) = \log(x+7)$	4 units right 7 units left
Vertical Translation Graph shifts up or down.	$f(x)+k$	$g(x) = \log x + 3$ $g(x) = \log x - 1$	3 units up 1 unit down
Reflection Graph flips over x- or y-axis.	$f(-x)$ $-f(x)$	$g(x) = \log(-x)$ $g(x) = -\log x$	in the y-axis in the x-axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y-axis.	$f(ax)$	$g(x) = \log(4x)$ $g(x) = \log\left(\frac{1}{3}x\right)$	shrink by a factor of $\frac{1}{4}$ stretch by a factor of 3
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis.	$a \cdot f(x)$	$g(x) = 5 \log x$ $g(x) = \frac{2}{3}\log x$	stretch by a factor of 5 shrink by a factor of $\frac{2}{3}$

EXAMPLE 4 Transforming Logarithmic Functions

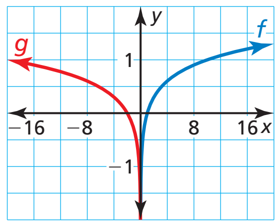
Describe the transformation of f represented by g. Then graph each function.

a. $f(x) = \log x$, $g(x) = \log\left(-\frac{1}{2}x\right)$ **b.** $f(x) = \log_{1/2} x$, $g(x) = 2\log_{1/2}(x+4)$

SOLUTION

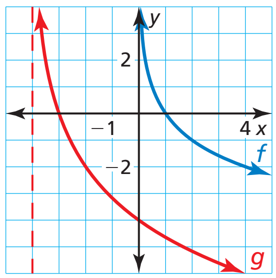
a. Notice that the function is of the form $g(x) = \log(ax)$, where $a = -\frac{1}{2}$.

▶ So, the graph of g is a reflection in the y-axis and a horizontal stretch by a factor of 2 of the graph of f.



b. Notice that the function is of the form $g(x) = a\log_{1/2}(x-h)$, where $a = 2$ and $h = -4$.

▶ So, the graph of g is a horizontal translation 4 units left and a vertical stretch by a factor of 2 of the graph of f.



STUDY TIP

In Example 4(b), notice in the graph that the horizontal translation also shifted the asymptote 4 units left, so the domain of g is $x > -4$.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Describe the transformation of f represented by g. Then graph each function.

5. $f(x) = \log_2 x$, $g(x) = -3 \log_2 x$

6. $f(x) = \log_{1/4} x$, $g(x) = \log_{1/4}(4x) - 5$

Writing Transformations of Graphs of Functions

EXAMPLE 5 Writing a Transformed Exponential Function

Let the graph of g be a reflection in the x-axis followed by a translation 4 units right of the graph of $f(x) = 2^x$. Write a rule for g.

SOLUTION

Step 1 First write a function h that represents the reflection of f.

$h(x) = -f(x)$ — Multiply the output by -1.

$= -2^x$ — Substitute 2^x for $f(x)$.

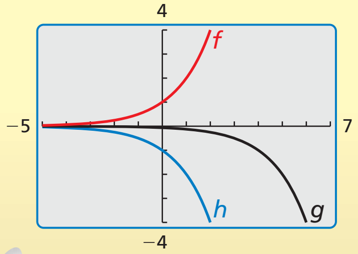
Step 2 Then write a function g that represents the translation of h.

$g(x) = h(x - 4)$ — Subtract 4 from the input.

$= -2^{x-4}$ — Replace x with $x - 4$ in $h(x)$.

▶ The transformed function is $g(x) = -2^{x-4}$.

Check



EXAMPLE 6 Writing a Transformed Logarithmic Function

Let the graph of g be a translation 2 units up followed by a vertical stretch by a factor of 2 of the graph of $f(x) = \log_{1/3} x$. Write a rule for g.

SOLUTION

Step 1 First write a function h that represents the translation of f.

$h(x) = f(x) + 2$ — Add 2 to the output.

$= \log_{1/3} x + 2$ — Substitute $\log_{1/3} x$ for $f(x)$.

Step 2 Then write a function g that represents the vertical stretch of h.

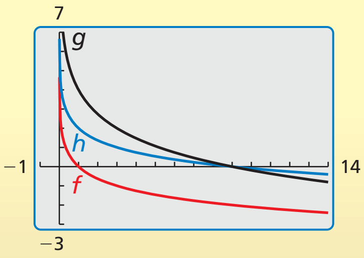
$g(x) = 2 \cdot h(x)$ — Multiply the output by 2.

$= 2 \cdot (\log_{1/3} x + 2)$ — Substitute $\log_{1/3} x + 2$ for $h(x)$.

$= 2 \log_{1/3} x + 4$ — Distributive Property

▶ The transformed function is $g(x) = 2 \log_{1/3} x + 4$.

Check



Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

7. Let the graph of g be a horizontal stretch by a factor of 3, followed by a translation 2 units up of the graph of $f(x) = e^{-x}$. Write a rule for g.

8. Let the graph of g be a reflection in the y-axis, followed by a translation 4 units to the left of the graph of $f(x) = \log x$. Write a rule for g.

6.4 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Given the function $f(x) = ab^{x-h} + k$, describe the effects of a, h, and k on the graph of the function.

2. **COMPLETE THE SENTENCE** The graph of $g(x) = \log_4(-x)$ is a reflection in the __________ of the graph of $f(x) = \log_4 x$.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, match the function with its graph. Explain your reasoning.

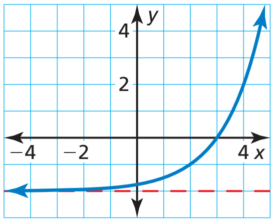
3. $f(x) = 2^{x+2} - 2$

4. $g(x) = 2^{x+2} + 2$

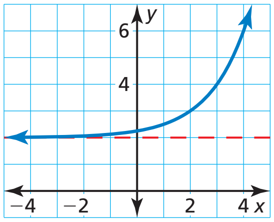
5. $h(x) = 2^{x-2} - 2$

6. $k(x) = 2^{x-2} + 2$

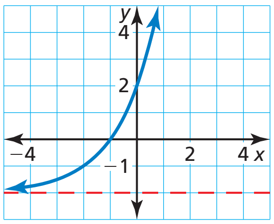
A.



B.



C.



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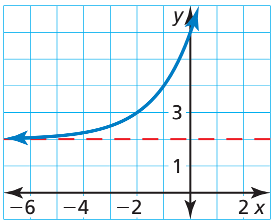


In Exercises 7–16, describe the transformation of f represented by g. Then graph each function. *(See Examples 1 and 2.)*

7. $f(x) = 3^x,\ g(x) = 3^x + 5$

8. $f(x) = 4^x,\ g(x) = 4^x - 8$

9. $f(x) = e^x,\ g(x) = e^x - 1$

10. $f(x) = e^x,\ g(x) = e^x + 4$

11. $f(x) = 2^x,\ g(x) = 2^{x-7}$

12. $f(x) = 5^x,\ g(x) = 5^{x+1}$

13. $f(x) = e^{-x},\ g(x) = e^{-x} + 6$

14. $f(x) = e^{-x},\ g(x) = e^{-x} - 9$

15. $f(x) = \left(\frac{1}{4}\right)^x,\ g(x) = \left(\frac{1}{4}\right)^{x-3} + 12$

16. $f(x) = \left(\frac{1}{3}\right)^x,\ g(x) = \left(\frac{1}{3}\right)^{x+2} - \frac{2}{3}$

In Exercises 17–24, describe the transformation of f represented by g. Then graph each function. *(See Example 3.)*

17. $f(x) = e^x,\ g(x) = e^{2x}$

18. $f(x) = e^x,\ g(x) = \frac{4}{3}e^x$

19. $f(x) = 2^x,\ g(x) = -2^{x-3}$

20. $f(x) = 4^x,\ g(x) = 4^{0.5x-5}$

21. $f(x) = e^{-x},\ g(x) = 3e^{-6x}$

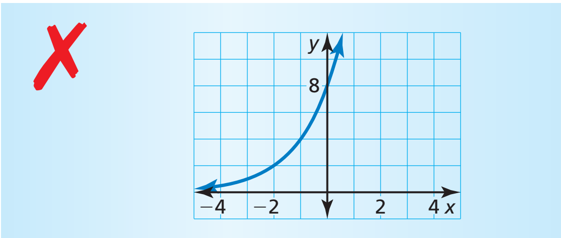
22. $f(x) = e^{-x},\ g(x) = e^{-5x} + 2$

23. $f(x) = \left(\frac{1}{2}\right)^x,\ g(x) = 6\left(\frac{1}{2}\right)^{x+5} - 2$

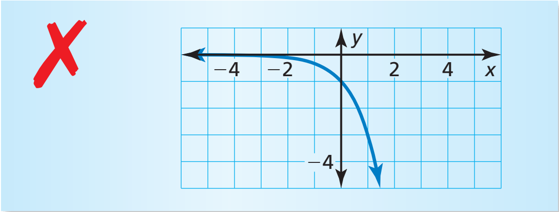
24. $f(x) = \left(\frac{3}{4}\right)^x,\ g(x) = -\left(\frac{3}{4}\right)^{x-7} + 1$

ERROR ANALYSIS In Exercises 25 and 26, describe and correct the error in graphing the function.

25. $f(x) = 2^x + 3$



26. $f(x) = 3^{-x}$



In Exercises 27–30, describe the transformation of *f* represented by *g*. Then graph each function. *(See Example 4.)*

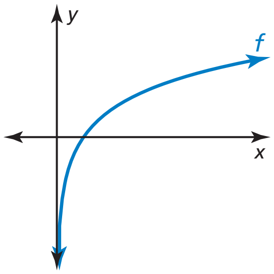
27. $f(x) = \log_4 x,\ g(x) = 3\log_4 x - 5$

28. $f(x) = \log_{1/3} x,\ g(x) = \log_{1/3}(-x) + 6$

29. $f(x) = \log_{1/5} x,\ g(x) = -\log_{1/5}(x - 7)$

30. $f(x) = \log_2 x,\ g(x) = \log_2(x + 2) - 3$

ANALYZING RELATIONSHIPS **In Exercises 31–34, match the function with the correct transformation of the graph of *f*. Explain your reasoning.**



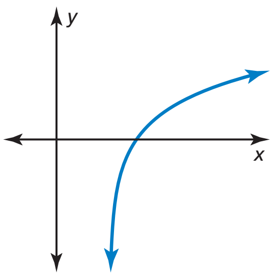
31. $y = f(x - 2)$

32. $y = f(x + 2)$

33. $y = 2f(x)$

34. $y = f(2x)$

A.



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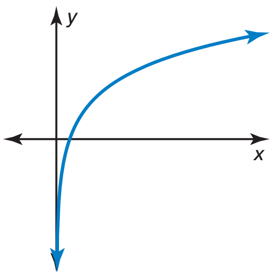

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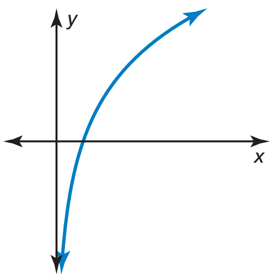

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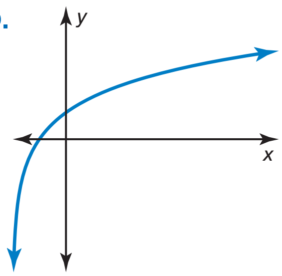


In Exercises 35–38, write a rule for *g* that represents the indicated transformations of the graph of *f*. *(See Example 5.)*

35. $f(x) = 5^x$; translation 2 units down, followed by a reflection in the y-axis

36. $f(x) = \left(\frac{2}{3}\right)^x$; reflection in the x-axis, followed by a vertical stretch by a factor of 6 and a translation 4 units left

37. $f(x) = e^x$; horizontal shrink by a factor of $\frac{1}{2}$, followed by a translation 5 units up

38. $f(x) = e^{-x}$; translation 4 units right and 1 unit down, followed by a vertical shrink by a factor of $\frac{1}{3}$

In Exercises 39–42, write a rule for *g* that represents the indicated transformation of the graph of *f*. *(See Example 6.)*

39. $f(x) = \log_6 x$; vertical stretch by a factor of 6, followed by a translation 5 units down

40. $f(x) = \log_5 x$; reflection in the x-axis, followed by a translation 9 units left

41. $f(x) = \log_{1/2} x$; translation 3 units left and 2 units up, followed by a reflection in the y-axis

42. $f(x) = \ln x$; translation 3 units right and 1 unit up, followed by a horizontal stretch by a factor of 8

JUSTIFYING STEPS **In Exercises 43 and 44, justify each step in writing a rule for *g* that represents the indicated transformations of the graph of *f*.**

43. $f(x) = \log_7 x$; reflection in the x-axis, followed by a translation 6 units down

$$h(x) = -f(x)$$
$$= -\log_7 x$$
$$g(x) = h(x) - 6$$
$$= -\log_7 x - 6$$

44. $f(x) = 8^x$; vertical stretch by a factor of 4, followed by a translation 1 unit up and 3 units left

$$h(x) = 4 \cdot f(x)$$
$$= 4 \cdot 8^x$$
$$g(x) = h(x + 3) + 1$$
$$= 4 \cdot 8^{x+3} + 1$$

USING STRUCTURE In Exercises 45–48, describe the transformation of the graph of f represented by the graph of g. Then give an equation of the asymptote.

45. $f(x) = e^x$, $g(x) = e^x + 4$

46. $f(x) = 3^x$, $g(x) = 3^{x-9}$

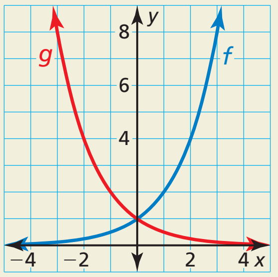
47. $f(x) = \ln x$, $g(x) = \ln(x + 6)$

48. $f(x) = \log_{1/5} x$, $g(x) = \log_{1/5} x + 13$

49. MODELING WITH MATHEMATICS The slope S of a beach is related to the average diameter d (in millimeters) of the sand particles on the beach by the equation $S = 0.159 + 0.118 \log d$. Describe the transformation of $f(d) = \log d$ represented by S. Then use the function to determine the slope of a beach for each sand type below.

Sand particle	Diameter (mm), d
fine sand	0.125
medium sand	0.25
coarse sand	0.5
very coarse sand	1

50. HOW DO YOU SEE IT? The graphs of $f(x) = b^x$ and $g(x) = \left(\frac{1}{b}\right)^x$ are shown for $b = 2$.



a. Use the graph to describe a transformation of the graph of f that results in the graph of g.

b. Does your answer in part (a) change when $0 < b < 1$? Explain.

51. MAKING AN ARGUMENT Your friend claims a single transformation of $f(x) = \log x$ can result in a function g whose graph never intersects the graph of f. Is your friend correct? Explain your reasoning.

52. THOUGHT PROVOKING Is it possible to transform the graph of $f(x) = e^x$ to obtain the graph of $g(x) = \ln x$? Explain your reasoning.

53. ABSTRACT REASONING Determine whether each statement is *always*, *sometimes*, or *never* true. Explain your reasoning.

a. A vertical translation of the graph of $f(x) = \log x$ changes the equation of the asymptote.

b. A vertical translation of the graph of $f(x) = e^x$ changes the equation of the asymptote.

c. A horizontal shrink of the graph of $f(x) = \log x$ does not change the domain.

d. The graph of $g(x) = ab^{x-h} + k$ does not intersect the x-axis.

54. PROBLEM SOLVING The amount P (in grams) of 100 grams of plutonium-239 that remains after t years can be modeled by $P = 100(0.99997)^t$.

a. Describe the domain and range of the function.

b. How much plutonium-239 is present after 12,000 years?

c. Describe the transformation of the function if the initial amount of plutonium were 550 grams.

d. Does the transformation in part (c) affect the domain and range of the function? Explain your reasoning.

55. CRITICAL THINKING Consider the graph of the function $h(x) = e^{-x-2}$. Describe the transformation of the graph of $f(x) = e^{-x}$ represented by the graph of h. Then describe the transformation of the graph of $g(x) = e^x$ represented by the graph of h. Justify your answers.

56. OPEN-ENDED Write a function of the form $y = ab^{x-h} + k$ whose graph has a y-intercept of 5 and an asymptote of $y = 2$.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Perform the indicated operation. *(Section 5.5)*

57. Let $f(x) = x^4$ and $g(x) = x^2$. Find $(fg)(x)$. Then evaluate the product when $x = 3$.

58. Let $f(x) = 4x^6$ and $g(x) = 2x^3$. Find $\left(\frac{f}{g}\right)(x)$. Then evaluate the quotient when $x = 5$.

59. Let $f(x) = 6x^3$ and $g(x) = 8x^3$. Find $(f + g)(x)$. Then evaluate the sum when $x = 2$.

60. Let $f(x) = 2x^2$ and $g(x) = 3x^2$. Find $(f - g)(x)$. Then evaluate the difference when $x = 6$.

6.1–6.4 What Did You Learn?

Core Vocabulary

exponential function, *p. 296*
exponential growth function, *p. 296*
growth factor, *p. 296*
asymptote, *p. 296*
exponential decay function, *p. 296*
decay factor, *p. 296*
natural base *e*, *p. 304*
logarithm of *y* with base *b*, *p. 310*
common logarithm, *p. 311*
natural logarithm, *p. 311*

Core Concepts

Section 6.1

Parent Function for Exponential Growth Functions, *p. 296*
Parent Function for Exponential Decay Functions, *p. 296*
Exponential Growth and Decay Models, *p. 297*
Compound Interest, *p. 299*

Section 6.2

The Natural Base *e*, *p. 304*
Natural Base Functions, *p. 305*
Continuously Compounded Interest, *p. 306*

Section 6.3

Definition of Logarithm with Base *b*, *p. 310*
Parent Graphs for Logarithmic Functions, *p. 313*

Section 6.4

Transforming Graphs of Exponential Functions, *p. 318*
Transforming Graphs of Logarithmic Functions, *p. 320*

Mathematical Practices

1. How did you check to make sure your answer was reasonable in Exercise 23 on page 300?
2. How can you justify your conclusions in Exercises 23–26 on page 307?
3. How did you monitor and evaluate your progress in Exercise 66 on page 315?

Study Skills

Forming a Weekly Study Group

- Select students who are just as dedicated to doing well in the math class as you are.
- Find a regular meeting place that has minimal distractions.
- Compare schedules and plan at least one time a week to meet, allowing at least 1.5 hours for study time.