4 Polynomial Functions

4.1 Graphing Polynomial Functions
4.2 Adding, Subtracting, and Multiplying Polynomials
4.3 Dividing Polynomials
4.4 Factoring Polynomials
4.5 Solving Polynomial Equations
4.6 The Fundamental Theorem of Algebra
4.7 Transformations of Polynomial Functions
4.8 Analyzing Graphs of Polynomial Functions
4.9 Modeling with Polynomial Functions

Quonset Hut *(p. 218)*

Zebra Mussels *(p. 203)*

Ruins of Caesarea *(p. 195)*

Basketball *(p. 178)*

Electric Vehicles *(p. 161)*

Maintaining Mathematical Proficiency

Simplifying Algebraic Expressions

Example 1 **Simplify the expression $9x + 4x$.**

$$9x + 4x = (9 + 4)x \qquad \text{Distributive Property}$$

$$= 13x \qquad \text{Add coefficients.}$$

Example 2 **Simplify the expression $2(x + 4) + 3(6 - x)$.**

$$2(x + 4) + 3(6 - x) = 2(x) + 2(4) + 3(6) + 3(-x) \qquad \text{Distributive Property}$$

$$= 2x + 8 + 18 - 3x \qquad \text{Multiply.}$$

$$= 2x - 3x + 8 + 18 \qquad \text{Group like terms.}$$

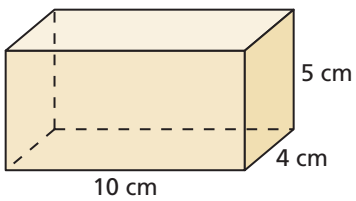
$$= -x + 26 \qquad \text{Combine like terms.}$$

Simplify the expression.

1. $6x - 4x$
2. $12m - m - 7m + 3$
3. $3(y + 2) - 4y$
4. $9x - 4(2x - 1)$
5. $-(z + 2) - 2(1 - z)$
6. $-x^2 + 5x + x^2$

Finding Volume

Example 3 **Find the volume of a rectangular prism with length 10 centimeters, width 4 centimeters, and height 5 centimeters.**



$$\text{Volume} = \ell wh \qquad \text{Write the volume formula.}$$

$$= (10)(4)(5) \qquad \text{Substitute 10 for } \ell\text{, 4 for } w\text{, and 5 for } h.$$

$$= 200 \qquad \text{Multiply.}$$

▶ The volume is 200 cubic centimeters.

Find the volume of the solid.

7. cube with side length 4 inches
8. sphere with radius 2 feet
9. rectangular prism with length 4 feet, width 2 feet, and height 6 feet
10. right cylinder with radius 3 centimeters and height 5 centimeters
11. **ABSTRACT REASONING** Does doubling the volume of a cube have the same effect on the side length? Explain your reasoning.

Mathematical Practices

Mathematically proficient students use technological tools to explore concepts.

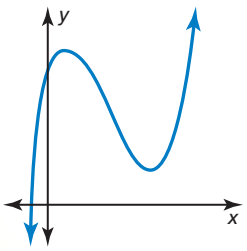
Using Technology to Explore Concepts

Core Concept

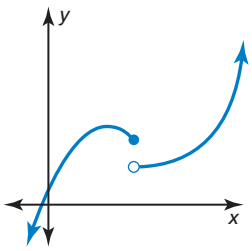
Continuous Functions

A function is *continuous* when its graph has no breaks, holes, or gaps.

Graph of a continuous function



Graph of a function that is not continuous



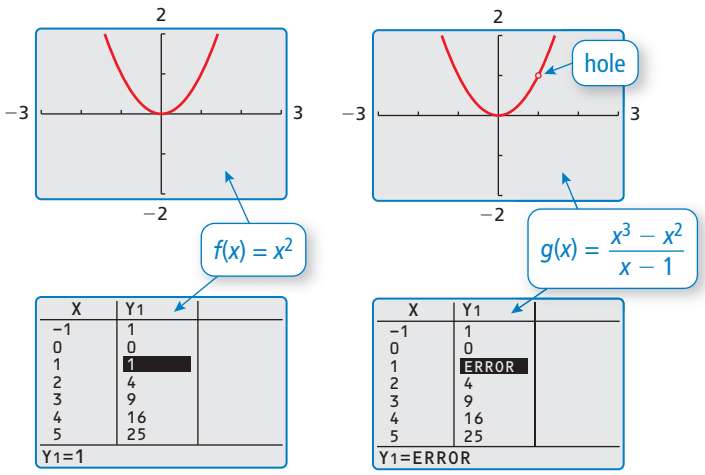
EXAMPLE 1 Determining Whether Functions Are Continuous

Use a graphing calculator to compare the two functions. What can you conclude? Which function is not continuous?

$f(x) = x^2$ $\qquad g(x) = \dfrac{x^3 - x^2}{x - 1}$

SOLUTION

The graphs appear to be identical, but g is not defined when $x = 1$. There is a *hole* in the graph of g at the point (1, 1). Using the *table* feature of a graphing calculator, you obtain an error for $g(x)$ when $x = 1$. So, g is not continuous.



Monitoring Progress

Use a graphing calculator to determine whether the function is continuous. Explain your reasoning.

1. $f(x) = \dfrac{x^2 - x}{x}$
2. $f(x) = x^3 - 3$
3. $f(x) = \sqrt{x^2 + 1}$
4. $f(x) = |x + 2|$
5. $f(x) = \dfrac{1}{x}$
6. $f(x) = \dfrac{1}{\sqrt{x^2 - 1}}$
7. $f(x) = x$
8. $f(x) = 2x - 3$
9. $f(x) = \dfrac{x}{x}$

4.1 Graphing Polynomial Functions

Essential Question What are some common characteristics of the graphs of cubic and quartic polynomial functions?

A *polynomial function* of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, is *cubic* when $n = 3$ and *quartic* when $n = 4$.

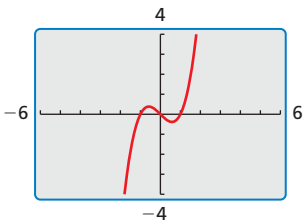
EXPLORATION 1 Identifying Graphs of Polynomial Functions

Work with a partner. Match each polynomial function with its graph. Explain your reasoning. Use a graphing calculator to verify your answers.

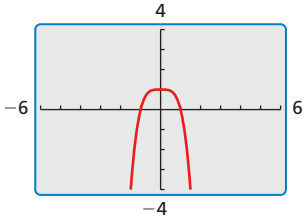
a. $f(x) = x^3 - x$ **b.** $f(x) = -x^3 + x$ **c.** $f(x) = -x^4 + 1$

d. $f(x) = x^4$ **e.** $f(x) = x^3$ **f.** $f(x) = x^4 - x^2$

A.



B.



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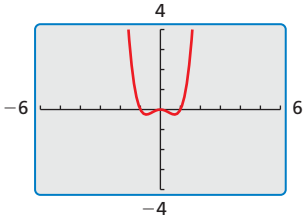
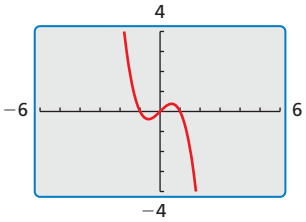
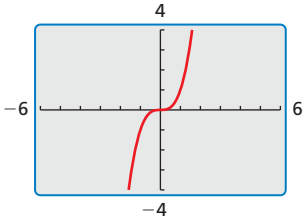


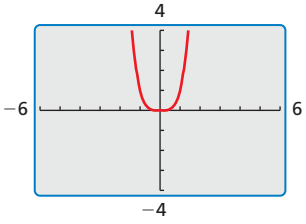
D.



E.



F.



EXPLORATION 2 Identifying *x*-Intercepts of Polynomial Graphs

Work with a partner. Each of the polynomial graphs in Exploration 1 has x-intercept(s) of -1, 0, or 1. Identify the x-intercept(s) of each graph. Explain how you can verify your answers.

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to justify your conclusions and communicate them to others.

Communicate Your Answer

3. What are some common characteristics of the graphs of cubic and quartic polynomial functions?

4. Determine whether each statement is *true* or *false*. Justify your answer.

a. When the graph of a cubic polynomial function rises to the left, it falls to the right.

b. When the graph of a quartic polynomial function falls to the left, it rises to the right.

4.1 Lesson

Core Vocabulary

polynomial, *p. 158*
polynomial function, *p. 158*
end behavior, *p. 159*

Previous
monomial
linear function
quadratic function

What You Will Learn

- Identify polynomial functions.
- Graph polynomial functions using tables and end behavior.

Polynomial Functions

Recall that a monomial is a number, a variable, or the product of a number and one or more variables with whole number exponents. A **polynomial** is a monomial or a sum of monomials. A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, the exponents are all whole numbers, and the coefficients are all real numbers. For this function, a_n is the leading coefficient, n is the degree, and a_0 is the constant term. A polynomial function is in *standard form* when its terms are written in descending order of exponents from left to right.

You are already familiar with some types of polynomial functions, such as linear and quadratic. Here is a summary of common types of polynomial functions.

Common Polynomial Functions

Degree	Type	Standard Form	Example
0	Constant	$f(x) = a_0$	$f(x) = -14$
1	Linear	$f(x) = a_1 x + a_0$	$f(x) = 5x - 7$
2	Quadratic	$f(x) = a_2 x^2 + a_1 x + a_0$	$f(x) = 2x^2 + x - 9$
3	Cubic	$f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$	$f(x) = x^3 - x^2 + 3x$
4	Quartic	$f(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$	$f(x) = x^4 + 2x - 1$

EXAMPLE 1 Identifying Polynomial Functions

Decide whether each function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

a. $f(x) = -2x^3 + 5x + 8$

b. $g(x) = -0.8x^3 + \sqrt{2}x^4 - 12$

c. $h(x) = -x^2 + 7x^{-1} + 4x$

d. $k(x) = x^2 + 3^x$

SOLUTION

a. The function is a polynomial function that is already written in standard form. It has degree 3 (cubic) and a leading coefficient of -2.

b. The function is a polynomial function written as $g(x) = \sqrt{2}x^4 - 0.8x^3 - 12$ in standard form. It has degree 4 (quartic) and a leading coefficient of $\sqrt{2}$.

c. The function is not a polynomial function because the term $7x^{-1}$ has an exponent that is not a whole number.

d. The function is not a polynomial function because the term 3^x does not have a variable base and an exponent that is a whole number.

Monitoring Progress

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Decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

1. $f(x) = 7 - 1.6x^2 - 5x$ **2.** $p(x) = x + 2x^{-2} + 9.5$ **3.** $q(x) = x^3 - 6x + 3x^4$

EXAMPLE 2 Evaluating a Polynomial Function

Evaluate $f(x) = 2x^4 - 8x^2 + 5x - 7$ when $x = 3$.

SOLUTION

$f(x) = 2x^4 - 8x^2 + 5x - 7$ — Write original equation.

$f(3) = 2(3)^4 - 8(3)^2 + 5(3) - 7$ — Substitute 3 for x.

$= 162 - 72 + 15 - 7$ — Evaluate powers and multiply.

$= 98$ — Simplify.

The **end behavior** of a function's graph is the behavior of the graph as x approaches positive infinity ($+\infty$) or negative infinity ($-\infty$). For the graph of a polynomial function, the end behavior is determined by the function's degree and the sign of its leading coefficient.

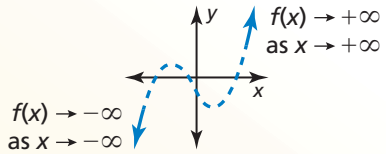
READING

The expression "$x \to +\infty$" is read as "x approaches positive infinity."

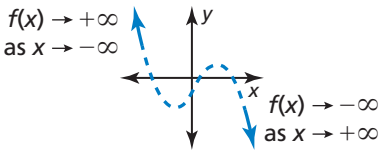
Core Concept

End Behavior of Polynomial Functions

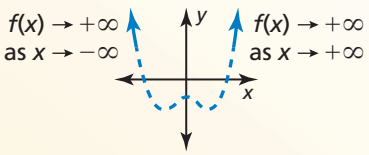
Degree: odd
Leading coefficient: positive



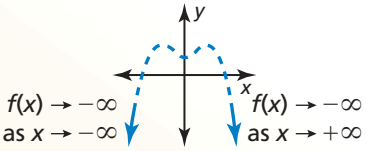
Degree: odd
Leading coefficient: negative



Degree: even
Leading coefficient: positive



Degree: even
Leading coefficient: negative



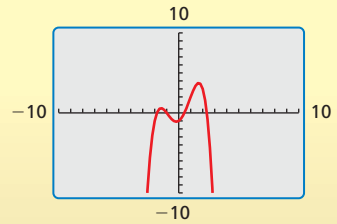
EXAMPLE 3 Describing End Behavior

Describe the end behavior of the graph of $f(x) = -0.5x^4 + 2.5x^2 + x - 1$.

SOLUTION

The function has degree 4 and leading coefficient -0.5. Because the degree is even and the leading coefficient is negative, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$. Check this by graphing the function on a graphing calculator, as shown.

Check



Monitoring Progress

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Evaluate the function for the given value of x.

4. $f(x) = -x^3 + 3x^2 + 9; x = 4$
5. $f(x) = 3x^5 - x^4 - 6x + 10; x = -2$
6. Describe the end behavior of the graph of $f(x) = 0.25x^3 - x^2 - 1$.

Graphing Polynomial Functions

To graph a polynomial function, first plot points to determine the shape of the graph's middle portion. Then connect the points with a smooth continuous curve and use what you know about end behavior to sketch the graph.

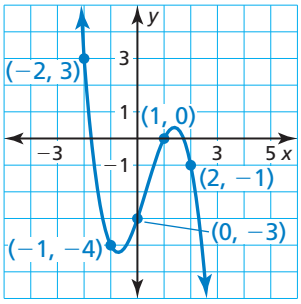
EXAMPLE 4 Graphing Polynomial Functions

Graph (a) $f(x) = -x^3 + x^2 + 3x - 3$ and (b) $f(x) = x^4 - x^3 - 4x^2 + 4$.

SOLUTION

a. To graph the function, make a table of values and plot the corresponding points. Connect the points with a smooth curve and check the end behavior.

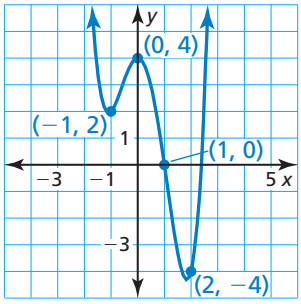
x	-2	-1	0	1	2
$f(x)$	3	-4	-3	0	-1



The degree is odd and the leading coefficient is negative. So, $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$.

b. To graph the function, make a table of values and plot the corresponding points. Connect the points with a smooth curve and check the end behavior.

x	-2	-1	0	1	2
$f(x)$	12	2	4	0	-4



The degree is even and the leading coefficient is positive. So, $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

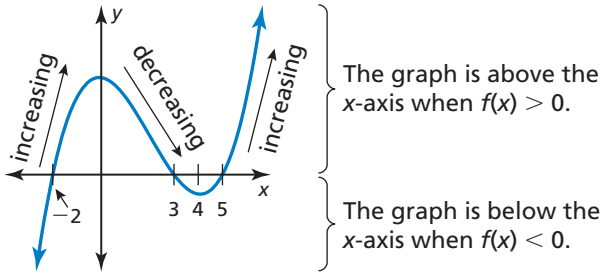
EXAMPLE 5 Sketching a Graph

Sketch a graph of the polynomial function f having these characteristics.

- f is increasing when $x < 0$ and $x > 4$.
- f is decreasing when $0 < x < 4$.
- $f(x) > 0$ when $-2 < x < 3$ and $x > 5$.
- $f(x) < 0$ when $x < -2$ and $3 < x < 5$.

Use the graph to describe the degree and leading coefficient of f.

SOLUTION



From the graph, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$. So, the degree is odd and the leading coefficient is positive.

EXAMPLE 6 Solving a Real-Life Problem

The estimated number V (in thousands) of electric vehicles in use in the United States can be modeled by the polynomial function

$$V(t) = 0.151280t^3 - 3.28234t^2 + 23.7565t - 2.041$$

where t represents the year, with $t = 1$ corresponding to 2001.

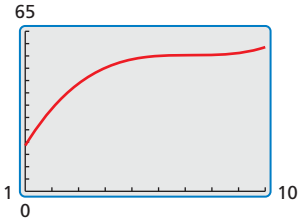
a. Use a graphing calculator to graph the function for the interval $1 \le t \le 10$. Describe the behavior of the graph on this interval.

b. What was the average rate of change in the number of electric vehicles in use from 2001 to 2010?

c. Do you think this model can be used for years before 2001 or after 2010? Explain your reasoning.

SOLUTION

a. Using a graphing calculator and a viewing window of $1 \le x \le 10$ and $0 \le y \le 65$, you obtain the graph shown.



▶ From 2001 to 2004, the numbers of electric vehicles in use increased. Around 2005, the growth in the numbers in use slowed and started to level off. Then the numbers in use started to increase again in 2009 and 2010.

b. The years 2001 and 2010 correspond to $t = 1$ and $t = 10$.

Average rate of change over $1 \le t \le 10$:

$$\frac{V(10) - V(1)}{10 - 1} = \frac{58.57 - 18.58444}{9} \approx 4.443$$

▶ The average rate of change from 2001 to 2010 is about 4.4 thousand electric vehicles per year.

c. Because the degree is odd and the leading coefficient is positive, $V(t) \to -\infty$ as $t \to -\infty$ and $V(t) \to +\infty$ as $t \to +\infty$. The end behavior indicates that the model has unlimited growth as t increases. While the model may be valid for a few years after 2010, in the long run, unlimited growth is not reasonable. Notice in 2000 that $V(0) = -2.041$. Because negative values of $V(t)$ do not make sense given the context (electric vehicles in use), the model should not be used for years before 2001.

Monitoring Progress

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Graph the polynomial function.

7. $f(x) = x^4 + x^2 - 3$

8. $f(x) = 4 - x^3$

9. $f(x) = x^3 - x^2 + x - 1$

10. Sketch a graph of the polynomial function f having these characteristics.

- f is decreasing when $x < -1.5$ and $x > 2.5$; f is increasing when $-1.5 < x < 2.5$.
- $f(x) > 0$ when $x < -3$ and $1 < x < 4$; $f(x) < 0$ when $-3 < x < 1$ and $x > 4$.

Use the graph to describe the degree and leading coefficient of f.

11. WHAT IF? Repeat Example 6 using the alternative model for electric vehicles of

$$V(t) = -0.0290900t^4 + 0.791260t^3 - 7.96583t^2 + 36.5561t - 12.025.$$

4.1 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Explain what is meant by the end behavior of a polynomial function.

2. **WHICH ONE DOESN'T BELONG?** Which function does *not* belong with the other three? Explain your reasoning.

$f(x) = 7x^5 + 3x^2 - 2x$

$g(x) = 3x^3 - 2x^8 + \frac{3}{4}$

$h(x) = -3x^4 + 5x^{-1} - 3x^2$

$k(x) = \sqrt{3}x + 8x^4 + 2x + 1$

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient. *(See Example 1.)*

3. $f(x) = -3x + 5x^3 - 6x^2 + 2$

4. $p(x) = \frac{1}{2}x^2 + 3x - 4x^3 + 6x^4 - 1$

5. $f(x) = 9x^4 + 8x^3 - 6x^{-2} + 2x$

6. $g(x) = \sqrt{3} - 12x + 13x^2$

7. $h(x) = \frac{5}{3}x^2 - \sqrt{7}x^4 + 8x^3 - \frac{1}{2} + x$

8. $h(x) = 3x^4 + 2x - \frac{5}{x} + 9x^3 - 7$

ERROR ANALYSIS **In Exercises 9 and 10, describe and correct the error in analyzing the function.**

9. $f(x) = 8x^3 - 7x^4 - 9x - 3x^2 + 11$

✗ *f is a polynomial function. The degree is 3 and f is a cubic function. The leading coefficient is 8.*

10. $f(x) = 2x^4 + 4x - 9\sqrt{x} + 3x^2 - 8$

✗ *f is a polynomial function. The degree is 4 and f is a quartic function. The leading coefficient is 2.*

In Exercises 11–16, evaluate the function for the given value of x. *(See Example 2.)*

11. $h(x) = -3x^4 + 2x^3 - 12x - 6; x = -2$

12. $f(x) = 7x^4 - 10x^2 + 14x - 26; x = -7$

13. $g(x) = x^6 - 64x^4 + x^2 - 7x - 51; x = 8$

14. $g(x) = -x^3 + 3x^2 + 5x + 1; x = -12$

15. $p(x) = 2x^3 + 4x^2 + 6x + 7; x = \frac{1}{2}$

16. $h(x) = 5x^3 - 3x^2 + 2x + 4; x = -\frac{1}{3}$

In Exercises 17–20, describe the end behavior of the graph of the function. *(See Example 3.)*

17. $h(x) = -5x^4 + 7x^3 - 6x^2 + 9x + 2$

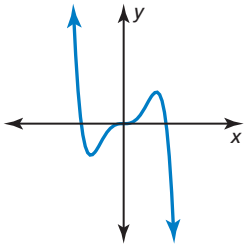
18. $g(x) = 7x^7 + 12x^5 - 6x^3 - 2x - 18$

19. $f(x) = -2x^4 + 12x^8 + 17 + 15x^2$

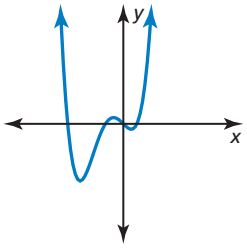
20. $f(x) = 11 - 18x^2 - 5x^5 - 12x^4 - 2x$

In Exercises 21 and 22, describe the degree and leading coefficient of the polynomial function using the graph.

21.



22.



23. USING STRUCTURE Determine whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

$$f(x) = 5x^3x + \frac{5}{2}x^3 - 9x^4 + \sqrt{2}x^2 + 4x - 1 - x^{-5}x^5 - 4$$

24. WRITING Let $f(x) = 13$. State the degree, type, and leading coefficient. Describe the end behavior of the function. Explain your reasoning.

In Exercises 25–32, graph the polynomial function. *(See Example 4.)*

25. $p(x) = 3 - x^4$

26. $g(x) = x^3 + x + 3$

27. $f(x) = 4x - 9 - x^3$

28. $p(x) = x^5 - 3x^3 + 2$

29. $h(x) = x^4 - 2x^3 + 3x$

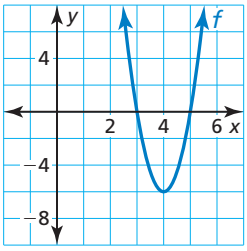
30. $h(x) = 5 + 3x^2 - x^4$

31. $g(x) = x^5 - 3x^4 + 2x - 4$

32. $p(x) = x^6 - 2x^5 - 2x^3 + x + 5$

ANALYZING RELATIONSHIPS In Exercises 33–36, describe the x-values for which (a) f is increasing or decreasing, (b) $f(x) > 0$, and (c) $f(x) < 0$.

33.



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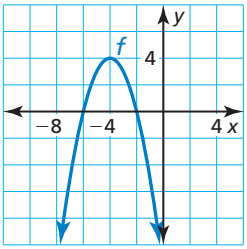

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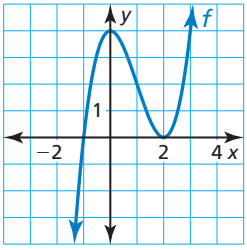

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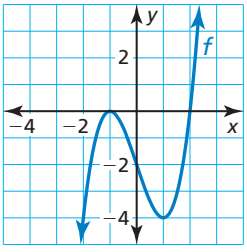


In Exercises 37–40, sketch a graph of the polynomial function f having the given characteristics. Use the graph to describe the degree and leading coefficient of the function f. *(See Example 5.)*

37.
- f is increasing when $x > 0.5$; f is decreasing when $x < 0.5$.
- $f(x) > 0$ when $x < -2$ and $x > 3$; $f(x) < 0$ when $-2 < x < 3$.

38.
- f is increasing when $-2 < x < 3$; f is decreasing when $x < -2$ and $x > 3$.
- $f(x) > 0$ when $x < -4$ and $1 < x < 5$; $f(x) < 0$ when $-4 < x < 1$ and $x > 5$.

39.
- f is increasing when $-2 < x < 0$ and $x > 2$; f is decreasing when $x < -2$ and $0 < x < 2$.
- $f(x) > 0$ when $x < -3$, $-1 < x < 1$, and $x > 3$; $f(x) < 0$ when $-3 < x < -1$ and $1 < x < 3$.

40.
- f is increasing when $x < -1$ and $x > 1$; f is decreasing when $-1 < x < 1$.
- $f(x) > 0$ when $-1.5 < x < 0$ and $x > 1.5$; $f(x) < 0$ when $x < -1.5$ and $0 < x < 1.5$.

41. MODELING WITH MATHEMATICS From 1980 to 2007 the number of drive-in theaters in the United States can be modeled by the function

$$d(t) = -0.141t^3 + 9.64t^2 - 232.5t + 2421$$

where $d(t)$ is the number of open theaters and t is the number of years after 1980. *(See Example 6.)*

a. Use a graphing calculator to graph the function for the interval $0 \le t \le 27$. Describe the behavior of the graph on this interval.

b. What is the average rate of change in the number of drive-in movie theaters from 1980 to 1995 and from 1995 to 2007? Interpret the average rates of change.

c. Do you think this model can be used for years before 1980 or after 2007? Explain.

42. PROBLEM SOLVING The weight of an ideal round-cut diamond can be modeled by

$$w = 0.00583d^3 - 0.0125d^2 + 0.022d - 0.01$$

where w is the weight of the diamond (in carats) and d is the diameter (in millimeters). According to the model, what is the weight of a diamond with a diameter of 12 millimeters?

43. ABSTRACT REASONING Suppose $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$. Describe the end behavior of $g(x) = -f(x)$. Justify your answer.

44. THOUGHT PROVOKING Write an even degree polynomial function such that the end behavior of f is given by $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$. Justify your answer by drawing the graph of your function.

45. USING TOOLS When using a graphing calculator to graph a polynomial function, explain how you know when the viewing window is appropriate.

46. MAKING AN ARGUMENT Your friend uses the table to speculate that the function f is an even degree polynomial and the function g is an odd degree polynomial. Is your friend correct? Explain your reasoning.

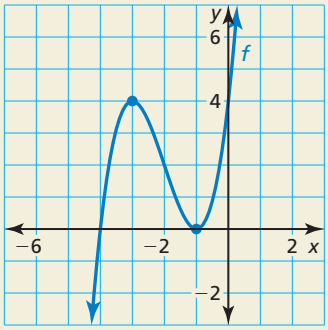
x	$f(x)$	$g(x)$
-8	4113	497
-2	21	5
0	1	1
2	13	-3
8	4081	-495

47. DRAWING CONCLUSIONS The graph of a function is symmetric with respect to the y-axis if for each point (a, b) on the graph, $(-a, b)$ is also a point on the graph. The graph of a function is symmetric with respect to the origin if for each point (a, b) on the graph, $(-a, -b)$ is also a point on the graph.

a. Use a graphing calculator to graph the function $y = x^n$ when $n = 1, 2, 3, 4, 5$, and 6. In each case, identify the symmetry of the graph.

b. Predict what symmetry the graphs of $y = x^{10}$ and $y = x^{11}$ each have. Explain your reasoning and then confirm your predictions by graphing.

48. HOW DO YOU SEE IT? The graph of a polynomial function is shown.



a. Describe the degree and leading coefficient of f.

b. Describe the intervals where the function is increasing and decreasing.

c. What is the constant term of the polynomial function?

49. REASONING A cubic polynomial function f has a leading coefficient of 2 and a constant term of -5. When $f(1) = 0$ and $f(2) = 3$, what is $f(-5)$? Explain your reasoning.

50. CRITICAL THINKING The weight y (in pounds) of a rainbow trout can be modeled by $y = 0.000304x^3$, where x is the length (in inches) of the trout.

a. Write a function that relates the weight y and length x of a rainbow trout when y is measured in kilograms and x is measured in centimeters. Use the fact that 1 kilogram $\approx$ 2.20 pounds and 1 centimeter $\approx$ 0.394 inch.

b. Graph the original function and the function from part (a) in the same coordinate plane. What type of transformation can you apply to the graph of $y = 0.000304x^3$ to produce the graph from part (a)?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the expression. *(Skills Review Handbook)*

51. $xy + x^2 + 2xy + y^2 - 3x^2$

52. $2h^3g + 3hg^3 + 7h^2g^2 + 5h^3g + 2hg^3$

53. $-wk + 3kz - 2kw + 9zk - kw$

54. $a^2(m - 7a^3) - m(a^2 - 10)$

55. $3x(xy - 4) + 3(4xy + 3) - xy(x^2y - 1)$

56. $cv(9 - 3c) + 2c(v - 4c) + 6c$

4.2 Adding, Subtracting, and Multiplying Polynomials

Essential Question

How can you cube a binomial?

EXPLORATION 1 Cubing Binomials

Work with a partner. Find each product. Show your steps.

a. $(x + 1)^3 = (x + 1)(x + 1)^2$ Rewrite as a product of first and second powers.

$= (x + 1)$ ____ Multiply second power.

$=$ ____ Multiply binomial and trinomial.

$=$ ____ Write in standard form, $ax^3 + bx^2 + cx + d$.

b. $(a + b)^3 = (a + b)(a + b)^2$ Rewrite as a product of first and second powers.

$= (a + b)$ ____ Multiply second power.

$=$ ____ Multiply binomial and trinomial.

$=$ ____ Write in standard form.

c. $(x - 1)^3 = (x - 1)(x - 1)^2$ Rewrite as a product of first and second powers.

$= (x - 1)$ ____ Multiply second power.

$=$ ____ Multiply binomial and trinomial.

$=$ ____ Write in standard form.

d. $(a - b)^3 = (a - b)(a - b)^2$ Rewrite as a product of first and second powers.

$= (a - b)$ ____ Multiply second power.

$=$ ____ Multiply binomial and trinomial.

$=$ ____ Write in standard form.

LOOKING FOR STRUCTURE

To be proficient in math, you need to look closely to discern a pattern or structure.

EXPLORATION 2 Generalizing Patterns for Cubing a Binomial

Work with a partner.

a. Use the results of Exploration 1 to describe a pattern for the coefficients of the terms when you expand the cube of a binomial. How is your pattern related to Pascal's Triangle, shown at the right?

		1		
	1		1	
1		2		1
	1		3		1
1		4		6		4		1

b. Use the results of Exploration 1 to describe a pattern for the exponents of the terms in the expansion of a cube of a binomial.

c. Explain how you can use the patterns you described in parts (a) and (b) to find the product $(2x - 3)^3$. Then find this product.

Communicate Your Answer

3. How can you cube a binomial?

4. Find each product.

a. $(x + 2)^3$ **b.** $(x - 2)^3$ **c.** $(2x - 3)^3$

d. $(x - 3)^3$ **e.** $(-2x + 3)^3$ **f.** $(3x - 5)^3$

4.2 Lesson

Core Vocabulary

Pascal's Triangle, *p. 169*

Previous
like terms
identity

What You Will Learn

- Add and subtract polynomials.
- Multiply polynomials.
- Use Pascal's Triangle to expand binomials.

Adding and Subtracting Polynomials

Recall that the set of integers is *closed* under addition and subtraction because every sum or difference results in an integer. To add or subtract polynomials, you add or subtract the coefficients of like terms. Because adding or subtracting polynomials results in a polynomial, the set of polynomials is closed under addition and subtraction.

EXAMPLE 1 Adding Polynomials Vertically and Horizontally

a. Add $3x^3 + 2x^2 - x - 7$ and $x^3 - 10x^2 + 8$ in a vertical format.

b. Add $9y^3 + 3y^2 - 2y + 1$ and $-5y^2 + y - 4$ in a horizontal format.

SOLUTION

a.
$$\begin{array}{r} 3x^3 + 2x^2 - x - 7 \\ +\ x^3 - 10x^2 \quad\ + 8 \\ \hline 4x^3 - 8x^2 - x + 1 \end{array}$$

b. $(9y^3 + 3y^2 - 2y + 1) + (-5y^2 + y - 4) = 9y^3 + 3y^2 - 5y^2 - 2y + y + 1 - 4$

$= 9y^3 - 2y^2 - y - 3$

To subtract one polynomial from another, add the opposite. To do this, change the sign of each term of the subtracted polynomial and then add the resulting like terms.

EXAMPLE 2 Subtracting Polynomials Vertically and Horizontally

a. Subtract $2x^3 + 6x^2 - x + 1$ from $8x^3 - 3x^2 - 2x + 9$ in a vertical format.

b. Subtract $3z^2 + z - 4$ from $2z^2 + 3z$ in a horizontal format.

SOLUTION

a. Align like terms, then add the opposite of the subtracted polynomial.

$$\begin{array}{r} 8x^3 - 3x^2 - 2x + 9 \\ -(2x^3 + 6x^2 - x + 1) \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{r} 8x^3 - 3x^2 - 2x + 9 \\ +\ -2x^3 - 6x^2 + x - 1 \\ \hline 6x^3 - 9x^2 - x + 8 \end{array}$$

b. Write the opposite of the subtracted polynomial, then add like terms.

$(2z^2 + 3z) - (3z^2 + z - 4) = 2z^2 + 3z - 3z^2 - z + 4$

$= -z^2 + 2z + 4$

COMMON ERROR

A common mistake is to forget to change signs correctly when subtracting one polynomial from another. Be sure to add the opposite of *every* term of the subtracted polynomial.

Monitoring Progress

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Find the sum or difference.

1. $(2x^2 - 6x + 5) + (7x^2 - x - 9)$
2. $(3t^3 + 8t^2 - t - 4) - (5t^3 - t^2 + 17)$

Multiplying Polynomials

To multiply two polynomials, you multiply each term of the first polynomial by each term of the second polynomial. As with addition and subtraction, the set of polynomials is closed under multiplication.

EXAMPLE 3 Multiplying Polynomials Vertically and Horizontally

a. Multiply $-x^2 + 2x + 4$ and $x - 3$ in a vertical format.

b. Multiply $y + 5$ and $3y^2 - 2y + 2$ in a horizontal format.

REMEMBER

Product of Powers Property

$a^m \cdot a^n = a^{m+n}$

a is a real number and m and n are integers.

SOLUTION

a.

$$\begin{array}{rr} & -x^2 + 2x + 4 \\ \times & x - 3 \\ \hline & 3x^2 - 6x - 12 \\ & -x^3 + 2x^2 + 4x \\ \hline & -x^3 + 5x^2 - 2x - 12 \end{array}$$

Multiply $-x^2 + 2x + 4$ by -3.

Multiply $-x^2 + 2x + 4$ by x.

Combine like terms.

b. $(y + 5)(3y^2 - 2y + 2) = (y + 5)3y^2 - (y + 5)2y + (y + 5)2$

$= 3y^3 + 15y^2 - 2y^2 - 10y + 2y + 10$

$= 3y^3 + 13y^2 - 8y + 10$

EXAMPLE 4 Multiplying Three Binomials

Multiply $x - 1$, $x + 4$, and $x + 5$ in a horizontal format.

SOLUTION

$(x - 1)(x + 4)(x + 5) = (x^2 + 3x - 4)(x + 5)$

$= (x^2 + 3x - 4)x + (x^2 + 3x - 4)5$

$= x^3 + 3x^2 - 4x + 5x^2 + 15x - 20$

$= x^3 + 8x^2 + 11x - 20$

Some binomial products occur so frequently that it is worth memorizing their patterns. You can verify these polynomial identities by multiplying.

COMMON ERROR

In general,

$(a \pm b)^2 \neq a^2 \pm b^2$

and

$(a \pm b)^3 \neq a^3 \pm b^3$.

Core Concept

Special Product Patterns

Sum and Difference	Example
$(a + b)(a - b) = a^2 - b^2$	$(x + 3)(x - 3) = x^2 - 9$

Square of a Binomial	Example
$(a + b)^2 = a^2 + 2ab + b^2$	$(y + 4)^2 = y^2 + 8y + 16$
$(a - b)^2 = a^2 - 2ab + b^2$	$(2t - 5)^2 = 4t^2 - 20t + 25$

Cube of a Binomial	Example
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	$(z + 3)^3 = z^3 + 9z^2 + 27z + 27$
$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$	$(m - 2)^3 = m^3 - 6m^2 + 12m - 8$

EXAMPLE 5 Proving a Polynomial Identity

a. Prove the polynomial identity for the cube of a binomial representing a sum:

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

b. Use the cube of a binomial in part (a) to calculate 11^3.

SOLUTION

a. Expand and simplify the expression on the left side of the equation.

$$\begin{aligned}(a + b)^3 &= (a + b)(a + b)(a + b)\\ &= (a^2 + 2ab + b^2)(a + b)\\ &= (a^2 + 2ab + b^2)a + (a^2 + 2ab + b^2)b\\ &= a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3\\ &= a^3 + 3a^2b + 3ab^2 + b^3 \checkmark\end{aligned}$$

▶ The simplified left side equals the right side of the original identity. So, the identity $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ is true.

b. To calculate 11^3 using the cube of a binomial, note that $11 = 10 + 1$.

$$\begin{aligned}11^3 &= (10 + 1)^3 && \text{Write 11 as } 10 + 1.\\ &= 10^3 + 3(10)^2(1) + 3(10)(1)^2 + 1^3 && \text{Cube of a binomial}\\ &= 1000 + 300 + 30 + 1 && \text{Expand.}\\ &= 1331 && \text{Simplify.}\end{aligned}$$

EXAMPLE 6 Using Special Product Patterns

REMEMBER

Power of a Product Property

$(ab)^m = a^m b^m$

a and b are real numbers and m is an integer.

Find each product.

a. $(4n + 5)(4n - 5)$ **b.** $(9y - 2)^2$ **c.** $(ab + 4)^3$

SOLUTION

$$\begin{aligned}\textbf{a. } (4n + 5)(4n - 5) &= (4n)^2 - 5^2 && \text{Sum and difference}\\ &= 16n^2 - 25 && \text{Simplify.}\\ \textbf{b. } (9y - 2)^2 &= (9y)^2 - 2(9y)(2) + 2^2 && \text{Square of a binomial}\\ &= 81y^2 - 36y + 4 && \text{Simplify.}\\ \textbf{c. } (ab + 4)^3 &= (ab)^3 + 3(ab)^2(4) + 3(ab)(4)^2 + 4^3 && \text{Cube of a binomial}\\ &= a^3b^3 + 12a^2b^2 + 48ab + 64 && \text{Simplify.}\end{aligned}$$

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Find the product.

3. $(4x^2 + x - 5)(2x + 1)$ **4.** $(y - 2)(5y^2 + 3y - 1)$

5. $(m - 2)(m - 1)(m + 3)$ **6.** $(3t - 2)(3t + 2)$

7. $(5a + 2)^2$ **8.** $(xy - 3)^3$

9. (a) Prove the polynomial identity for the cube of a binomial representing a difference: $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

(b) Use the cube of a binomial in part (a) to calculate 9^3.

Pascal's Triangle

Consider the expansion of the binomial $(a + b)^n$ for whole number values of n. When you arrange the coefficients of the variables in the expansion of $(a + b)^n$, you will see a special pattern called **Pascal's Triangle**. Pascal's Triangle is named after French mathematician Blaise Pascal (1623−1662).

Core Concept

Pascal's Triangle

In Pascal's Triangle, the first and last numbers in each row are 1. Every number other than 1 is the sum of the closest two numbers in the row directly above it. The numbers in Pascal's Triangle are the same numbers that are the coefficients of binomial expansions, as shown in the first six rows.

	n	$(a + b)^n$	Binomial Expansion	Pascal's Triangle
0th row	0	$(a + b)^0 =$	1	1
1st row	1	$(a + b)^1 =$	$1a + 1b$	1 1
2nd row	2	$(a + b)^2 =$	$1a^2 + 2ab + 1b^2$	1 2 1
3rd row	3	$(a + b)^3 =$	$1a^3 + 3a^2b + 3ab^2 + 1b^3$	1 3 3 1
4th row	4	$(a + b)^4 =$	$1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$	1 4 6 4 1
5th row	5	$(a + b)^5 =$	$1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$	1 5 10 10 5 1

In general, the nth row in Pascal's Triangle gives the coefficients of $(a + b)^n$. Here are some other observations about the expansion of $(a + b)^n$.

1. An expansion has $n + 1$ terms.
2. The power of a begins with n, decreases by 1 in each successive term, and ends with 0.
3. The power of b begins with 0, increases by 1 in each successive term, and ends with n.
4. The sum of the powers of each term is n.

EXAMPLE 7 Using Pascal's Triangle to Expand Binomials

Use Pascal's Triangle to expand (a) $(x - 2)^5$ and (b) $(3y + 1)^3$.

SOLUTION

a. The coefficients from the fifth row of Pascal's Triangle are 1, 5, 10, 10, 5, and 1.

$$(x - 2)^5 = 1x^5 + 5x^4(-2) + 10x^3(-2)^2 + 10x^2(-2)^3 + 5x(-2)^4 + 1(-2)^5$$
$$= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$$

b. The coefficients from the third row of Pascal's Triangle are 1, 3, 3, and 1.

$$(3y + 1)^3 = 1(3y)^3 + 3(3y)^2(1) + 3(3y)(1)^2 + 1(1)^3$$
$$= 27y^3 + 27y^2 + 9y + 1$$

Monitoring Progress

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10. Use Pascal's Triangle to expand (a) $(z + 3)^4$ and (b) $(2t - 1)^5$.

4.2 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Describe three different methods to expand $(x + 3)^3$.

2. **WRITING** Is $(a + b)(a - b) = a^2 - b^2$ an identity? Explain your reasoning.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, find the sum. *(See Example 1.)*

3. $(3x^2 + 4x - 1) + (-2x^2 - 3x + 2)$

4. $(-5x^2 + 4x - 2) + (-8x^2 + 2x + 1)$

5. $(12x^5 - 3x^4 + 2x - 5) + (8x^4 - 3x^3 + 4x + 1)$

6. $(8x^4 + 2x^2 - 1) + (3x^3 - 5x^2 + 7x + 1)$

7. $(7x^6 + 2x^5 - 3x^2 + 9x) + (5x^5 + 8x^3 - 6x^2 + 2x - 5)$

8. $(9x^4 - 3x^3 + 4x^2 + 5x + 7) + (11x^4 - 4x^2 - 11x - 9)$

In Exercises 9–14, find the difference. *(See Example 2.)*

9. $(3x^3 - 2x^2 + 4x - 8) - (5x^3 + 12x^2 - 3x - 4)$

10. $(7x^4 - 9x^3 - 4x^2 + 5x + 6) - (2x^4 + 3x^3 - x^2 + x - 4)$

11. $(5x^6 - 2x^4 + 9x^3 + 2x - 4) - (7x^5 - 8x^4 + 2x - 11)$

12. $(4x^5 - 7x^3 - 9x^2 + 18) - (14x^5 - 8x^4 + 11x^2 + x)$

13. $(8x^5 + 6x^3 - 2x^2 + 10x) - (9x^5 - x^3 - 13x^2 + 4)$

14. $(11x^4 - 9x^2 + 3x + 11) - (2x^4 + 6x^3 + 2x - 9)$

15. **MODELING WITH MATHEMATICS** During a recent period of time, the numbers (in thousands) of males M and females F that attend degree-granting institutions in the United States can be modeled by

$M = 19.7t^2 + 310.5t + 7539.6$

$F = 28t^2 + 368t + 10127.8$

where t is time in years. Write a polynomial to model the total number of people attending degree-granting institutions. Interpret its constant term.

16. **MODELING WITH MATHEMATICS** A farmer plants a garden that contains corn and pumpkins. The total area (in square feet) of the garden is modeled by the expression $2x^2 + 5x + 4$. The area of the corn is modeled by the expression $x^2 - 3x + 2$. Write an expression that models the area of the pumpkins.

In Exercises 17–24, find the product. *(See Example 3.)*

17. $7x^3(5x^2 + 3x + 1)$

18. $-4x^5(11x^3 + 2x^2 + 9x + 1)$

19. $(5x^2 - 4x + 6)(-2x + 3)$

20. $(-x - 3)(2x^2 + 5x + 8)$

21. $(x^2 - 2x - 4)(x^2 - 3x - 5)$

22. $(3x^2 + x - 2)(-4x^2 - 2x - 1)$

23. $(3x^3 - 9x + 7)(x^2 - 2x + 1)$

24. $(4x^2 - 8x - 2)(x^4 + 3x^2 + 4x)$

ERROR ANALYSIS **In Exercises 25 and 26, describe and correct the error in performing the operation.**

25. ✗
$$(x^2 - 3x + 4) - (x^3 + 7x - 2)$$
$$= x^2 - 3x + 4 - x^3 + 7x - 2$$
$$= -x^3 + x^2 + 4x + 2$$

26. ✗
$$(2x - 7)^3 = (2x)^3 - 7^3$$
$$= 8x^3 - 343$$

In Exercises 27–32, find the product of the binomials. *(See Example 4.)*

27. $(x - 3)(x + 2)(x + 4)$

28. $(x - 5)(x + 2)(x - 6)$

29. $(x - 2)(3x + 1)(4x - 3)$

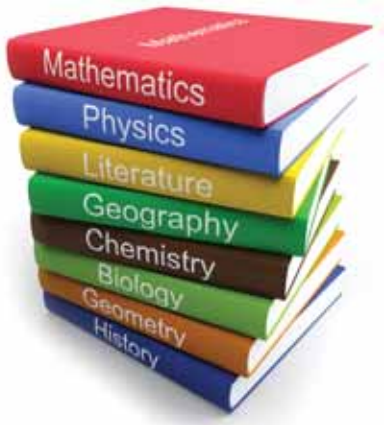
30. $(2x + 5)(x - 2)(3x + 4)$

31. $(3x - 4)(5 - 2x)(4x + 1)$

32. $(4 - 5x)(1 - 2x)(3x + 2)$

33. REASONING Prove the polynomial identity $(a + b)(a - b) = a^2 - b^2$. Then give an example of two whole numbers greater than 10 that can be multiplied using mental math and the given identity. Justify your answers. *(See Example 5.)*

34. NUMBER SENSE You have been asked to order textbooks for your class. You need to order 29 textbooks that cost \$31 each. Explain how you can use the polynomial identity $(a + b)(a - b) = a^2 - b^2$ and mental math to find the total cost of the textbooks.



In Exercises 35–42, find the product. *(See Example 6.)*

35. $(x - 9)(x + 9)$

36. $(m + 6)^2$

37. $(3c - 5)^2$

38. $(2y - 5)(2y + 5)$

39. $(7h + 4)^2$

40. $(9g - 4)^2$

41. $(2k + 6)^3$

42. $(4n - 3)^3$

In Exercises 43–48, use Pascal's Triangle to expand the binomial. *(See Example 7.)*

43. $(2t + 4)^3$

44. $(6m + 2)^2$

45. $(2q - 3)^4$

46. $(g + 2)^5$

47. $(yz + 1)^5$

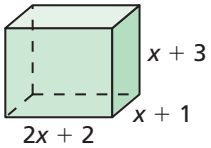
48. $(np - 1)^4$

49. COMPARING METHODS Find the product of the expression $(a^2 + 4b^2)^2(3a^2 - b^2)^2$ using two different methods. Which method do you prefer? Explain.

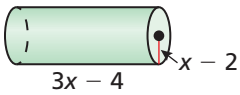
50. THOUGHT PROVOKING Adjoin one or more polygons to the rectangle to form a single new polygon whose perimeter is double that of the rectangle. Find the perimeter of the new polygon.

MATHEMATICAL CONNECTIONS In Exercises 51 and 52, write an expression for the volume of the figure as a polynomial in standard form.

51. $V = \ell wh$



52. $V = \pi r^2 h$



53. MODELING WITH MATHEMATICS Two people make three deposits into their bank accounts earning the same simple interest rate r.

Person A

Account No. 2-5384100608

Date	Transaction	Amount
01/01/2012	Deposit	\$2000.00
01/01/2013	Deposit	\$3000.00
01/01/2014	Deposit	\$1000.00

Person B

Account No. 1-5233032905

Date	Transaction	Amount
01/01/2012	Deposit	\$5000.00
01/01/2013	Deposit	\$1000.00
01/01/2014	Deposit	\$4000.00

Person A's account is worth

$$2000(1 + r)^3 + 3000(1 + r)^2 + 1000(1 + r)$$

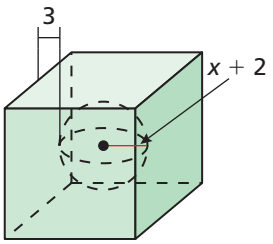
on January 1, 2015.

a. Write a polynomial for the value of Person B's account on January 1, 2015.

b. Write the total value of the two accounts as a polynomial in standard form. Then interpret the coefficients of the polynomial.

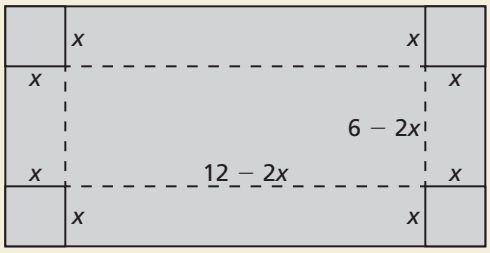
c. Suppose their interest rate is 0.05. What is the total value of the two accounts on January 1, 2015?

54. PROBLEM SOLVING The sphere is centered in the cube. Find an expression for the volume of the cube outside the sphere.



55. MAKING AN ARGUMENT Your friend claims the sum of two binomials is always a binomial and the product of two binomials is always a trinomial. Is your friend correct? Explain your reasoning.

56. HOW DO YOU SEE IT? You make a tin box by cutting x-inch-by-x-inch pieces of tin off the corners of a rectangle and folding up each side. The plan for your box is shown.



a. What are the dimensions of the original piece of tin?

b. Write a function that represents the volume of the box. Without multiplying, determine its degree.

USING TOOLS In Exercises 57–60, use a graphing calculator to make a conjecture about whether the two functions are equivalent. Explain your reasoning.

57. $f(x) = (2x - 3)^3$; $g(x) = 8x^3 - 36x^2 + 54x - 27$

58. $h(x) = (x + 2)^5$;
$k(x) = x^5 + 10x^4 + 40x^3 + 80x^2 + 64x$

59. $f(x) = (-x - 3)^4$;
$g(x) = x^4 + 12x^3 + 54x^2 + 108x + 80$

60. $f(x) = (-x + 5)^3$; $g(x) = -x^3 + 15x^2 - 75x + 125$

61. REASONING Copy Pascal's Triangle and add rows for $n = 6$, 7, 8, 9, and 10. Use the new rows to expand $(x + 3)^7$ and $(x - 5)^9$.

62. ABSTRACT REASONING You are given the function $f(x) = (x + a)(x + b)(x + c)(x + d)$. When $f(x)$ is written in standard form, show that the coefficient of x^3 is the sum of a, b, c, and d, and the constant term is the product of a, b, c, and d.

63. DRAWING CONCLUSIONS Let $g(x) = 12x^4 + 8x + 9$ and $h(x) = 3x^5 + 2x^3 - 7x + 4$.

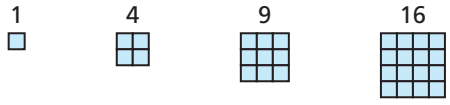
a. What is the degree of the polynomial $g(x) + h(x)$?

b. What is the degree of the polynomial $g(x) - h(x)$?

c. What is the degree of the polynomial $g(x) \cdot h(x)$?

d. In general, if $g(x)$ and $h(x)$ are polynomials such that $g(x)$ has degree m and $h(x)$ has degree n, and $m > n$, what are the degrees of $g(x) + h(x)$, $g(x) - h(x)$, and $g(x) \cdot h(x)$?

64. FINDING A PATTERN In this exercise, you will explore the sequence of square numbers. The first four square numbers are represented below.



a. Find the differences between consecutive square numbers. Explain what you notice.

b. Show how the polynomial identity $(n + 1)^2 - n^2 = 2n + 1$ models the differences between square numbers.

c. Prove the polynomial identity in part (b).

65. CRITICAL THINKING Recall that a Pythagorean triple is a set of positive integers a, b, and c such that $a^2 + b^2 = c^2$. The numbers 3, 4, and 5 form a Pythagorean triple because $3^2 + 4^2 = 5^2$. You can use the polynomial identity $(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$ to generate other Pythagorean triples.

a. Prove the polynomial identity is true by showing that the simplified expressions for the left and right sides are the same.

b. Use the identity to generate the Pythagorean triple when $x = 6$ and $y = 5$.

c. Verify that your answer in part (b) satisfies $a^2 + b^2 = c^2$.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Perform the operation. Write the answer in standard form. *(Section 3.2)*

66. $(3 - 2i) + (5 + 9i)$

67. $(12 + 3i) - (7 - 8i)$

68. $(7i)(-3i)$

69. $(4 + i)(2 - i)$

4.3 Dividing Polynomials

Essential Question How can you use the factors of a cubic polynomial to solve a division problem involving the polynomial?

EXPLORATION 1 Dividing Polynomials

Work with a partner. Match each division statement with the graph of the related cubic polynomial $f(x)$. Explain your reasoning. Use a graphing calculator to verify your answers.

a. $\frac{f(x)}{x} = (x - 1)(x + 2)$

b. $\frac{f(x)}{x - 1} = (x - 1)(x + 2)$

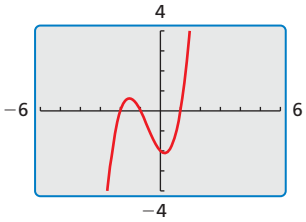
c. $\frac{f(x)}{x + 1} = (x - 1)(x + 2)$

d. $\frac{f(x)}{x - 2} = (x - 1)(x + 2)$

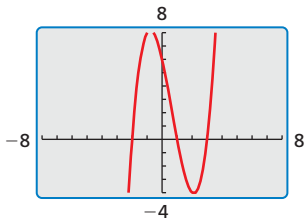
e. $\frac{f(x)}{x + 2} = (x - 1)(x + 2)$

f. $\frac{f(x)}{x - 3} = (x - 1)(x + 2)$

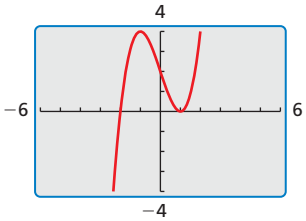
A.



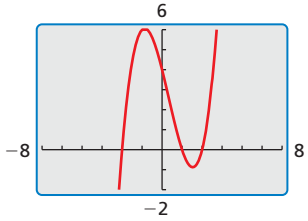
B.



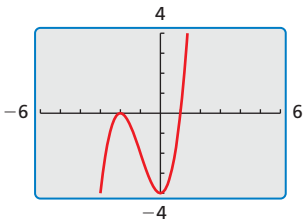
C.



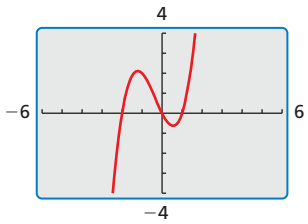
D.



E.



F.



REASONING ABSTRACTLY

To be proficient in math, you need to understand a situation abstractly and represent it symbolically.

EXPLORATION 2 Dividing Polynomials

Work with a partner. Use the results of Exploration 1 to find each quotient. Write your answers in standard form. Check your answers by multiplying.

a. $(x^3 + x^2 - 2x) \div x$

b. $(x^3 - 3x + 2) \div (x - 1)$

c. $(x^3 + 2x^2 - x - 2) \div (x + 1)$

d. $(x^3 - x^2 - 4x + 4) \div (x - 2)$

e. $(x^3 + 3x^2 - 4) \div (x + 2)$

f. $(x^3 - 2x^2 - 5x + 6) \div (x - 3)$

Communicate Your Answer

3. How can you use the factors of a cubic polynomial to solve a division problem involving the polynomial?

4.3 Lesson

What You Will Learn

- Use long division to divide polynomials by other polynomials.
- Use synthetic division to divide polynomials by binomials of the form $x - k$.
- Use the Remainder Theorem.

Core Vocabulary

polynomial long division, *p. 174*
synthetic division, *p. 175*

Previous
long division
divisor
quotient
remainder
dividend

Long Division of Polynomials

When you divide a polynomial $f(x)$ by a nonzero polynomial divisor $d(x)$, you get a quotient polynomial $q(x)$ and a remainder polynomial $r(x)$.

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

The degree of the remainder must be less than the degree of the divisor. When the remainder is 0, the divisor *divides evenly* into the dividend. Also, the degree of the divisor is less than or equal to the degree of the dividend $f(x)$. One way to divide polynomials is called **polynomial long division**.

EXAMPLE 1 Using Polynomial Long Division

Divide $2x^4 + 3x^3 + 5x - 1$ by $x^2 + 3x + 2$.

SOLUTION

Write polynomial division in the same format you use when dividing numbers. Include a "0" as the coefficient of x^2 in the dividend. At each stage, divide the term with the highest power in what is left of the dividend by the first term of the divisor. This gives the next term of the quotient.

$$\begin{array}{rl} & 2x^2 - 3x + 5 \quad \leftarrow \text{quotient} \\ x^2 + 3x + 2 \overline{)} & 2x^4 + 3x^3 + 0x^2 + 5x - 1 \\ & \underline{2x^4 + 6x^3 + 4x^2} \qquad \text{Multiply divisor by } \frac{2x^4}{x^2} = 2x^2. \\ & -3x^3 - 4x^2 + 5x \qquad \text{Subtract. Bring down next term.} \\ & \underline{-3x^3 - 9x^2 - 6x} \qquad \text{Multiply divisor by } \frac{-3x^3}{x^2} = -3x. \\ & 5x^2 + 11x - 1 \qquad \text{Subtract. Bring down next term.} \\ & \underline{5x^2 + 15x + 10} \qquad \text{Multiply divisor by } \frac{5x^2}{x^2} = 5. \\ & -4x - 11 \quad \leftarrow \text{remainder} \end{array}$$

$$\frac{2x^4 + 3x^3 + 5x - 1}{x^2 + 3x + 2} = 2x^2 - 3x + 5 + \frac{-4x - 11}{x^2 + 3x + 2}$$

COMMON ERROR

The expression added to the quotient in the result of a long division problem is $\frac{r(x)}{d(x)}$, not $r(x)$.

Check

You can check the result of a division problem by multiplying the quotient by the divisor and adding the remainder. The result should be the dividend.

$$\begin{aligned} &(2x^2 - 3x + 5)(x^2 + 3x + 2) + (-4x - 11) \\ &\quad = (2x^2)(x^2 + 3x + 2) - (3x)(x^2 + 3x + 2) + (5)(x^2 + 3x + 2) - 4x - 11 \\ &\quad = 2x^4 + 6x^3 + 4x^2 - 3x^3 - 9x^2 - 6x + 5x^2 + 15x + 10 - 4x - 11 \\ &\quad = 2x^4 + 3x^3 + 5x - 1 \checkmark \end{aligned}$$

Monitoring Progress

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Divide using polynomial long division.

1. $(x^3 - x^2 - 2x + 8) \div (x - 1)$
2. $(x^4 + 2x^2 - x + 5) \div (x^2 - x + 1)$

Synthetic Division

There is a shortcut for dividing polynomials by binomials of the form $x - k$. This shortcut is called **synthetic division**. This method is shown in the next example.

EXAMPLE 2 Using Synthetic Division

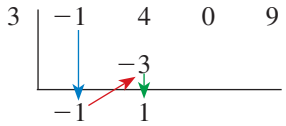
Divide $-x^3 + 4x^2 + 9$ by $x - 3$.

SOLUTION

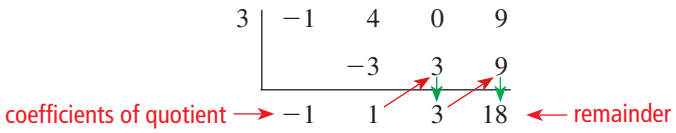
Step 1 Write the coefficients of the dividend in order of descending exponents. Include a "0" for the missing x-term. Because the divisor is $x - 3$, use $k = 3$. Write the k-value to the left of the vertical bar.

k-value → 3 | −1 4 0 9 ← coefficients of $-x^3 + 4x^2 + 9$

Step 2 Bring down the leading coefficient. Multiply the leading coefficient by the k-value. Write the product under the second coefficient. Add.



Step 3 Multiply the previous sum by the k-value. Write the product under the third coefficient. Add. Repeat this process for the remaining coefficient. The first three numbers in the bottom row are the coefficients of the quotient, and the last number is the remainder.



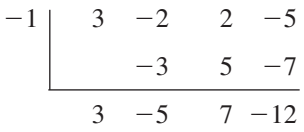
▶ $\dfrac{-x^3 + 4x^2 + 9}{x - 3} = -x^2 + x + 3 + \dfrac{18}{x - 3}$

EXAMPLE 3 Using Synthetic Division

Divide $3x^3 - 2x^2 + 2x - 5$ by $x + 1$.

SOLUTION

Use synthetic division. Because the divisor is $x + 1 = x - (-1)$, $k = -1$.



▶ $\dfrac{3x^3 - 2x^2 + 2x - 5}{x + 1} = 3x^2 - 5x + 7 - \dfrac{12}{x + 1}$

STUDY TIP

Note that dividing polynomials does not always result in a polynomial. This means that the set of polynomials is *not* closed under division.

Monitoring Progress

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Divide using synthetic division.

3. $(x^3 - 3x^2 - 7x + 6) \div (x - 2)$

4. $(2x^3 - x - 7) \div (x + 3)$

The Remainder Theorem

The remainder in the synthetic division process has an important interpretation. When you divide a polynomial $f(x)$ by $d(x) = x - k$, the result is

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$ Polynomial division

$$\frac{f(x)}{x - k} = q(x) + \frac{r(x)}{x - k}$$ Substitute $x - k$ for $d(x)$.

$$f(x) = (x - k)q(x) + r(x).$$ Multiply both sides by $x - k$.

Because either $r(x) = 0$ *or* the degree of $r(x)$ is less than the degree of $x - k$, you know that $r(x)$ is a constant function. So, let $r(x) = r$, where r is a real number, and evaluate $f(x)$ when $x = k$.

$$f(k) = (k - k)q(k) + r$$ Substitute k for x and r for $r(x)$.

$$f(k) = r$$ Simplify.

This result is stated in the *Remainder Theorem*.

Core Concept

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$.

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. So, to evaluate $f(x)$ when $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$.

EXAMPLE 4 Evaluating a Polynomial

Use synthetic division to evaluate $f(x) = 5x^3 - x^2 + 13x + 29$ when $x = -4$.

SOLUTION

$$\begin{array}{r|rrrr} -4 & 5 & -1 & 13 & 29 \\ & & -20 & 84 & -388 \\ \hline & 5 & -21 & 97 & -359 \end{array}$$

▶ The remainder is -359. So, you can conclude from the Remainder Theorem that $f(-4) = -359$.

Check

Check this by substituting $x = -4$ in the original function.

$$f(-4) = 5(-4)^3 - (-4)^2 + 13(-4) + 29$$
$$= -320 - 16 - 52 + 29$$
$$= -359 \checkmark$$

Monitoring Progress

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Use synthetic division to evaluate the function for the indicated value of x.

5. $f(x) = 4x^2 - 10x - 21; x = 5$

6. $f(x) = 5x^4 + 2x^3 - 20x - 6; x = 2$

4.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Explain the Remainder Theorem in your own words. Use an example in your explanation.

2. **VOCABULARY** What form must the divisor have to make synthetic division an appropriate method for dividing a polynomial? Provide examples to support your claim.

3. **VOCABULARY** Write the polynomial divisor, dividend, and quotient functions represented by the synthetic division shown at the right.

$$\begin{array}{r|rrrr} -3 & 1 & -2 & -9 & 18 \\ & & -3 & 15 & -18 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

4. **WRITING** Explain what the colored numbers represent in the synthetic division in Exercise 3.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, divide using polynomial long division. *(See Example 1.)*

5. $(x^2 + x - 17) \div (x - 4)$

6. $(3x^2 - 14x - 5) \div (x - 5)$

7. $(x^3 + x^2 + x + 2) \div (x^2 - 1)$

8. $(7x^3 + x^2 + x) \div (x^2 + 1)$

9. $(5x^4 - 2x^3 - 7x^2 - 39) \div (x^2 + 2x - 4)$

10. $(4x^4 + 5x - 4) \div (x^2 - 3x - 2)$

In Exercises 11–18, divide using synthetic division. *(See Examples 2 and 3.)*

11. $(x^2 + 8x + 1) \div (x - 4)$

12. $(4x^2 - 13x - 5) \div (x - 2)$

13. $(2x^2 - x + 7) \div (x + 5)$

14. $(x^3 - 4x + 6) \div (x + 3)$

15. $(x^2 + 9) \div (x - 3)$

16. $(3x^3 - 5x^2 - 2) \div (x - 1)$

17. $(x^4 - 5x^3 - 8x^2 + 13x - 12) \div (x - 6)$

18. $(x^4 + 4x^3 + 16x - 35) \div (x + 5)$

ANALYZING RELATIONSHIPS In Exercises 19–22, match the equivalent expressions. Justify your answers.

19. $(x^2 + x - 3) \div (x - 2)$

20. $(x^2 - x - 3) \div (x - 2)$

21. $(x^2 - x + 3) \div (x - 2)$

22. $(x^2 + x + 3) \div (x - 2)$

A. $x + 1 - \frac{1}{x - 2}$ **B.** $x + 3 + \frac{9}{x - 2}$

C. $x + 1 + \frac{5}{x - 2}$ **D.** $x + 3 + \frac{3}{x - 2}$

ERROR ANALYSIS In Exercises 23 and 24, describe and correct the error in using synthetic division to divide $x^3 - 5x + 3$ by $x - 2$.

23.

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -5 & 3 \\ & & 2 & 4 & -2 \\ \hline & 1 & 2 & -1 & 1 \end{array}$$

$$\frac{x^3 - 5x + 3}{x - 2} = x^3 + 2x^2 - x + 1$$

24.

$$\begin{array}{r|rrr} 2 & 1 & -5 & 3 \\ & & 2 & -6 \\ \hline & 1 & -3 & -3 \end{array}$$

$$\frac{x^3 - 5x + 3}{x - 2} = x^2 - 3x - \frac{3}{x - 2}$$

In Exercises 25–32, use synthetic division to evaluate the function for the indicated value of x. *(See Example 4.)*

25. $f(x) = -x^2 - 8x + 30; x = -1$

26. $f(x) = 3x^2 + 2x - 20; x = 3$

27. $f(x) = x^3 - 2x^2 + 4x + 3; x = 2$

28. $f(x) = x^3 + x^2 - 3x + 9; x = -4$

29. $f(x) = x^3 - 6x + 1; x = 6$

30. $f(x) = x^3 - 9x - 7; x = 10$

31. $f(x) = x^4 + 6x^2 - 7x + 1; x = 3$

32. $f(x) = -x^4 - x^3 - 2; x = 5$

33. MAKING AN ARGUMENT You use synthetic division to divide $f(x)$ by $(x - a)$ and find that the remainder equals 15. Your friend concludes that $f(15) = a$. Is your friend correct? Explain your reasoning.

34. THOUGHT PROVOKING A polygon has an area represented by $A = 4x^2 + 8x + 4$. The figure has at least one dimension equal to $2x + 2$. Draw the figure and label its dimensions.

35. USING TOOLS The total attendance A (in thousands) at NCAA women's basketball games and the number T of NCAA women's basketball teams over a period of time can be modeled by

$$A = -1.95x^3 + 70.1x^2 - 188x + 2150$$
$$T = 14.8x + 725$$

where x is in years and $0 < x < 18$. Write a function for the average attendance per team over this period of time.

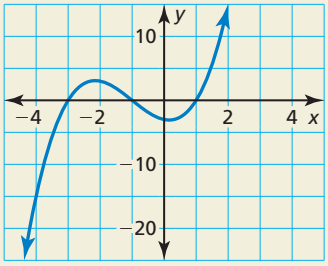
36. COMPARING METHODS The profit P (in millions of dollars) for a DVD manufacturer can be modeled by $P = -6x^3 + 72x$, where x is the number (in millions) of DVDs produced. Use synthetic division to show that the company yields a profit of \$96 million when 2 million DVDs are produced. Is there an easier method? Explain.

37. CRITICAL THINKING What is the value of k such that $(x^3 - x^2 + kx - 30) \div (x - 5)$ has a remainder of zero?

Ⓐ -14 Ⓑ -2

Ⓒ 26 Ⓓ 32

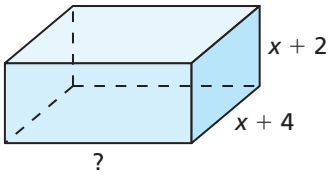
38. HOW DO YOU SEE IT? The graph represents the polynomial function $f(x) = x^3 + 3x^2 - x - 3$.



a. The expression $f(x) \div (x - k)$ has a remainder of -15. What is the value of k?

b. Use the graph to compare the remainders of $(x^3 + 3x^2 - x - 3) \div (x + 3)$ and $(x^3 + 3x^2 - x - 3) \div (x + 1)$.

39. MATHEMATICAL CONNECTIONS The volume V of the rectangular prism is given by $V = 2x^3 + 17x^2 + 46x + 40$. Find an expression for the missing dimension.



40. USING STRUCTURE You divide two polynomials and obtain the result $5x^2 - 13x + 47 - \frac{102}{x + 2}$. What is the dividend? How did you find it?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find the zero(s) of the function. *(Sections 3.1 and 3.2)*

41. $f(x) = x^2 - 6x + 9$

42. $g(x) = 3(x + 6)(x - 2)$

43. $g(x) = x^2 + 14x + 49$

44. $h(x) = 4x^2 + 36$

4.4 Factoring Polynomials

Essential Question How can you factor a polynomial?

EXPLORATION 1 Factoring Polynomials

Work with a partner. Match each polynomial equation with the graph of its related polynomial function. Use the x-intercepts of the graph to write each polynomial in factored form. Explain your reasoning.

a. $x^2 + 5x + 4 = 0$

b. $x^3 - 2x^2 - x + 2 = 0$

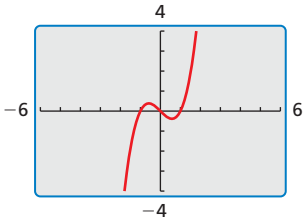
c. $x^3 + x^2 - 2x = 0$

d. $x^3 - x = 0$

e. $x^4 - 5x^2 + 4 = 0$

f. $x^4 - 2x^3 - x^2 + 2x = 0$

A.



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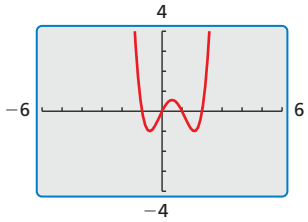

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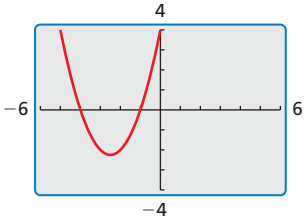

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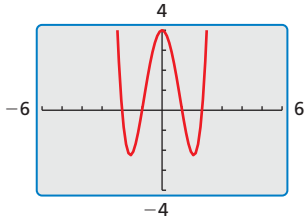

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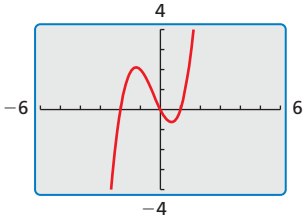

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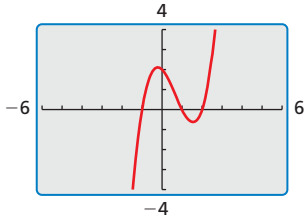

MAKING SENSE OF PROBLEMS

To be proficient in math, you need to check your answers to problems and continually ask yourself, "Does this make sense?"

EXPLORATION 2 Factoring Polynomials

Work with a partner. Use the x-intercepts of the graph of the polynomial function to write each polynomial in factored form. Explain your reasoning. Check your answers by multiplying.

a. $f(x) = x^2 - x - 2$

b. $f(x) = x^3 - x^2 - 2x$

c. $f(x) = x^3 - 2x^2 - 3x$

d. $f(x) = x^3 - 3x^2 - x + 3$

e. $f(x) = x^4 + 2x^3 - x^2 - 2x$

f. $f(x) = x^4 - 10x^2 + 9$

Communicate Your Answer

3. How can you factor a polynomial?

4. What information can you obtain about the graph of a polynomial function written in factored form?

4.4 Lesson

Core Vocabulary

factored completely, *p. 180*
factor by grouping, *p. 181*
quadratic form, *p. 181*

Previous
zero of a function
synthetic division

What You Will Learn

- Factor polynomials.
- Use the Factor Theorem.

Factoring Polynomials

Previously, you factored quadratic polynomials. You can also factor polynomials with degree greater than 2. Some of these polynomials can be *factored completely* using techniques you have previously learned. A factorable polynomial with integer coefficients is **factored completely** when it is written as a product of unfactorable polynomials with integer coefficients.

EXAMPLE 1 Finding a Common Monomial Factor

Factor each polynomial completely.

a. $x^3 - 4x^2 - 5x$ **b.** $3y^5 - 48y^3$ **c.** $5z^4 + 30z^3 + 45z^2$

SOLUTION

a. $x^3 - 4x^2 - 5x = x(x^2 - 4x - 5)$ Factor common monomial.

$= x(x - 5)(x + 1)$ Factor trinomial.

b. $3y^5 - 48y^3 = 3y^3(y^2 - 16)$ Factor common monomial.

$= 3y^3(y - 4)(y + 4)$ Difference of Two Squares Pattern

c. $5z^4 + 30z^3 + 45z^2 = 5z^2(z^2 + 6z + 9)$ Factor common monomial.

$= 5z^2(z + 3)^2$ Perfect Square Trinomial Pattern

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Factor the polynomial completely.

1. $x^3 - 7x^2 + 10x$ **2.** $3n^7 - 75n^5$ **3.** $8m^5 - 16m^4 + 8m^3$

In part (b) of Example 1, the special factoring pattern for the difference of two squares was used to factor the expression completely. There are also factoring patterns that you can use to factor the sum or difference of two *cubes*.

Core Concept

Special Factoring Patterns

Sum of Two Cubes	Example
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	$64x^3 + 1 = (4x)^3 + 1^3$ $= (4x + 1)(16x^2 - 4x + 1)$

Difference of Two Cubes	Example
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	$27x^3 - 8 = (3x)^3 - 2^3$ $= (3x - 2)(9x^2 + 6x + 4)$

EXAMPLE 2 Factoring the Sum or Difference of Two Cubes

Factor (a) $x^3 - 125$ and (b) $16s^5 + 54s^2$ completely.

SOLUTION

a. $x^3 - 125 = x^3 - 5^3$ — Write as $a^3 - b^3$.

$= (x - 5)(x^2 + 5x + 25)$ — Difference of Two Cubes Pattern

b. $16s^5 + 54s^2 = 2s^2(8s^3 + 27)$ — Factor common monomial.

$= 2s^2[(2s)^3 + 3^3]$ — Write $8s^3 + 27$ as $a^3 + b^3$.

$= 2s^2(2s + 3)(4s^2 - 6s + 9)$ — Sum of Two Cubes Pattern

For some polynomials, you can **factor by grouping** pairs of terms that have a common monomial factor. The pattern for factoring by grouping is shown below.

$$ra + rb + sa + sb = r(a + b) + s(a + b)$$
$$= (r + s)(a + b)$$

EXAMPLE 3 Factoring by Grouping

Factor $z^3 + 5z^2 - 4z - 20$ completely.

SOLUTION

$z^3 + 5z^2 - 4z - 20 = z^2(z + 5) - 4(z + 5)$ — Factor by grouping.

$= (z^2 - 4)(z + 5)$ — Distributive Property

$= (z - 2)(z + 2)(z + 5)$ — Difference of Two Squares Pattern

An expression of the form $au^2 + bu + c$, where u is an algebraic expression, is said to be in **quadratic form**. The factoring techniques you have studied can sometimes be used to factor such expressions.

LOOKING FOR STRUCTURE

The expression $16x^4 - 81$ is in quadratic form because it can be written as $u^2 - 81$ where $u = 4x^2$.

EXAMPLE 4 Factoring Polynomials in Quadratic Form

Factor (a) $16x^4 - 81$ and (b) $3p^8 + 15p^5 + 18p^2$ completely.

SOLUTION

a. $16x^4 - 81 = (4x^2)^2 - 9^2$ — Write as $a^2 - b^2$.

$= (4x^2 + 9)(4x^2 - 9)$ — Difference of Two Squares Pattern

$= (4x^2 + 9)(2x - 3)(2x + 3)$ — Difference of Two Squares Pattern

b. $3p^8 + 15p^5 + 18p^2 = 3p^2(p^6 + 5p^3 + 6)$ — Factor common monomial.

$= 3p^2(p^3 + 3)(p^3 + 2)$ — Factor trinomial in quadratic form.

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Factor the polynomial completely.

4. $a^3 + 27$

5. $6z^5 - 750z^2$

6. $x^3 + 4x^2 - x - 4$

7. $3y^3 + y^2 + 9y + 3$

8. $-16n^4 + 625$

9. $5w^6 - 25w^4 + 30w^2$

The Factor Theorem

When dividing polynomials in the previous section, the examples had nonzero remainders. Suppose the remainder is 0 when a polynomial $f(x)$ is divided by $x - k$. Then,

$$\frac{f(x)}{x-k} = q(x) + \frac{0}{x-k} = q(x)$$

where $q(x)$ is the quotient polynomial. Therefore, $f(x) = (x - k) \cdot q(x)$, so that $x - k$ is a factor of $f(x)$. This result is summarized by the *Factor Theorem*, which is a special case of the Remainder Theorem.

READING

In other words, $x - k$ is a factor of $f(x)$ if and only if k is a zero of f.

Core Concept

The Factor Theorem

A polynomial $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$.

STUDY TIP

In part (b), notice that direct substitution would have resulted in more difficult computations than synthetic division.

EXAMPLE 5 Determining Whether a Linear Binomial Is a Factor

Determine whether (a) $x - 2$ is a factor of $f(x) = x^2 + 2x - 4$ and (b) $x + 5$ is a factor of $f(x) = 3x^4 + 15x^3 - x^2 + 25$.

SOLUTION

a. Find $f(2)$ by direct substitution.

$$\begin{aligned} f(2) &= 2^2 + 2(2) - 4 \\ &= 4 + 4 - 4 \\ &= 4 \end{aligned}$$

▶ Because $f(2) \neq 0$, the binomial $x - 2$ is not a factor of $f(x) = x^2 + 2x - 4$.

b. Find $f(-5)$ by synthetic division.

-5	3	15	-1	0	25
		-15	0	5	-25
	3	0	-1	5	0

▶ Because $f(-5) = 0$, the binomial $x + 5$ is a factor of $f(x) = 3x^4 + 15x^3 - x^2 + 25$.

EXAMPLE 6 Factoring a Polynomial

Show that $x + 3$ is a factor of $f(x) = x^4 + 3x^3 - x - 3$. Then factor $f(x)$ completely.

SOLUTION

Show that $f(-3) = 0$ by synthetic division.

-3	1	3	0	-1	-3
		-3	0	0	3
	1	0	0	-1	0

ANOTHER WAY

Notice that you can factor $f(x)$ by grouping.

$$\begin{aligned} f(x) &= x^3(x + 3) - 1(x + 3) \\ &= (x^3 - 1)(x + 3) \\ &= (x + 3)(x - 1) \cdot (x^2 + x + 1) \end{aligned}$$

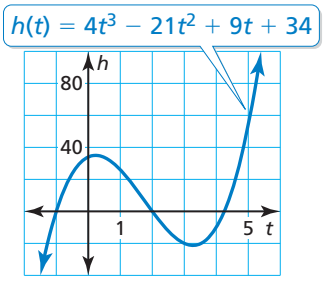
Because $f(-3) = 0$, you can conclude that $x + 3$ is a factor of $f(x)$ by the Factor Theorem. Use the result to write $f(x)$ as a product of two factors and then factor completely.

$$\begin{aligned} f(x) &= x^4 + 3x^3 - x - 3 && \text{Write original polynomial.} \\ &= (x + 3)(x^3 - 1) && \text{Write as a product of two factors.} \\ &= (x + 3)(x - 1)(x^2 + x + 1) && \text{Difference of Two Cubes Pattern} \end{aligned}$$

Because the x-intercepts of the graph of a function are the zeros of the function, you can use the graph to approximate the zeros. You can check the approximations using the Factor Theorem.

EXAMPLE 7 Real-Life Application

During the first 5 seconds of a roller coaster ride, the function $h(t) = 4t^3 - 21t^2 + 9t + 34$ represents the height h (in feet) of the roller coaster after t seconds. How long is the roller coaster at or below ground level in the first 5 seconds?



SOLUTION

1. **Understand the Problem** You are given a function rule that represents the height of a roller coaster. You are asked to determine how long the roller coaster is at or below ground during the first 5 seconds of the ride.
2. **Make a Plan** Use a graph to estimate the zeros of the function and check using the Factor Theorem. Then use the zeros to describe where the graph lies below the t-axis.
3. **Solve the Problem** From the graph, two of the zeros appear to be -1 and 2. The third zero is between 4 and 5.

   Step 1 Determine whether -1 is a zero using synthetic division.

$$\begin{array}{r|rrrr} -1 & 4 & -21 & 9 & 34 \\ & & -4 & 25 & -34 \\ \hline & 4 & -25 & 34 & 0 \end{array}$$

$h(-1) = 0$, so -1 is a zero of h and $t + 1$ is a factor of $h(t)$.

STUDY TIP

You could also check that 2 is a zero using the original function, but using the quotient polynomial helps you find the remaining factor.

   Step 2 Determine whether 2 is a zero. If 2 is also a zero, then $t - 2$ is a factor of the resulting quotient polynomial. Check using synthetic division.

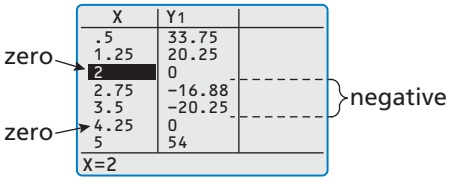
$$\begin{array}{r|rrr} 2 & 4 & -25 & 34 \\ & & 8 & -34 \\ \hline & 4 & -17 & 0 \end{array}$$

The remainder is 0, so $t - 2$ is a factor of $h(t)$ and 2 is a zero of h.

   So, $h(t) = (t + 1)(t - 2)(4t - 17)$. The factor $4t - 17$ indicates that the zero between 4 and 5 is $\frac{17}{4}$, or 4.25.

   The zeros are -1, 2, and 4.25. Only $t = 2$ and $t = 4.25$ occur in the first 5 seconds. The graph shows that the roller coaster is at or below ground level for $4.25 - 2 = 2.25$ seconds.

4. **Look Back** Use a table of values to verify the positive zeros and heights between the zeros.



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10. Determine whether $x - 4$ is a factor of $f(x) = 2x^2 + 5x - 12$.

11. Show that $x - 6$ is a factor of $f(x) = x^3 - 5x^2 - 6x$. Then factor $f(x)$ completely.

12. In Example 7, does your answer change when you first determine whether 2 is a zero and then whether -1 is a zero? Justify your answer.

4.4 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The expression $9x^4 - 49$ is in _____ form because it can be written as $u^2 - 49$ where $u =$ _____.
2. **VOCABULARY** Explain when you should try factoring a polynomial by grouping.
3. **WRITING** How do you know when a polynomial is factored completely?
4. **WRITING** Explain the Factor Theorem and why it is useful.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, factor the polynomial completely. *(See Example 1.)*

5. $x^3 - 2x^2 - 24x$
6. $4k^5 - 100k^3$
7. $3p^5 - 192p^3$
8. $2m^6 - 24m^5 + 64m^4$
9. $2q^4 + 9q^3 - 18q^2$
10. $3r^6 - 11r^5 - 20r^4$
11. $10w^{10} - 19w^9 + 6w^8$
12. $18v^9 + 33v^8 + 14v^7$

In Exercises 13–20, factor the polynomial completely. *(See Example 2.)*

13. $x^3 + 64$
14. $y^3 + 512$
15. $g^3 - 343$
16. $c^3 - 27$
17. $3h^9 - 192h^6$
18. $9n^6 - 6561n^3$
19. $16t^7 + 250t^4$
20. $135z^{11} - 1080z^8$

ERROR ANALYSIS **In Exercises 21 and 22, describe and correct the error in factoring the polynomial.**

21. ✗
$$3x^3 + 27x = 3x(x^2 + 9)$$
$$= 3x(x + 3)(x - 3)$$

22. ✗
$$x^9 + 8x^3 = (x^3)^3 + (2x)^3$$
$$= (x^3 + 2x)[(x^3)^2 - (x^3)(2x) + (2x)^2]$$
$$= (x^3 + 2x)(x^6 - 2x^4 + 4x^2)$$

In Exercises 23–30, factor the polynomial completely. *(See Example 3.)*

23. $y^3 - 5y^2 + 6y - 30$
24. $m^3 - m^2 + 7m - 7$
25. $3a^3 + 18a^2 + 8a + 48$
26. $2k^3 - 20k^2 + 5k - 50$
27. $x^3 - 8x^2 - 4x + 32$
28. $z^3 - 5z^2 - 9z + 45$
29. $4q^3 - 16q^2 - 9q + 36$
30. $16n^3 + 32n^2 - n - 2$

In Exercises 31–38, factor the polynomial completely. *(See Example 4.)*

31. $49k^4 - 9$
32. $4m^4 - 25$
33. $c^4 + 9c^2 + 20$
34. $y^4 - 3y^2 - 28$
35. $16z^4 - 81$
36. $81a^4 - 256$
37. $3r^8 + 3r^5 - 60r^2$
38. $4n^{12} - 32n^7 + 48n^2$

In Exercises 39–44, determine whether the binomial is a factor of the polynomial. *(See Example 5.)*

39. $f(x) = 2x^3 + 5x^2 - 37x - 60;\ x - 4$
40. $g(x) = 3x^3 - 28x^2 + 29x + 140;\ x + 7$
41. $h(x) = 6x^5 - 15x^4 - 9x^3;\ x + 3$
42. $g(x) = 8x^5 - 58x^4 + 60x^3 + 140;\ x - 6$
43. $h(x) = 6x^4 - 6x^3 - 84x^2 + 144x;\ x + 4$
44. $t(x) = 48x^4 + 36x^3 - 138x^2 - 36x;\ x + 2$

In Exercises 45–50, show that the binomial is a factor of the polynomial. Then factor the polynomial completely. *(See Example 6.)*

45. $g(x) = x^3 - x^2 - 20x;\ x + 4$

46. $t(x) = x^3 - 5x^2 - 9x + 45;\ x - 5$

47. $f(x) = x^4 - 6x^3 - 8x + 48;\ x - 6$

48. $s(x) = x^4 + 4x^3 - 64x - 256;\ x + 4$

49. $r(x) = x^3 - 37x + 84;\ x + 7$

50. $h(x) = x^3 - x^2 - 24x - 36;\ x + 2$

ANALYZING RELATIONSHIPS **In Exercises 51–54, match the function with the correct graph. Explain your reasoning.**

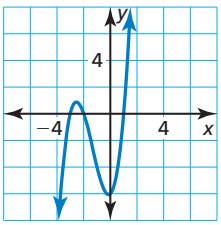
51. $f(x) = (x - 2)(x - 3)(x + 1)$

52. $g(x) = x(x + 2)(x + 1)(x - 2)$

53. $h(x) = (x + 2)(x + 3)(x - 1)$

54. $k(x) = x(x - 2)(x - 1)(x + 2)$

A.



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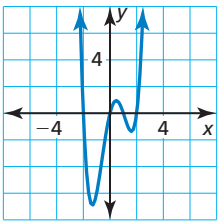

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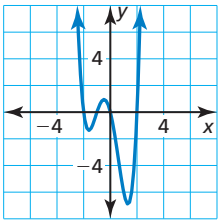

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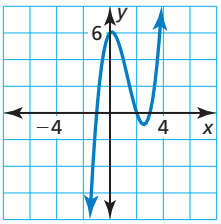
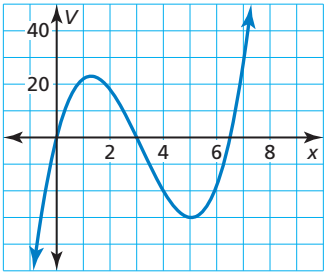
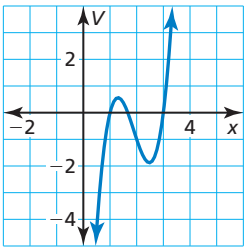


55. **MODELING WITH MATHEMATICS** The volume (in cubic inches) of a shipping box is modeled by $V = 2x^3 - 19x^2 + 39x$, where x is the length (in inches). Determine the values of x for which the model makes sense. Explain your reasoning. *(See Example 7.)*



56. **MODELING WITH MATHEMATICS** The volume (in cubic inches) of a rectangular birdcage can be modeled by $V = 3x^3 - 17x^2 + 29x - 15$, where x is the length (in inches). Determine the values of x for which the model makes sense. Explain your reasoning.



USING STRUCTURE **In Exercises 57–64, use the method of your choice to factor the polynomial completely. Explain your reasoning.**

57. $a^6 + a^5 - 30a^4$

58. $8m^3 - 343$

59. $z^3 - 7z^2 - 9z + 63$

60. $2p^8 - 12p^5 + 16p^2$

61. $64r^3 + 729$

62. $5x^5 - 10x^4 - 40x^3$

63. $16n^4 - 1$

64. $9k^3 - 24k^2 + 3k - 8$

65. **REASONING** Determine whether each polynomial is factored completely. If not, factor completely.

a. $7z^4(2z^2 - z - 6)$

b. $(2 - n)(n^2 + 6n)(3n - 11)$

c. $3(4y - 5)(9y^2 - 6y - 4)$

66. **PROBLEM SOLVING** The profit P (in millions of dollars) for a T-shirt manufacturer can be modeled by $P = -x^3 + 4x^2 + x$, where x is the number (in millions) of T-shirts produced. Currently the company produces 4 million T-shirts and makes a profit of \$4 million. What lesser number of T-shirts could the company produce and still make the same profit?

67. **PROBLEM SOLVING** The profit P (in millions of dollars) for a shoe manufacturer can be modeled by $P = -21x^3 + 46x$, where x is the number (in millions) of shoes produced. The company now produces 1 million shoes and makes a profit of \$25 million, but it would like to cut back production. What lesser number of shoes could the company produce and still make the same profit?

68. THOUGHT PROVOKING Find a value of k such that $\frac{f(x)}{x-k}$ has a remainder of 0. Justify your answer.

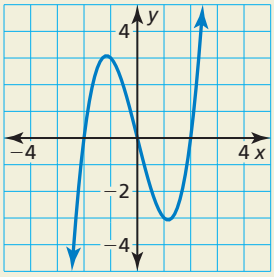
$$f(x) = x^3 - 3x^2 - 4x$$

69. COMPARING METHODS You are taking a test where calculators are not permitted. One question asks you to evaluate $g(7)$ for the function $g(x) = x^3 - 7x^2 - 4x + 28$. You use the Factor Theorem and synthetic division and your friend uses direct substitution. Whose method do you prefer? Explain your reasoning.

70. MAKING AN ARGUMENT You divide $f(x)$ by $(x - a)$ and find that the remainder does not equal 0. Your friend concludes that $f(x)$ cannot be factored. Is your friend correct? Explain your reasoning.

71. CRITICAL THINKING What is the value of k such that $x - 7$ is a factor of $h(x) = 2x^3 - 13x^2 - kx + 105$? Justify your answer.

72. HOW DO YOU SEE IT? Use the graph to write an equation of the cubic function in factored form. Explain your reasoning.



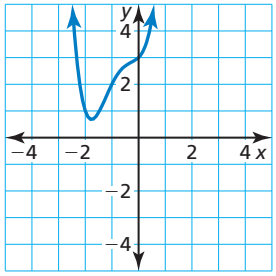
73. ABSTRACT REASONING Factor each polynomial completely.

a. $7ac^2 + bc^2 - 7ad^2 - bd^2$

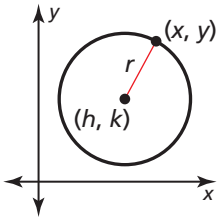
b. $x^{2n} - 2x^n + 1$

c. $a^5b^2 - a^2b^4 + 2a^4b - 2ab^3 + a^3 - b^2$

74. REASONING The graph of the function $f(x) = x^4 + 3x^3 + 2x^2 + x + 3$ is shown. Can you use the Factor Theorem to factor $f(x)$? Explain.



75. MATHEMATICAL CONNECTIONS The standard equation of a circle with radius r and center (h, k) is $(x - h)^2 + (y - k)^2 = r^2$. Rewrite each equation of a circle in standard form. Identify the center and radius of the circle. Then graph the circle.



a. $x^2 + 6x + 9 + y^2 = 25$

b. $x^2 - 4x + 4 + y^2 = 9$

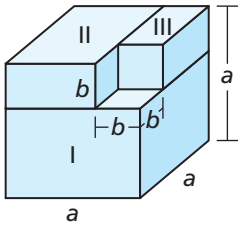
c. $x^2 - 8x + 16 + y^2 + 2y + 1 = 36$

76. CRITICAL THINKING Use the diagram to complete parts (a)–(c).

a. Explain why $a^3 - b^3$ is equal to the sum of the volumes of the solids I, II, and III.

b. Write an algebraic expression for the volume of each of the three solids. Leave your expressions in factored form.

c. Use the results from part (a) and part (b) to derive the factoring pattern $a^3 - b^3$.



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the quadratic equation by factoring. *(Section 3.1)*

77. $x^2 - x - 30 = 0$

78. $2x^2 - 10x - 72 = 0$

79. $3x^2 - 11x + 10 = 0$

80. $9x^2 - 28x + 3 = 0$

Solve the quadratic equation by completing the square. *(Section 3.3)*

81. $x^2 - 12x + 36 = 144$

82. $x^2 - 8x - 11 = 0$

83. $3x^2 + 30x + 63 = 0$

84. $4x^2 + 36x - 4 = 0$

4.1–4.4 What Did You Learn?

Core Vocabulary

polynomial, *p. 158*
polynomial function, *p. 158*
end behavior, *p. 159*
Pascal's Triangle, *p. 169*
polynomial long division, *p. 174*
synthetic division, *p. 175*
factored completely, *p. 180*
factor by grouping, *p. 181*
quadratic form, *p. 181*

Core Concepts

Section 4.1

Common Polynomial Functions, *p. 158*
End Behavior of Polynomial Functions, *p. 159*
Graphing Polynomial Functions, *p. 160*

Section 4.2

Operations with Polynomials, *p. 166*
Special Product Patterns, *p. 167*
Pascal's Triangle, *p. 169*

Section 4.3

Polynomial Long Division, *p. 174*
Synthetic Division, *p. 175*
The Remainder Theorem, *p. 176*

Section 4.4

Factoring Polynomials, *p. 180*
Special Factoring Patterns, *p. 180*
The Factor Theorem, *p. 182*

Mathematical Practices

1. Describe the entry points you used to analyze the function in Exercise 43 on page 164.
2. Describe how you maintained oversight in the process of factoring the polynomial in Exercise 49 on page 185.

Study Skills

Keeping Your Mind Focused

- When you sit down at your desk, review your notes from the last class.
- Repeat in your mind what you are writing in your notes.
- When a mathematical concept is particularly difficult, ask your teacher for another example.

4.1–4.4 Quiz

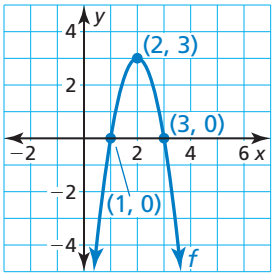
Decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient. *(Section 4.1)*

1. $f(x) = 5 + 2x^2 - 3x^4 - 2x - x^3$

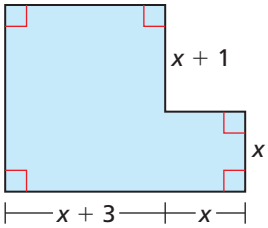
2. $g(x) = \frac{1}{4}x^3 + 2x - 3x^2 + 1$

3. $h(x) = 3 - 6x^3 + 4x^{-2} + 6x$

4. Describe the x-values for which (a) f is increasing or decreasing, (b) $f(x) > 0$, and (c) $f(x) < 0$. *(Section 4.1)*



5. Write an expression for the area and perimeter for the figure shown. *(Section 4.2)*



Perform the indicated operation. *(Section 4.2)*

6. $(7x^2 - 4) - (3x^2 - 5x + 1)$

7. $(x^2 - 3x + 2)(3x - 1)$

8. $(x - 1)(x + 3)(x - 4)$

9. Use Pascal's Triangle to expand $(x + 2)^5$. *(Section 4.2)*

10. Divide $4x^4 - 2x^3 + x^2 - 5x + 8$ by $x^2 - 2x - 1$. *(Section 4.3)*

Factor the polynomial completely. *(Section 4.4)*

11. $a^3 - 2a^2 - 8a$

12. $8m^3 + 27$

13. $z^3 + z^2 - 4z - 4$

14. $49b^4 - 64$

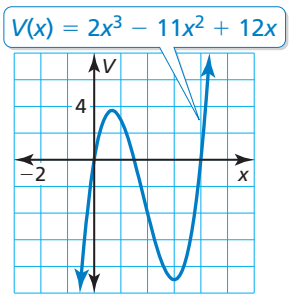
15. Show that $x + 5$ is a factor of $f(x) = x^3 - 2x^2 - 23x + 60$. Then factor $f(x)$ completely. *(Section 4.4)*

16. The estimated price P (in cents) of stamps in the United States can be modeled by the polynomial function $P(t) = 0.007t^3 - 0.16t^2 + 1t + 17$, where t represents the number of years since 1990. *(Section 4.1)*

a. Use a graphing calculator to graph the function for the interval $0 \le t \le 20$. Describe the behavior of the graph on this interval.

b. What was the average rate of change in the price of stamps from 1990 to 2010?

17. The volume V (in cubic feet) of a rectangular wooden crate is modeled by the function $V(x) = 2x^3 - 11x^2 + 12x$, where x is the width (in feet) of the crate. Determine the values of x for which the model makes sense. Explain your reasoning. *(Section 4.4)*



4.5 Solving Polynomial Equations

Essential Question How can you determine whether a polynomial equation has a repeated solution?

EXPLORATION 1 Cubic Equations and Repeated Solutions

Work with a partner. Some cubic equations have three distinct solutions. Others have repeated solutions. Match each cubic polynomial equation with the graph of its related polynomial function. Then solve each equation. For those equations that have repeated solutions, describe the behavior of the related function near the repeated zero using the graph or a table of values.

a. $x^3 - 6x^2 + 12x - 8 = 0$

b. $x^3 + 3x^2 + 3x + 1 = 0$

c. $x^3 - 3x + 2 = 0$

d. $x^3 + x^2 - 2x = 0$

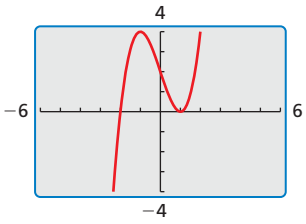
e. $x^3 - 3x - 2 = 0$

f. $x^3 - 3x^2 + 2x = 0$

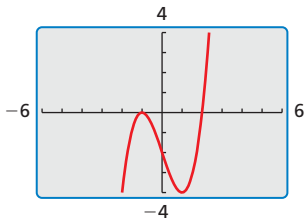
USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technological tools to explore and deepen your understanding of concepts.

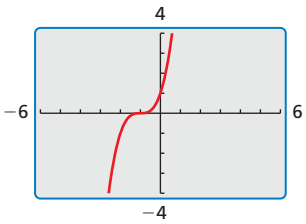
A.



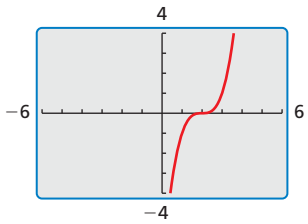
B.



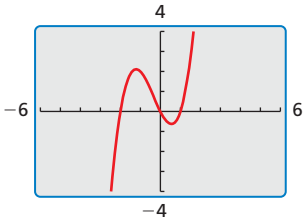
C.



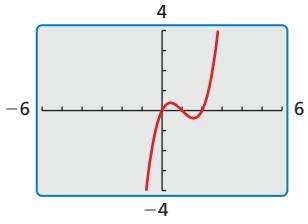
D.



E.



F.



EXPLORATION 2 Quartic Equations and Repeated Solutions

Work with a partner. Determine whether each quartic equation has repeated solutions using the graph of the related quartic function or a table of values. Explain your reasoning. Then solve each equation.

a. $x^4 - 4x^3 + 5x^2 - 2x = 0$

b. $x^4 - 2x^3 - x^2 + 2x = 0$

c. $x^4 - 4x^3 + 4x^2 = 0$

d. $x^4 + 3x^3 = 0$

Communicate Your Answer

3. How can you determine whether a polynomial equation has a repeated solution?

4. Write a cubic or a quartic polynomial equation that is different from the equations in Explorations 1 and 2 and has a repeated solution.

4.5 Lesson

Core Vocabulary

repeated solution, *p. 190*

Previous

roots of an equation
real numbers
conjugates

What You Will Learn

- Find solutions of polynomial equations and zeros of polynomial functions.
- Use the Rational Root Theorem.
- Use the Irrational Conjugates Theorem.

Finding Solutions and Zeros

You have used the Zero-Product Property to solve factorable quadratic equations. You can extend this technique to solve some higher-degree polynomial equations.

EXAMPLE 1 Solving a Polynomial Equation by Factoring

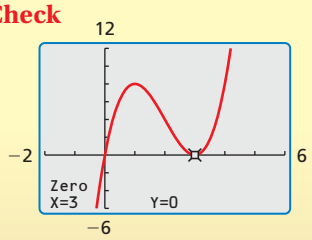
Solve $2x^3 - 12x^2 + 18x = 0$.

SOLUTION

$2x^3 - 12x^2 + 18x = 0$	Write the equation.
$2x(x^2 - 6x + 9) = 0$	Factor common monomial.
$2x(x - 3)^2 = 0$	Perfect Square Trinomial Pattern
$2x = 0$ or $(x - 3)^2 = 0$	Zero-Product Property
$x = 0$ or $x = 3$	Solve for x.

- The solutions, or roots, are $x = 0$ and $x = 3$.

Check



In Example 1, the factor $x - 3$ appears more than once. This creates a **repeated solution** of $x = 3$. Note that the graph of the related function touches the x-axis (but does not cross the x-axis) at the repeated zero $x = 3$, and crosses the x-axis at the zero $x = 0$. This concept can be generalized as follows.

- When a factor $x - k$ of $f(x)$ is raised to an odd power, the graph of f *crosses* the x-axis at $x = k$.
- When a factor $x - k$ of $f(x)$ is raised to an even power, the graph of f *touches* the x-axis (but does not cross the x-axis) at $x = k$.

STUDY TIP

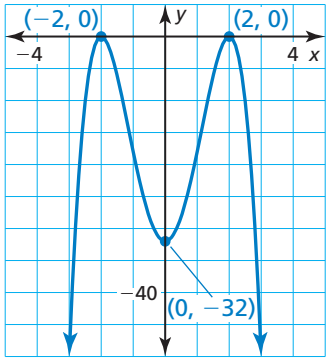
Because the factor $x - 3$ appears twice, the root $x = 3$ has a *multiplicity* of 2.

EXAMPLE 2 Finding Zeros of a Polynomial Function

Find the zeros of $f(x) = -2x^4 + 16x^2 - 32$. Then sketch a graph of the function.

SOLUTION

$0 = -2x^4 + 16x^2 - 32$	Set $f(x)$ equal to 0.
$0 = -2(x^4 - 8x^2 + 16)$	Factor out -2.
$0 = -2(x^2 - 4)(x^2 - 4)$	Factor trinomial in quadratic form.
$0 = -2(x + 2)(x - 2)(x + 2)(x - 2)$	Difference of Two Squares Pattern
$0 = -2(x + 2)^2(x - 2)^2$	Rewrite using exponents.



Because both factors $x + 2$ and $x - 2$ are raised to an even power, the graph of f touches the x-axis at the zeros $x = -2$ and $x = 2$.

By analyzing the original function, you can determine that the y-intercept is -32. Because the degree is even and the leading coefficient is negative, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$. Use these characteristics to sketch a graph of the function.

Monitoring Progress

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Solve the equation.

1. $4x^4 - 40x^2 + 36 = 0$

2. $2x^5 + 24x = 14x^3$

Find the zeros of the function. Then sketch a graph of the function.

3. $f(x) = 3x^4 - 6x^2 + 3$

4. $f(x) = x^3 + x^2 - 6x$

The Rational Root Theorem

The solutions of the equation $64x^3 + 152x^2 - 62x - 105 = 0$ are $-\frac{5}{2}$, $-\frac{3}{4}$, and $\frac{7}{8}$. Notice that the numerators (5, 3, and 7) of the zeros are factors of the constant term, -105. Also notice that the denominators (2, 4, and 8) are factors of the leading coefficient, 64. These observations are generalized by the *Rational Root Theorem*.

Core Concept

The Rational Root Theorem

If $f(x) = a_n x^n + \cdots + a_1 x + a_0$ has *integer* coefficients, then every rational solution of $f(x) = 0$ has the following form:

$$\frac{p}{q} = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$$

STUDY TIP

Notice that you can use the Rational Root Theorem to list possible zeros of polynomial functions.

The Rational Root Theorem can be a starting point for finding solutions of polynomial equations. However, the theorem lists only *possible* solutions. In order to find the *actual* solutions, you must test values from the list of possible solutions.

EXAMPLE 3 Using the Rational Root Theorem

Find all real solutions of $x^3 - 8x^2 + 11x + 20 = 0$.

SOLUTION

ANOTHER WAY

You can use direct substitution to test possible solutions, but synthetic division helps you identify other factors of the polynomial.

The polynomial $f(x) = x^3 - 8x^2 + 11x + 20$ is not easily factorable. Begin by using the Rational Root Theorem.

Step 1 List the possible rational solutions. The leading coefficient of $f(x)$ is 1 and the constant term is 20. So, the possible rational solutions of $f(x) = 0$ are

$$x = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{4}{1}, \pm\frac{5}{1}, \pm\frac{10}{1}, \pm\frac{20}{1}.$$

Step 2 Test possible solutions using synthetic division until a solution is found.

Test $x = 1$:

1	1	−8	11	20
		1	−7	4
	1	−7	4	24

$f(1) \neq 0$, so $x - 1$ is not a factor of $f(x)$.

Test $x = -1$:

−1	1	−8	11	20
		−1	9	−20
	1	−9	20	0

$f(-1) = 0$, so $x + 1$ is a factor of $f(x)$.

Step 3 Factor completely using the result of the synthetic division.

$(x + 1)(x^2 - 9x + 20) = 0$ Write as a product of factors.

$(x + 1)(x - 4)(x - 5) = 0$ Factor the trinomial.

▶ So, the solutions are $x = -1$, $x = 4$, and $x = 5$.

In Example 3, the leading coefficient of the polynomial is 1. When the leading coefficient is not 1, the list of possible rational solutions or zeros can increase dramatically. In such cases, the search can be shortened by using a graph.

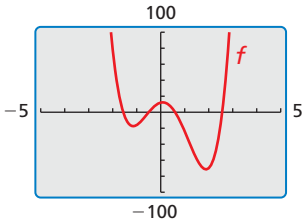
EXAMPLE 4 Finding Zeros of a Polynomial Function

Find all real zeros of $f(x) = 10x^4 - 11x^3 - 42x^2 + 7x + 12$.

SOLUTION

Step 1 List the possible rational zeros of f: $\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{6}{1}, \pm\frac{12}{1}, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{4}{5}, \pm\frac{6}{5}, \pm\frac{12}{5}, \pm\frac{1}{10}, \pm\frac{3}{10}$

Step 2 Choose reasonable values from the list above to test using the graph of the function. For f, the values $x = -\frac{3}{2}$, $x = -\frac{1}{2}$, $x = \frac{3}{5}$, and $x = \frac{12}{5}$ are reasonable based on the graph shown at the right.



Step 3 Test the values using synthetic division until a zero is found.

$$-\frac{3}{2}\left|\begin{array}{rrrrr} 10 & -11 & -42 & 7 & 12 \\ & -15 & 39 & \frac{9}{2} & -\frac{69}{4} \\ \hline 10 & -26 & -3 & \frac{23}{2} & -\frac{21}{4} \end{array}\right. \qquad -\frac{1}{2}\left|\begin{array}{rrrrr} 10 & -11 & -42 & 7 & 12 \\ & -5 & 8 & 17 & -12 \\ \hline 10 & -16 & -34 & 24 & 0 \end{array}\right.$$

$-\frac{1}{2}$ is a zero.

Step 4 Factor out a binomial using the result of the synthetic division.

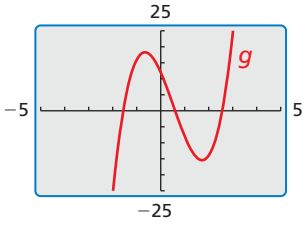
$$f(x) = \left(x + \frac{1}{2}\right)(10x^3 - 16x^2 - 34x + 24)$$ Write as a product of factors.

$$= \left(x + \frac{1}{2}\right)(2)(5x^3 - 8x^2 - 17x + 12)$$ Factor 2 out of the second factor.

$$= (2x + 1)(5x^3 - 8x^2 - 17x + 12)$$ Multiply the first factor by 2.

Step 5 Repeat the steps above for $g(x) = 5x^3 - 8x^2 - 17x + 12$. Any zero of g will also be a zero of f. The possible rational zeros of g are:

$$x = \pm1, \pm2, \pm3, \pm4, \pm6, \pm12, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{4}{5}, \pm\frac{6}{5}, \pm\frac{12}{5}$$



The graph of g shows that $\frac{3}{5}$ may be a zero. Synthetic division shows that $\frac{3}{5}$ *is* a zero and $g(x) = \left(x - \frac{3}{5}\right)(5x^2 - 5x - 20) = (5x - 3)(x^2 - x - 4)$. It follows that:

$$f(x) = (2x + 1) \cdot g(x) = (2x + 1)(5x - 3)(x^2 - x - 4)$$

Step 6 Find the remaining zeros of f by solving $x^2 - x - 4 = 0$.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2(1)}$$ Substitute 1 for a, -1 for b, and -4 for c in the Quadratic Formula.

$$x = \frac{1 \pm \sqrt{17}}{2}$$ Simplify.

▶ The real zeros of f are $-\frac{1}{2}, \frac{3}{5}, \frac{1 + \sqrt{17}}{2} \approx 2.56$, and $\frac{1 - \sqrt{17}}{2} \approx -1.56$.

Monitoring Progress

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5. Find all real solutions of $x^3 - 5x^2 - 2x + 24 = 0$.

6. Find all real zeros of $f(x) = 3x^4 - 2x^3 - 37x^2 + 24x + 12$.

The Irrational Conjugates Theorem

In Example 4, notice that the irrational zeros are *conjugates* of the form $a + \sqrt{b}$ and $a - \sqrt{b}$. This illustrates the theorem below.

Core Concept

The Irrational Conjugates Theorem

Let f be a polynomial function with rational coefficients, and let a and b be rational numbers such that $\sqrt{b}$ is irrational. If $a + \sqrt{b}$ is a zero of f, then $a - \sqrt{b}$ is also a zero of f.

EXAMPLE 5 Using Zeros to Write a Polynomial Function

Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 3 and $2 + \sqrt{5}$.

SOLUTION

Because the coefficients are rational and $2 + \sqrt{5}$ is a zero, $2 - \sqrt{5}$ must also be a zero by the Irrational Conjugates Theorem. Use the three zeros and the Factor Theorem to write $f(x)$ as a product of three factors.

$$\begin{aligned}
f(x) &= (x - 3)\left[x - (2 + \sqrt{5})\right]\left[x - (2 - \sqrt{5})\right] && \text{Write } f(x) \text{ in factored form.}\\
&= (x - 3)\left[(x - 2) - \sqrt{5}\right]\left[(x - 2) + \sqrt{5}\right] && \text{Regroup terms.}\\
&= (x - 3)\left[(x - 2)^2 - 5\right] && \text{Multiply.}\\
&= (x - 3)\left[(x^2 - 4x + 4) - 5\right] && \text{Expand binomial.}\\
&= (x - 3)(x^2 - 4x - 1) && \text{Simplify.}\\
&= x^3 - 4x^2 - x - 3x^2 + 12x + 3 && \text{Multiply.}\\
&= x^3 - 7x^2 + 11x + 3 && \text{Combine like terms.}
\end{aligned}$$

Check

You can check this result by evaluating f at each of its three zeros.

$$f(3) = 3^3 - 7(3)^2 + 11(3) + 3 = 27 - 63 + 33 + 3 = 0 \checkmark$$

$$\begin{aligned}
f(2 + \sqrt{5}) &= (2 + \sqrt{5})^3 - 7(2 + \sqrt{5})^2 + 11(2 + \sqrt{5}) + 3\\
&= 38 + 17\sqrt{5} - 63 - 28\sqrt{5} + 22 + 11\sqrt{5} + 3\\
&= 0 \checkmark
\end{aligned}$$

Because $f(2 + \sqrt{5}) = 0$, by the Irrational Conjugates Theorem $f(2 - \sqrt{5}) = 0$. ✓

Monitoring Progress

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7. Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 4 and $1 - \sqrt{5}$.

4.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. COMPLETE THE SENTENCE If a polynomial function f has integer coefficients, then every rational solution of $f(x) = 0$ has the form $\frac{p}{q}$, where p is a factor of the _____ and q is a factor of the _____.

2. DIFFERENT WORDS, SAME QUESTION Which is different? Find "both" answers.

Find the y-intercept of the graph of $y = x^3 - 2x^2 - x + 2$.

Find the x-intercepts of the graph of $y = x^3 - 2x^2 - x + 2$.

Find all the real solutions of $x^3 - 2x^2 - x + 2 = 0$.

Find the real zeros of $f(x) = x^3 - 2x^2 - x + 2$.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–12, solve the equation. *(See Example 1.)*

3. $z^3 - z^2 - 12z = 0$

4. $a^3 - 4a^2 + 4a = 0$

5. $2x^4 - 4x^3 = -2x^2$

6. $v^3 - 2v^2 - 16v = -32$

7. $5w^3 = 50w$

8. $9m^5 = 27m^3$

9. $2c^4 - 6c^3 = 12c^2 - 36c$

10. $p^4 + 40 = 14p^2$

11. $12n^2 + 48n = -n^3 - 64$

12. $y^3 - 27 = 9y^2 - 27y$

In Exercises 13–20, find the zeros of the function. Then sketch a graph of the function. *(See Example 2.)*

13. $h(x) = x^4 + x^3 - 6x^2$

14. $f(x) = x^4 - 18x^2 + 81$

15. $p(x) = x^6 - 11x^5 + 30x^4$

16. $g(x) = -2x^5 + 2x^4 + 40x^3$

17. $g(x) = -4x^4 + 8x^3 + 60x^2$

18. $h(x) = -x^3 - 2x^2 + 15x$

19. $h(x) = -x^3 - x^2 + 9x + 9$

20. $p(x) = x^3 - 5x^2 - 4x + 20$

21. USING EQUATIONS According to the Rational Root Theorem, which is *not* a possible solution of the equation $2x^4 - 5x^3 + 10x^2 - 9 = 0$?

Ⓐ -9 Ⓑ $-\frac{1}{2}$ Ⓒ $\frac{5}{2}$ Ⓓ 3

22. USING EQUATIONS According to the Rational Root Theorem, which is *not* a possible zero of the function $f(x) = 40x^5 - 42x^4 - 107x^3 + 107x^2 + 33x - 36$?

Ⓐ $-\frac{2}{3}$ Ⓑ $-\frac{3}{8}$ Ⓒ $\frac{3}{4}$ Ⓓ $\frac{4}{5}$

ERROR ANALYSIS **In Exercises 23 and 24, describe and correct the error in listing the possible rational zeros of the function.**

23.

$f(x) = x^3 + 5x^2 - 9x - 45$

Possible rational zeros of f:

$1, 3, 5, 9, 15, 45$

24.

$f(x) = 3x^3 + 13x^2 - 41x + 8$

Possible rational zeros of f:

$\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm\frac{3}{8}$

In Exercises 25–32, find all the real solutions of the equation. *(See Example 3.)*

25. $x^3 + x^2 - 17x + 15 = 0$

26. $x^3 - 2x^2 - 5x + 6 = 0$

27. $x^3 - 10x^2 + 19x + 30 = 0$

28. $x^3 + 4x^2 - 11x - 30 = 0$

29. $x^3 - 6x^2 - 7x + 60 = 0$

30. $x^3 - 16x^2 + 55x + 72 = 0$

31. $2x^3 - 3x^2 - 50x - 24 = 0$

32. $3x^3 + x^2 - 38x + 24 = 0$

In Exercises 33–38, find all the real zeros of the function. *(See Example 4.)*

33. $f(x) = x^3 - 2x^2 - 23x + 60$

34. $g(x) = x^3 - 28x - 48$

35. $h(x) = x^3 + 10x^2 + 31x + 30$

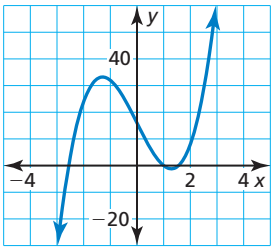
36. $f(x) = x^3 - 14x^2 + 55x - 42$

37. $p(x) = 2x^3 - x^2 - 27x + 36$

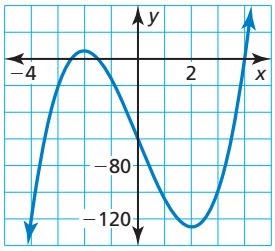
38. $g(x) = 3x^3 - 25x^2 + 58x - 40$

USING TOOLS In Exercises 39 and 40, use the graph to shorten the list of possible rational zeros of the function. Then find all real zeros of the function.

39. $f(x) = 4x^3 - 20x + 16$



40. $f(x) = 4x^3 - 49x - 60$



In Exercises 41–46, write a polynomial function f of least degree that has a leading coefficient of 1 and the given zeros. *(See Example 5.)*

41. $-2, 3, 6$

42. $-4, -2, 5$

43. $-2, 1 + \sqrt{7}$

44. $4, 6 - \sqrt{7}$

45. $-6, 0, 3 - \sqrt{5}$

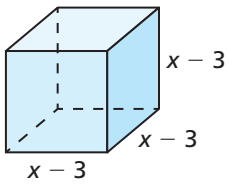
46. $0, 5, -5 + \sqrt{8}$

47. COMPARING METHODS Solve the equation $x^3 - 4x^2 - 9x + 36 = 0$ using two different methods. Which method do you prefer? Explain your reasoning.

48. REASONING Is it possible for a cubic function to have more than three real zeros? Explain.

49. PROBLEM SOLVING At a factory, molten glass is poured into molds to make paperweights. Each mold is a rectangular prism with a height 3 centimeters greater than the length of each side of its square base. Each mold holds 112 cubic centimeters of glass. What are the dimensions of the mold?

50. MATHEMATICAL CONNECTIONS The volume of the cube shown is 8 cubic centimeters.



a. Write a polynomial equation that you can use to find the value of x.

b. Identify the possible rational solutions of the equation in part (a).

c. Use synthetic division to find a rational solution of the equation. Show that no other real solutions exist.

d. What are the dimensions of the cube?

51. PROBLEM SOLVING Archaeologists discovered a huge hydraulic concrete block at the ruins of Caesarea with a volume of 945 cubic meters. The block is x meters high by $12x - 15$ meters long by $12x - 21$ meters wide. What are the dimensions of the block?

52. MAKING AN ARGUMENT Your friend claims that when a polynomial function has a leading coefficient of 1 and the coefficients are all integers, every possible rational zero is an integer. Is your friend correct? Explain your reasoning.

53. MODELING WITH MATHEMATICS During a 10-year period, the amount (in millions of dollars) of athletic equipment E sold domestically can be modeled by $E(t) = -20t^3 + 252t^2 - 280t + 21{,}614$, where t is in years.

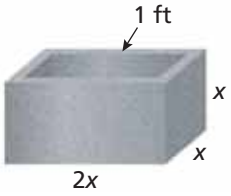
a. Write a polynomial equation to find the year when about \$24,014,000,000 of athletic equipment is sold.

b. List the possible whole-number solutions of the equation in part (a). Consider the domain when making your list of possible solutions.

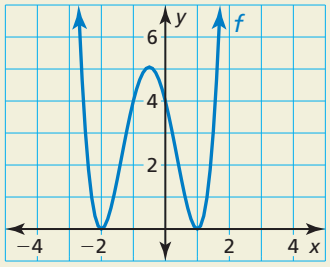
c. Use synthetic division to find when \$24,014,000,000 of athletic equipment is sold.

54. THOUGHT PROVOKING Write a third or fourth degree polynomial function that has zeros at $\pm\frac{3}{4}$. Justify your answer.

55. MODELING WITH MATHEMATICS You are designing a marble basin that will hold a fountain for a city park. The sides and bottom of the basin should be 1 foot thick. Its outer length should be twice its outer width and outer height. What should the outer dimensions of the basin be if it is to hold 36 cubic feet of water?



56. HOW DO YOU SEE IT? Use the information in the graph to answer the questions.



a. What are the real zeros of the function f?

b. Write an equation of the quartic function in factored form.

57. REASONING Determine the value of k for each equation so that the given x-value is a solution.

a. $x^3 - 6x^2 - 7x + k = 0; x = 4$

b. $2x^3 + 7x^2 - kx - 18 = 0; x = -6$

c. $kx^3 - 35x^2 + 19x + 30 = 0; x = 5$

58. WRITING EQUATIONS Write a polynomial function g of least degree that has rational coefficients, a leading coefficient of 1, and the zeros $-2 + \sqrt{7}$ and $3 + \sqrt{2}$.

In Exercises 59–62, solve $f(x) = g(x)$ by graphing and algebraic methods.

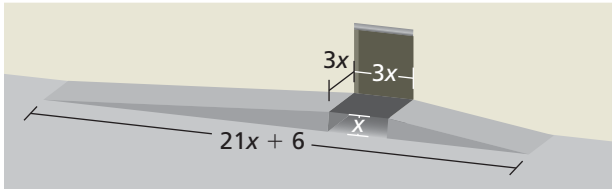
59. $f(x) = x^3 + x^2 - x - 1; g(x) = -x + 1$

60. $f(x) = x^4 - 5x^3 + 2x^2 + 8x; g(x) = -x^2 + 6x - 8$

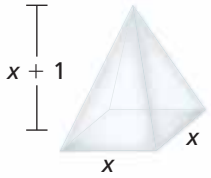
61. $f(x) = x^3 - 4x^2 + 4x; g(x) = -2x + 4$

62. $f(x) = x^4 + 2x^3 - 11x^2 - 12x + 36;$
$g(x) = -x^2 - 6x - 9$

63. MODELING WITH MATHEMATICS You are building a pair of ramps for a loading platform. The left ramp is twice as long as the right ramp. If 150 cubic feet of concrete are used to build the ramps, what are the dimensions of each ramp?



64. MODELING WITH MATHEMATICS Some ice sculptures are made by filling a mold and then freezing it. You are making an ice mold for a school dance. It is to be shaped like a pyramid with a height 1 foot greater than the length of each side of its square base. The volume of the ice sculpture is 4 cubic feet. What are the dimensions of the mold?



65. ABSTRACT REASONING Let a_n be the leading coefficient of a polynomial function f and a_0 be the constant term. If a_n has r factors and a_0 has s factors, what is the greatest number of possible rational zeros of f that can be generated by the Rational Zero Theorem? Explain your reasoning.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient. *(Section 4.1)*

66. $h(x) = -3x^2 + 2x - 9 + \sqrt{4}x^3$

67. $g(x) = 2x^3 - 7x^2 - 3x^{-1} + x$

68. $f(x) = \frac{1}{3}x^2 + 2x^3 - 4x^4 - \sqrt{3}$

69. $p(x) = 2x - 5x^3 + 9x^2 + \sqrt[4]{x} + 1$

Find the zeros of the function. *(Section 3.2)*

70. $f(x) = 7x^2 + 42$

71. $g(x) = 9x^2 + 81$

72. $h(x) = 5x^2 + 40$

73. $f(x) = 8x^2 - 1$

4.6 The Fundamental Theorem of Algebra

Essential Question How can you determine whether a polynomial equation has imaginary solutions?

EXPLORATION 1 Cubic Equations and Imaginary Solutions

Work with a partner. Match each cubic polynomial equation with the graph of its related polynomial function. Then find *all* solutions. Make a conjecture about how you can use a graph or table of values to determine the number and types of solutions of a cubic polynomial equation.

a. $x^3 - 3x^2 + x + 5 = 0$

b. $x^3 - 2x^2 - x + 2 = 0$

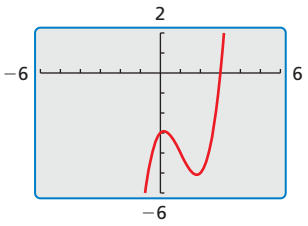
c. $x^3 - x^2 - 4x + 4 = 0$

d. $x^3 + 5x^2 + 8x + 6 = 0$

e. $x^3 - 3x^2 + x - 3 = 0$

f. $x^3 - 3x^2 + 2x = 0$

A.



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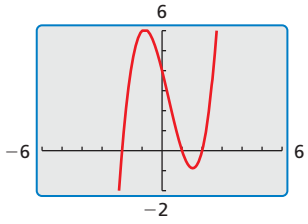

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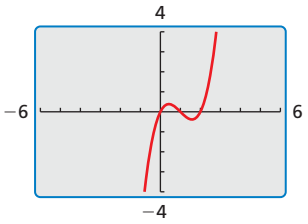

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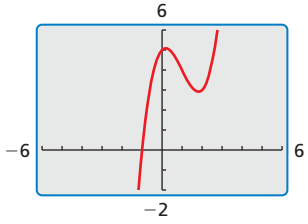

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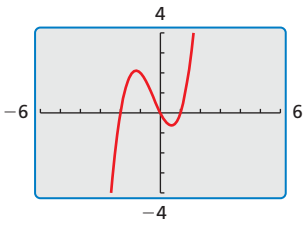

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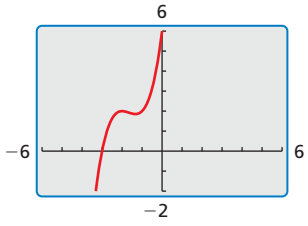


USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technology to enable you to visualize results and explore consequences.

EXPLORATION 2 Quartic Equations and Imaginary Solutions

Work with a partner. Use the graph of the related quartic function, or a table of values, to determine whether each quartic equation has imaginary solutions. Explain your reasoning. Then find *all* solutions.

a. $x^4 - 2x^3 - x^2 + 2x = 0$

b. $x^4 - 1 = 0$

c. $x^4 + x^3 - x - 1 = 0$

d. $x^4 - 3x^3 + x^2 + 3x - 2 = 0$

Communicate Your Answer

3. How can you determine whether a polynomial equation has imaginary solutions?

4. Is it possible for a cubic equation to have three imaginary solutions? Explain your reasoning.

4.6 Lesson

Core Vocabulary

complex conjugates, *p. 199*

Previous

repeated solution
degree of a polynomial
solution of an equation
zero of a function
conjugates

What You Will Learn

- Use the Fundamental Theorem of Algebra.
- Find conjugate pairs of complex zeros of polynomial functions.
- Use Descartes's Rule of Signs.

The Fundamental Theorem of Algebra

The table shows several polynomial equations and their solutions, including repeated solutions. Notice that for the last equation, the repeated solution $x = -1$ is counted twice.

Equation	Degree	Solution(s)	Number of solutions
$2x - 1 = 0$	1	$\frac{1}{2}$	1
$x^2 - 2 = 0$	2	$\pm\sqrt{2}$	2
$x^3 - 8 = 0$	3	$2, -1 \pm i\sqrt{3}$	3
$x^3 + x^2 - x - 1 = 0$	3	$-1, -1, 1$	3

In the table, note the relationship between the degree of the polynomial $f(x)$ and the number of solutions of $f(x) = 0$. This relationship is generalized by the *Fundamental Theorem of Algebra*, first proven by German mathematician Carl Friedrich Gauss (1777–1855).

Core Concept

The Fundamental Theorem of Algebra

Theorem If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has at least one solution in the set of complex numbers.

Corollary If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has exactly n solutions provided each solution repeated twice is counted as two solutions, each solution repeated three times is counted as three solutions, and so on.

STUDY TIP

The statements "the polynomial equation $f(x) = 0$ has exactly n solutions" and "the polynomial function f has exactly n zeros" are equivalent.

The corollary to the Fundamental Theorem of Algebra also means that an nth-degree polynomial function f has exactly n zeros.

EXAMPLE 1 Finding the Number of Solutions or Zeros

a. How many solutions does the equation $x^3 + 3x^2 + 16x + 48 = 0$ have?

b. How many zeros does the function $f(x) = x^4 + 6x^3 + 12x^2 + 8x$ have?

SOLUTION

a. Because $x^3 + 3x^2 + 16x + 48 = 0$ is a polynomial equation of degree 3, it has three solutions. (The solutions are -3, $4i$, and $-4i$.)

b. Because $f(x) = x^4 + 6x^3 + 12x^2 + 8x$ is a polynomial function of degree 4, it has four zeros. (The zeros are -2, -2, -2, and 0.)

EXAMPLE 2 Finding the Zeros of a Polynomial Function

Find all zeros of $f(x) = x^5 + x^3 - 2x^2 - 12x - 8$.

SOLUTION

Step 1 Find the rational zeros of f. Because f is a polynomial function of degree 5, it has five zeros. The possible rational zeros are ± 1, ± 2, ± 4, and ± 8. Using synthetic division, you can determine that -1 is a zero repeated twice and 2 is also a zero.

Step 2 Write $f(x)$ in factored form. Dividing $f(x)$ by its known factors $x + 1$, $x + 1$, and $x - 2$ gives a quotient of $x^2 + 4$. So,

$$f(x) = (x + 1)^2(x - 2)(x^2 + 4).$$

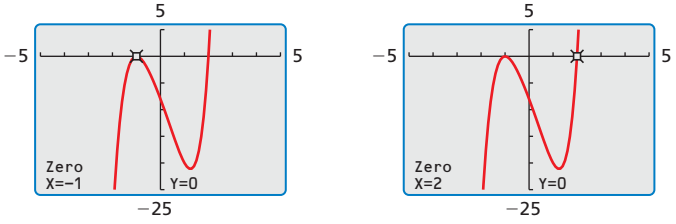
Step 3 Find the complex zeros of f. Solving $x^2 + 4 = 0$, you get $x = \pm 2i$. This means $x^2 + 4 = (x + 2i)(x - 2i)$.

$$f(x) = (x + 1)^2(x - 2)(x + 2i)(x - 2i)$$

From the factorization, there are five zeros. The zeros of f are

$$-1, -1, 2, -2i, \text{ and } 2i.$$

The graph of f and the real zeros are shown. Notice that only the *real* zeros appear as x-intercepts. Also, the graph of f touches the x-axis at the repeated zero $x = -1$ and crosses the x-axis at $x = 2$.



STUDY TIP

Notice that you can use imaginary numbers to write $(x^2 + 4)$ as $(x + 2i)(x - 2i)$. In general, $(a^2 + b^2) = (a + bi)(a - bi)$.

Monitoring Progress

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1. How many solutions does the equation $x^4 + 7x^2 - 144 = 0$ have?
2. How many zeros does the function $f(x) = x^3 - 5x^2 - 8x + 48$ have?

Find all zeros of the polynomial function.

3. $f(x) = x^3 + 7x^2 + 16x + 12$
4. $f(x) = x^5 - 3x^4 + 5x^3 - x^2 - 6x + 4$

Complex Conjugates

Pairs of complex numbers of the forms $a + bi$ and $a - bi$, where $b \neq 0$, are called **complex conjugates**. In Example 2, notice that the zeros $2i$ and $-2i$ are complex conjugates. This illustrates the next theorem.

Core Concept

The Complex Conjugates Theorem

If f is a polynomial function with real coefficients, and $a + bi$ is an imaginary zero of f, then $a - bi$ is also a zero of f.

EXAMPLE 3 Using Zeros to Write a Polynomial Function

Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 2 and $3 + i$.

SOLUTION

Because the coefficients are rational and $3 + i$ is a zero, $3 - i$ must also be a zero by the Complex Conjugates Theorem. Use the three zeros and the Factor Theorem to write $f(x)$ as a product of three factors.

$$f(x) = (x - 2)[x - (3 + i)][x - (3 - i)]$$ Write $f(x)$ in factored form.

$$= (x - 2)[(x - 3) - i][(x - 3) + i]$$ Regroup terms.

$$= (x - 2)[(x - 3)^2 - i^2]$$ Multiply.

$$= (x - 2)[(x^2 - 6x + 9) - (-1)]$$ Expand binomial and use $i^2 = -1$.

$$= (x - 2)(x^2 - 6x + 10)$$ Simplify.

$$= x^3 - 6x^2 + 10x - 2x^2 + 12x - 20$$ Multiply.

$$= x^3 - 8x^2 + 22x - 20$$ Combine like terms.

Check

You can check this result by evaluating f at each of its three zeros.

$f(2) = (2)^3 - 8(2)^2 + 22(2) - 20 = 8 - 32 + 44 - 20 = 0$ ✓

$f(3 + i) = (3 + i)^3 - 8(3 + i)^2 + 22(3 + i) - 20$

$= 18 + 26i - 64 - 48i + 66 + 22i - 20$

$= 0$ ✓

Because $f(3 + i) = 0$, by the Complex Conjugates Theorem $f(3 - i) = 0$. ✓

Monitoring Progress

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Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

5. $-1, 4i$ **6.** $3, 1 + i\sqrt{5}$ **7.** $\sqrt{2}, 1 - 3i$ **8.** $2, 2i, 4 - \sqrt{6}$

Descartes's Rule of Signs

French mathematician René Descartes (1596–1650) found the following relationship between the coefficients of a polynomial function and the number of positive and negative zeros of the function.

Core Concept

Descartes's Rule of Signs

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial function with real coefficients.

- The number of *positive real zeros* of f is equal to the number of changes in sign of the coefficients of $f(x)$ or is less than this by an even number.
- The number of *negative real zeros* of f is equal to the number of changes in sign of the coefficients of $f(-x)$ or is less than this by an even number.

EXAMPLE 4 Using Descartes's Rule of Signs

Determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for $f(x) = x^6 - 2x^5 + 3x^4 - 10x^3 - 6x^2 - 8x - 8$.

SOLUTION

$$f(x) = x^6 - 2x^5 + 3x^4 - 10x^3 - 6x^2 - 8x - 8.$$

The coefficients in $f(x)$ have 3 sign changes, so f has 3 or 1 positive real zero(s).

$$f(-x) = (-x)^6 - 2(-x)^5 + 3(-x)^4 - 10(-x)^3 - 6(-x)^2 - 8(-x) - 8$$
$$= x^6 + 2x^5 + 3x^4 + 10x^3 - 6x^2 + 8x - 8$$

The coefficients in $f(-x)$ have 3 sign changes, so f has 3 or 1 negative zero(s).

▶ The possible numbers of zeros for f are summarized in the table below.

Positive real zeros	Negative real zeros	Imaginary zeros	Total zeros
3	3	0	6
3	1	2	6
1	3	2	6
1	1	4	6

EXAMPLE 5 Real-Life Application

A tachometer measures the speed (in revolutions per minute, or RPMs) at which an engine shaft rotates. For a certain boat, the speed x (in hundreds of RPMs) of the engine shaft and the speed s (in miles per hour) of the boat are modeled by

$$s(x) = 0.00547x^3 - 0.225x^2 + 3.62x - 11.0.$$

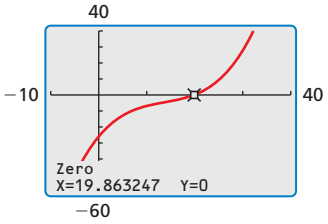
What is the tachometer reading when the boat travels 15 miles per hour?

SOLUTION

Substitute 15 for $s(x)$ in the function. You can rewrite the resulting equation as

$$0 = 0.00547x^3 - 0.225x^2 + 3.62x - 26.0.$$

The related function to this equation is $f(x) = 0.00547x^3 - 0.225x^2 + 3.62x - 26.0$. By Descartes's Rule of Signs, you know f has 3 or 1 positive real zero(s). In the context of speed, negative real zeros and imaginary zeros do not make sense, so you do not need to check for them. To approximate the positive real zeros of f, use a graphing calculator. From the graph, there is 1 real zero, $x \approx 19.9$.



▶ The tachometer reading is about 1990 RPMs.

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Determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for the function.

9. $f(x) = x^3 + 9x - 25$

10. $f(x) = 3x^4 - 7x^3 + x^2 - 13x + 8$

11. WHAT IF? In Example 5, what is the tachometer reading when the boat travels 20 miles per hour?

4.6 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The expressions $5 + i$ and $5 - i$ are _____.

2. **WRITING** How many solutions does the polynomial equation $(x + 8)^3(x - 1) = 0$ have? Explain.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, identify the number of solutions or zeros. *(See Example 1.)*

3. $x^4 + 2x^3 - 4x^2 + x = 0$

4. $5y^3 - 3y^2 + 8y = 0$

5. $9t^6 - 14t^3 + 4t - 1 = 0$

6. $f(z) = -7z^4 + z^2 - 25$

7. $g(s) = 4s^5 - s^3 + 2s^7 - 2$

8. $h(x) = 5x^4 + 7x^8 - x^{12}$

In Exercises 9–16, find all zeros of the polynomial function. *(See Example 2.)*

9. $f(x) = x^4 - 6x^3 + 7x^2 + 6x - 8$

10. $f(x) = x^4 + 5x^3 - 7x^2 - 29x + 30$

11. $g(x) = x^4 - 9x^2 - 4x + 12$

12. $h(x) = x^3 + 5x^2 - 4x - 20$

13. $g(x) = x^4 + 4x^3 + 7x^2 + 16x + 12$

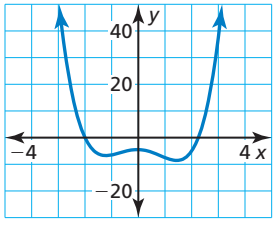
14. $h(x) = x^4 - x^3 + 7x^2 - 9x - 18$

15. $g(x) = x^5 + 3x^4 - 4x^3 - 2x^2 - 12x - 16$

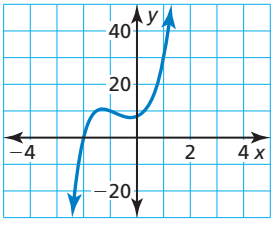
16. $f(x) = x^5 - 20x^3 + 20x^2 - 21x + 20$

ANALYZING RELATIONSHIPS In Exercises 17–20, determine the number of imaginary zeros for the function with the given degree and graph. Explain your reasoning.

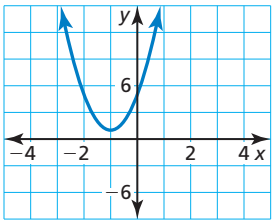
17. Degree: 4



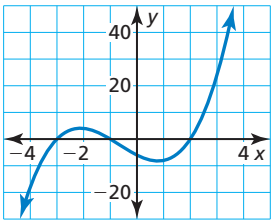
18. Degree: 5



19. Degree: 2



20. Degree: 3



In Exercises 21–28, write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros. *(See Example 3.)*

21. $-5, -1, 2$

22. $-2, 1, 3$

23. $3, 4 + i$

24. $2, 5 - i$

25. $4, -\sqrt{5}$

26. $3i, 2 - i$

27. $2, 1 + i, 2 - \sqrt{3}$

28. $3, 4 + 2i, 1 + \sqrt{7}$

ERROR ANALYSIS In Exercises 29 and 30, describe and correct the error in writing a polynomial function with rational coefficients and the given zero(s).

29. Zeros: $2, 1 + i$

✗
$$\begin{aligned} f(x) &= (x - 2)[x - (1 + i)] \\ &= x(x - 1 - i) - 2(x - 1 - i) \\ &= x^2 - x - ix - 2x + 2 + 2i \\ &= x^2 - (3 + i)x + (2 + 2i) \end{aligned}$$

30. Zero: $2 + i$

✗
$$\begin{aligned} f(x) &= [x - (2 + i)][x + (2 + i)] \\ &= (x - 2 - i)(x + 2 + i) \\ &= x^2 + 2x + ix - 2x - 4 - 2i - ix - 2i - i^2 \\ &= x^2 - 4i - 3 \end{aligned}$$

31. OPEN-ENDED Write a polynomial function of degree 6 with zeros 1, 2, and $-i$. Justify your answer.

32. REASONING Two zeros of $f(x) = x^3 - 6x^2 - 16x + 96$ are 4 and -4. Explain why the third zero must also be a real number.

In Exercises 33–40, determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for the function. *(See Example 4.)*

33. $g(x) = x^4 - x^2 - 6$

34. $g(x) = -x^3 + 5x^2 + 12$

35. $g(x) = x^3 - 4x^2 + 8x + 7$

36. $g(x) = x^5 - 2x^3 - x^2 + 6$

37. $g(x) = x^5 - 3x^3 + 8x - 10$

38. $g(x) = x^5 + 7x^4 - 4x^3 - 3x^2 + 9x - 15$

39. $g(x) = x^6 + x^5 - 3x^4 + x^3 + 5x^2 + 9x - 18$

40. $g(x) = x^7 + 4x^4 - 10x + 25$

41. REASONING Which is *not* a possible classification of zeros for $f(x) = x^5 - 4x^3 + 6x^2 + 2x - 6$? Explain.

Ⓐ three positive real zeros, two negative real zeros, and zero imaginary zeros

Ⓑ three positive real zeros, zero negative real zeros, and two imaginary zeros

Ⓒ one positive real zero, four negative real zeros, and zero imaginary zeros

Ⓓ one positive real zero, two negative real zeros, and two imaginary zeros

42. USING STRUCTURE Use Descartes's Rule of Signs to determine which function has at least 1 positive real zero.

Ⓐ $f(x) = x^4 + 2x^3 - 9x^2 - 2x - 8$

Ⓑ $f(x) = x^4 + 4x^3 + 8x^2 + 16x + 16$

Ⓒ $f(x) = -x^4 - 5x^2 - 4$

Ⓓ $f(x) = x^4 + 4x^3 + 7x^2 + 12x + 12$

43. MODELING WITH MATHEMATICS From 1890 to 2000, the American Indian, Eskimo, and Aleut population P (in thousands) can be modeled by the function $P = 0.004t^3 - 0.24t^2 + 4.9t + 243$, where t is the number of years since 1890. In which year did the population first reach 722,000? *(See Example 5.)*

44. MODELING WITH MATHEMATICS Over a period of 14 years, the number N of inland lakes infested with zebra mussels in a certain state can be modeled by

$$N = -0.0284t^4 + 0.5937t^3 - 2.464t^2 + 8.33t - 2.5$$

where t is time (in years). In which year did the number of infested inland lakes first reach 120?

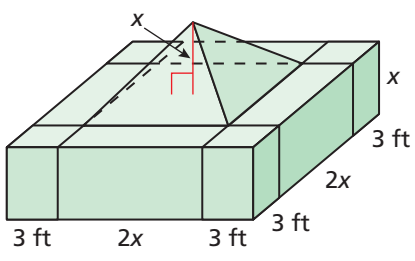
45. MODELING WITH MATHEMATICS For the 12 years that a grocery store has been open, its annual revenue R (in millions of dollars) can be modeled by the function

$$R = 0.0001(-t^4 + 12t^3 - 77t^2 + 600t + 13{,}650)$$

where t is the number of years since the store opened. In which year(s) was the revenue \$1.5 million?

46. MAKING AN ARGUMENT Your friend claims that $2 - i$ is a complex zero of the polynomial function $f(x) = x^3 - 2x^2 + 2x + 5i$, but that its conjugate is *not* a zero. You claim that both $2 - i$ and its conjugate *must* be zeros by the Complex Conjugates Theorem. Who is correct? Justify your answer.

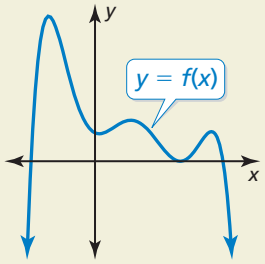
47. MATHEMATICAL CONNECTIONS A solid monument with the dimensions shown is to be built using 1000 cubic feet of marble. What is the value of x?



48. THOUGHT PROVOKING Write and graph a polynomial function of degree 5 that has all positive or negative real zeros. Label each x-intercept. Then write the function in standard form.

49. WRITING The graph of the constant polynomial function $f(x) = 2$ is a line that does not have any x-intercepts. Does the function contradict the Fundamental Theorem of Algebra? Explain.

50. HOW DO YOU SEE IT? The graph represents a polynomial function of degree 6.



a. How many positive real zeros does the function have? negative real zeros? imaginary zeros?

b. Use Descartes's Rule of Signs and your answers in part (a) to describe the possible sign changes in the coefficients of $f(x)$.

51. FINDING A PATTERN Use a graphing calculator to graph the function $f(x) = (x + 3)^n$ for $n = 2, 3, 4, 5, 6$, and 7.

a. Compare the graphs when n is even and n is odd.

b. Describe the behavior of the graph near the zero $x = -3$ as n increases.

c. Use your results from parts (a) and (b) to describe the behavior of the graph of $g(x) = (x - 4)^{20}$ near $x = 4$.

52. DRAWING CONCLUSIONS Find the zeros of each function.

$$f(x) = x^2 - 5x + 6$$
$$g(x) = x^3 - 7x + 6$$
$$h(x) = x^4 + 2x^3 + x^2 + 8x - 12$$
$$k(x) = x^5 - 3x^4 - 9x^3 + 25x^2 - 6x$$

a. Describe the relationship between the sum of the zeros of a polynomial function and the coefficients of the polynomial function.

b. Describe the relationship between the product of the zeros of a polynomial function and the coefficients of the polynomial function.

53. PROBLEM SOLVING You want to save money so you can buy a used car in four years. At the end of each summer, you deposit \$1000 earned from summer jobs into your bank account. The table shows the value of your deposits over the four-year period. In the table, g is the growth factor $1 + r$, where r is the annual interest rate expressed as a decimal.

Deposit	Year 1	Year 2	Year 3	Year 4
1st Deposit	1000	$1000g$	$1000g^2$	$1000g^3$
2nd Deposit	–	1000		
3rd Deposit	–	–	1000	
4th Deposit	–	–	–	1000

a. Copy and complete the table.

b. Write a polynomial function that gives the value v of your account at the end of the fourth summer in terms of g.

c. You want to buy a car that costs about \$4300. What growth factor do you need to obtain this amount? What annual interest rate do you need?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Describe the transformation of $f(x) = x^2$ represented by g. Then graph each function. *(Section 2.1)*

54. $g(x) = -3x^2$

55. $g(x) = (x - 4)^2 + 6$

56. $g(x) = -(x - 1)^2$

57. $g(x) = 5(x + 4)^2$

Write a function g whose graph represents the indicated transformation of the graph of f. *(Sections 1.2 and 2.1)*

58. $f(x) = x$; vertical shrink by a factor of $\frac{1}{3}$ and a reflection in the y-axis

59. $f(x) = |x + 1| - 3$; horizontal stretch by a factor of 9

60. $f(x) = x^2$; reflection in the x-axis, followed by a translation 2 units right and 7 units up

4.7 Transformations of Polynomial Functions

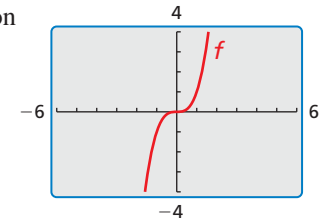
Essential Question How can you transform the graph of a polynomial function?

EXPLORATION 1 Transforming the Graph of a Cubic Function

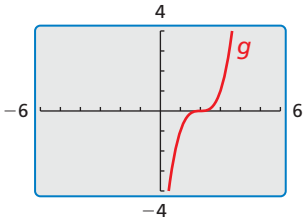
Work with a partner. The graph of the cubic function

$$f(x) = x^3$$

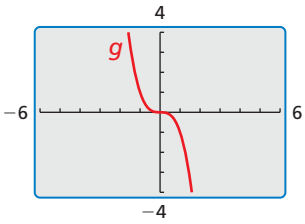
is shown. The graph of each cubic function g represents a transformation of the graph of f. Write a rule for g. Use a graphing calculator to verify your answers.



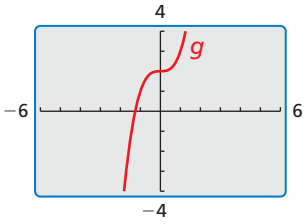
a.



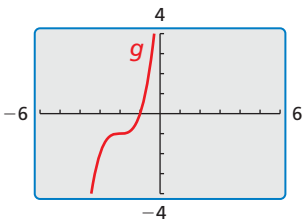
b.



c.



d.

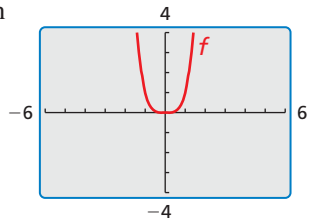


EXPLORATION 2 Transforming the Graph of a Quartic Function

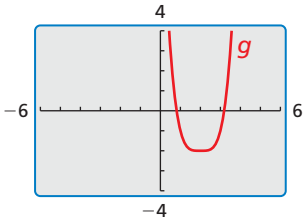
Work with a partner. The graph of the quartic function

$$f(x) = x^4$$

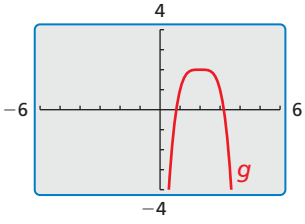
is shown. The graph of each quartic function g represents a transformation of the graph of f. Write a rule for g. Use a graphing calculator to verify your answers.



a.



b.



LOOKING FOR STRUCTURE

To be proficient in math, you need to see complicated things, such as some algebraic expressions, as being single objects or as being composed of several objects.

Communicate Your Answer

3. How can you transform the graph of a polynomial function?
4. Describe the transformation of $f(x) = x^4$ represented by $g(x) = (x + 1)^4 + 3$. Then graph g.

4.7 Lesson

Core Vocabulary

Previous
polynomial function
transformations

What You Will Learn

- Describe transformations of polynomial functions.
- Write transformations of polynomial functions.

Describing Transformations of Polynomial Functions

You can transform graphs of polynomial functions in the same way you transformed graphs of linear functions, absolute value functions, and quadratic functions. Examples of transformations of the graph of $f(x) = x^4$ are shown below.

Core Concept

Transformation	$f(x)$ Notation	Examples	
Horizontal Translation Graph shifts left or right.	$f(x - h)$	$g(x) = (x - 5)^4$ $g(x) = (x + 2)^4$	5 units right 2 units left
Vertical Translation Graph shifts up or down.	$f(x) + k$	$g(x) = x^4 + 1$ $g(x) = x^4 - 4$	1 unit up 4 units down
Reflection Graph flips over x- or y-axis.	$f(-x)$ $-f(x)$	$g(x) = (-x)^4 = x^4$ $g(x) = -x^4$	over y-axis over x-axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y-axis.	$f(ax)$	$g(x) = (2x)^4$ $g(x) = \left(\frac{1}{2}x\right)^4$	shrink by a factor of $\frac{1}{2}$ stretch by a factor of 2
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis.	$a \cdot f(x)$	$g(x) = 8x^4$ $g(x) = \frac{1}{4}x^4$	stretch by a factor of 8 shrink by a factor of $\frac{1}{4}$

EXAMPLE 1 Translating a Polynomial Function

Describe the transformation of $f(x) = x^3$ represented by $g(x) = (x + 5)^3 + 2$. Then graph each function.

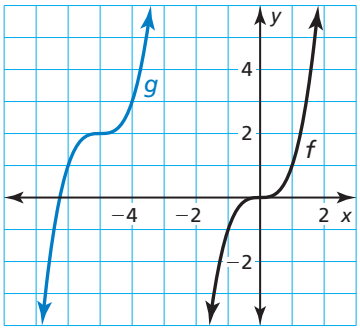
SOLUTION

Notice that the function is of the form $g(x) = (x - h)^3 + k$. Rewrite the function to identify h and k.

$$g(x) = (x - (-5))^3 + 2$$

(h = -5, k = 2)

- Because $h = -5$ and $k = 2$, the graph of g is a translation 5 units left and 2 units up of the graph of f.



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1. Describe the transformation of $f(x) = x^4$ represented by $g(x) = (x - 3)^4 - 1$. Then graph each function.

EXAMPLE 2 Transforming Polynomial Functions

Describe the transformation of f represented by g. Then graph each function.

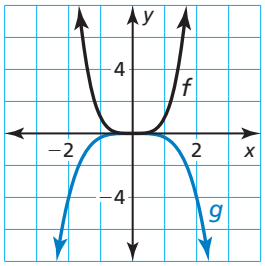
a. $f(x) = x^4$, $g(x) = -\frac{1}{4}x^4$

b. $f(x) = x^5$, $g(x) = (2x)^5 - 3$

SOLUTION

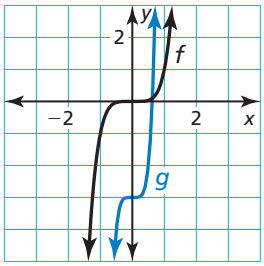
a. Notice that the function is of the form $g(x) = -ax^4$, where $a = \frac{1}{4}$.

▶ So, the graph of g is a reflection in the x-axis and a vertical shrink by a factor of $\frac{1}{4}$ of the graph of f.



b. Notice that the function is of the form $g(x) = (ax)^5 + k$, where $a = 2$ and $k = -3$.

▶ So, the graph of g is a horizontal shrink by a factor of $\frac{1}{2}$ and a translation 3 units down of the graph of f.



Monitoring Progress

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2. Describe the transformation of $f(x) = x^3$ represented by $g(x) = 4(x + 2)^3$. Then graph each function.

Writing Transformations of Polynomial Functions

EXAMPLE 3 Writing Transformed Polynomial Functions

Let $f(x) = x^3 + x^2 + 1$. Write a rule for g and then graph each function. Describe the graph of g as a transformation of the graph of f.

a. $g(x) = f(-x)$

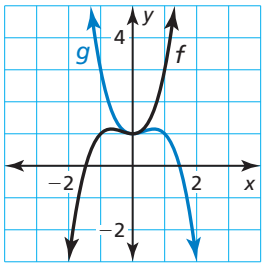
b. $g(x) = 3f(x)$

SOLUTION

a. $g(x) = f(-x)$

$= (-x)^3 + (-x)^2 + 1$

$= -x^3 + x^2 + 1$

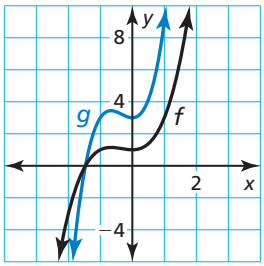


▶ The graph of g is a reflection in the y-axis of the graph of f.

b. $g(x) = 3f(x)$

$= 3(x^3 + x^2 + 1)$

$= 3x^3 + 3x^2 + 3$



▶ The graph of g is a vertical stretch by a factor of 3 of the graph of f.

REMEMBER

Vertical stretches and shrinks do not change the x-intercept(s) of a graph. You can observe this using the graph in Example 3(b).

EXAMPLE 4 Writing a Transformed Polynomial Function

Let the graph of g be a vertical stretch by a factor of 2, followed by a translation 3 units up of the graph of $f(x) = x^4 - 2x^2$. Write a rule for g.

SOLUTION

Step 1 First write a function h that represents the vertical stretch of f.

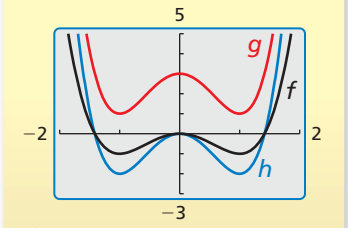
$$\begin{aligned} h(x) &= 2 \cdot f(x) && \text{Multiply the output by 2.} \\ &= 2(x^4 - 2x^2) && \text{Substitute } x^4 - 2x^2 \text{ for } f(x). \\ &= 2x^4 - 4x^2 && \text{Distributive Property} \end{aligned}$$

Step 2 Then write a function g that represents the translation of h.

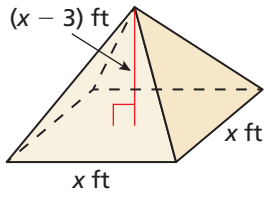
$$\begin{aligned} g(x) &= h(x) + 3 && \text{Add 3 to the output.} \\ &= 2x^4 - 4x^2 + 3 && \text{Substitute } 2x^4 - 4x^2 \text{ for } h(x). \end{aligned}$$

▶ The transformed function is $g(x) = 2x^4 - 4x^2 + 3$.

Check



EXAMPLE 5 Modeling with Mathematics



The function $V(x) = \frac{1}{3}x^3 - x^2$ represents the volume (in cubic feet) of the square pyramid shown. The function $W(x) = V(3x)$ represents the volume (in cubic feet) when x is measured in yards. Write a rule for W. Find and interpret $W(10)$.

SOLUTION

1. **Understand the Problem** You are given a function V whose inputs are in feet and whose outputs are in cubic feet. You are given another function W whose inputs are in yards and whose outputs are in cubic feet. The horizontal shrink shown by $W(x) = V(3x)$ makes sense because there are 3 feet in 1 yard. You are asked to write a rule for W and interpret the output for a given input.
2. **Make a Plan** Write the transformed function $W(x)$ and then find $W(10)$.
3. **Solve the Problem**

$$\begin{aligned} W(x) &= V(3x) \\ &= \tfrac{1}{3}(3x)^3 - (3x)^2 && \text{Replace } x \text{ with } 3x \text{ in } V(x). \\ &= 9x^3 - 9x^2 && \text{Simplify.} \end{aligned}$$

Next, find $W(10)$.

$$W(10) = 9(10)^3 - 9(10)^2 = 9000 - 900 = 8100$$

▶ When x is 10 yards, the volume of the pyramid is 8100 cubic feet.

4. **Look Back** Because $W(10) = V(30)$, you can check that your solution is correct by verifying that $V(30) = 8100$.

$$V(30) = \tfrac{1}{3}(30)^3 - (30)^2 = 9000 - 900 = 8100 \checkmark$$

Monitoring Progress

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3. Let $f(x) = x^5 - 4x + 6$ and $g(x) = -f(x)$. Write a rule for g and then graph each function. Describe the graph of g as a transformation of the graph of f.
4. Let the graph of g be a horizontal stretch by a factor of 2, followed by a translation 3 units to the right of the graph of $f(x) = 8x^3 + 3$. Write a rule for g.
5. **WHAT IF?** In Example 5, the height of the pyramid is $6x$, and the volume (in cubic feet) is represented by $V(x) = 2x^3$. Write a rule for W. Find and interpret $W(7)$.

4.7 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The graph of $f(x) = (x + 2)^3$ is a _____ translation of the graph of $f(x) = x^3$.

2. **VOCABULARY** Describe how the vertex form of quadratic functions is similar to the form $f(x) = a(x - h)^3 + k$ for cubic functions.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, describe the transformation of f represented by g. Then graph each function. *(See Example 1.)*

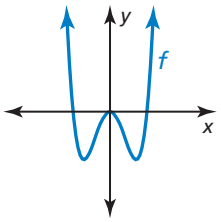
3. $f(x) = x^4$, $g(x) = x^4 + 3$

4. $f(x) = x^4$, $g(x) = (x - 5)^4$

5. $f(x) = x^5$, $g(x) = (x - 2)^5 - 1$

6. $f(x) = x^6$, $g(x) = (x + 1)^6 - 4$

ANALYZING RELATIONSHIPS In Exercises 7–10, match the function with the correct transformation of the graph of f. Explain your reasoning.

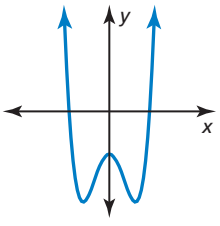


7. $y = f(x - 2)$

8. $y = f(x + 2) + 2$

9. $y = f(x - 2) + 2$

10. $y = f(x) - 2$

A.


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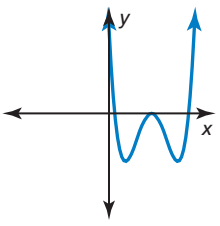

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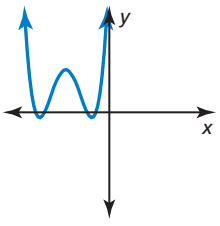

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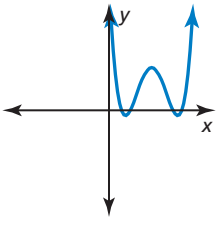


In Exercises 11–16, describe the transformation of f represented by g. Then graph each function. *(See Example 2.)*

11. $f(x) = x^4$, $g(x) = -2x^4$

12. $f(x) = x^6$, $g(x) = -3x^6$

13. $f(x) = x^3$, $g(x) = 5x^3 + 1$

14. $f(x) = x^4$, $g(x) = \frac{1}{2}x^4 + 1$

15. $f(x) = x^5$, $g(x) = \frac{3}{4}(x + 4)^5$

16. $f(x) = x^4$, $g(x) = (2x)^4 - 3$

In Exercises 17–20, write a rule for g and then graph each function. Describe the graph of g as a transformation of the graph of f. *(See Example 3.)*

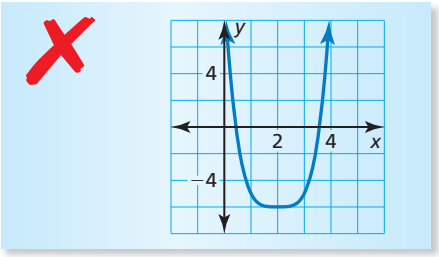
17. $f(x) = x^4 + 1$, $g(x) = f(x + 2)$

18. $f(x) = x^5 - 2x + 3$, $g(x) = 3f(x)$

19. $f(x) = 2x^3 - 2x^2 + 6$, $g(x) = -\frac{1}{2}f(x)$

20. $f(x) = x^4 + x^3 - 1$, $g(x) = f(-x) - 5$

21. **ERROR ANALYSIS** Describe and correct the error in graphing the function $g(x) = (x + 2)^4 - 6$.



22. ERROR ANALYSIS Describe and correct the error in describing the transformation of the graph of $f(x) = x^5$ represented by the graph of $g(x) = (3x)^5 - 4$.

> ✗ The graph of g is a horizontal shrink by a factor of 3, followed by a translation 4 units down of the graph of f.

In Exercises 23–26, write a rule for g that represents the indicated transformations of the graph of f. *(See Example 4.)*

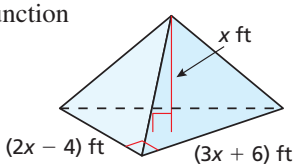
23. $f(x) = x^3 - 6$; translation 3 units left, followed by a reflection in the y-axis

24. $f(x) = x^4 + 2x + 6$; vertical stretch by a factor of 2, followed by a translation 4 units right

25. $f(x) = x^3 + 2x^2 - 9$; horizontal shrink by a factor of $\frac{1}{3}$ and a translation 2 units up, followed by a reflection in the x-axis

26. $f(x) = 2x^5 - x^3 + x^2 + 4$; reflection in the y-axis and a vertical stretch by a factor of 3, followed by a translation 1 unit down

27. MODELING WITH MATHEMATICS The volume V (in cubic feet) of the pyramid is given by $V(x) = x^3 - 4x$. The function $W(x) = V(3x)$ gives the volume (in cubic feet) of the pyramid when x is measured in yards. Write a rule for W. Find and interpret $W(5)$. *(See Example 5.)*



28. MAKING AN ARGUMENT The volume of a cube with side length x is given by $V(x) = x^3$. Your friend claims that when you divide the volume in half, the volume decreases by a greater amount than when you divide each side length in half. Is your friend correct? Justify your answer.

29. OPEN-ENDED Describe two transformations of the graph of $f(x) = x^5$ where the order in which the transformations are performed is important. Then describe two transformations where the order is *not* important. Explain your reasoning.

30. THOUGHT PROVOKING Write and graph a transformation of the graph of $f(x) = x^5 - 3x^4 + 2x - 4$ that results in a graph with a y-intercept of -2.

31. PROBLEM SOLVING A portion of the path that a hummingbird flies while feeding can be modeled by the function

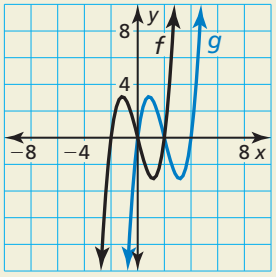
$$f(x) = -\frac{1}{5}x(x - 4)^2(x - 7), 0 \le x \le 7$$

where x is the horizontal distance (in meters) and $f(x)$ is the height (in meters). The hummingbird feeds each time it is at ground level.

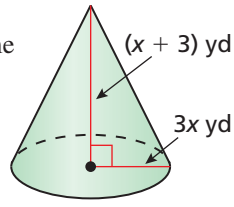
a. At what distances does the hummingbird feed?

b. A second hummingbird feeds 2 meters farther away than the first hummingbird and flies twice as high. Write a function to model the path of the second hummingbird.

32. HOW DO YOU SEE IT? Determine the real zeros of each function. Then describe the transformation of the graph of f that results in the graph of g.



33. MATHEMATICAL CONNECTIONS Write a function V for the volume (in cubic yards) of the right circular cone shown. Then write a function W that gives the volume (in cubic yards) of the cone when x is measured in feet. Find and interpret $W(3)$.



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find the minimum value or maximum value of the function. Describe the domain and range of the function, and where the function is increasing and decreasing. *(Section 2.2)*

34. $h(x) = (x + 5)^2 - 7$

35. $f(x) = 4 - x^2$

36. $f(x) = 3(x - 10)(x + 4)$

37. $g(x) = -(x + 2)(x + 8)$

38. $h(x) = \frac{1}{2}(x - 1)^2 - 3$

39. $f(x) = -2x^2 + 4x - 1$

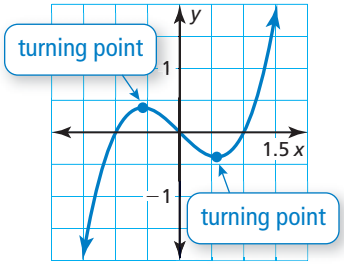
4.8 Analyzing Graphs of Polynomial Functions

Essential Question

How many turning points can the graph of a polynomial function have?

A *turning point* of the graph of a polynomial function is a point on the graph at which the function changes from

- increasing to decreasing, or
- decreasing to increasing.



EXPLORATION 1 Approximating Turning Points

Work with a partner. Match each polynomial function with its graph. Explain your reasoning. Then use a graphing calculator to approximate the coordinates of the turning points of the graph of the function. Round your answers to the nearest hundredth.

a. $f(x) = 2x^2 + 3x - 4$

b. $f(x) = x^2 + 3x + 2$

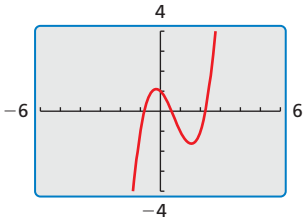
c. $f(x) = x^3 - 2x^2 - x + 1$

d. $f(x) = -x^3 + 5x - 2$

e. $f(x) = x^4 - 3x^2 + 2x - 1$

f. $f(x) = -2x^5 - x^2 + 5x + 3$

A.



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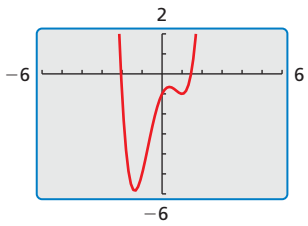

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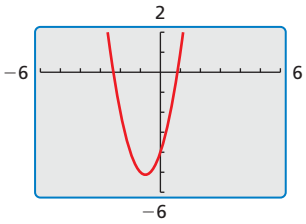

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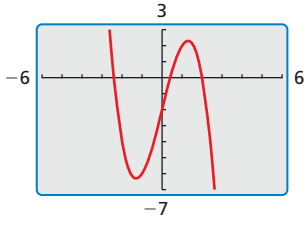

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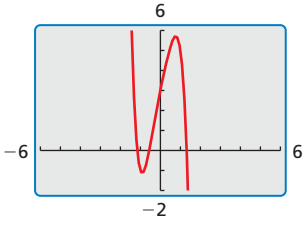
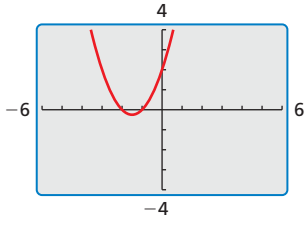


F.



ATTENDING TO PRECISION

To be proficient in math, you need to express numerical answers with a degree of precision appropriate for the problem context.

Communicate Your Answer

2. How many turning points can the graph of a polynomial function have?

3. Is it possible to sketch the graph of a cubic polynomial function that has *no* turning points? Justify your answer.

4.8 Lesson

Core Vocabulary

local maximum, *p. 214*
local minimum, *p. 214*
even function, *p. 215*
odd function, *p. 215*

Previous
end behavior
increasing
decreasing
symmetric about the *y*-axis

What You Will Learn

- Use *x*-intercepts to graph polynomial functions.
- Use the Location Principle to identify zeros of polynomial functions.
- Find turning points and identify local maximums and local minimums of graphs of polynomial functions.
- Identify even and odd functions.

Graphing Polynomial Functions

In this chapter, you have learned that zeros, factors, solutions, and *x*-intercepts are closely related concepts. Here is a summary of these relationships.

Concept Summary

Zeros, Factors, Solutions, and Intercepts

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial function. The following statements are equivalent.

Zero: k is a zero of the polynomial function f.

Factor: $x - k$ is a factor of the polynomial $f(x)$.

Solution: k is a solution (or root) of the polynomial equation $f(x) = 0$.

***x*-Intercept:** If k is a real number, then k is an x-intercept of the graph of the polynomial function f. The graph of f passes through $(k, 0)$.

EXAMPLE 1 Using *x*-Intercepts to Graph a Polynomial Function

Graph the function

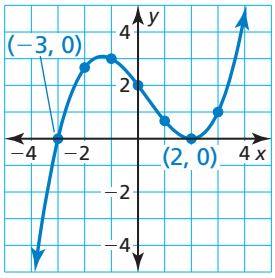
$$f(x) = \frac{1}{6}(x + 3)(x - 2)^2.$$

SOLUTION

Step 1 Plot the x-intercepts. Because -3 and 2 are zeros of f, plot $(-3, 0)$ and $(2, 0)$.

Step 2 Plot points between and beyond the x-intercepts.

x	-2	-1	0	1	3
y	$\frac{8}{3}$	3	2	$\frac{2}{3}$	1



Step 3 Determine end behavior. Because $f(x)$ has three factors of the form $x - k$ and a constant factor of $\frac{1}{6}$, f is a cubic function with a positive leading coefficient. So, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

Step 4 Draw the graph so that it passes through the plotted points and has the appropriate end behavior.

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Graph the function.

1. $f(x) = \frac{1}{2}(x + 1)(x - 4)^2$

2. $f(x) = \frac{1}{4}(x + 2)(x - 1)(x - 3)$

The Location Principle

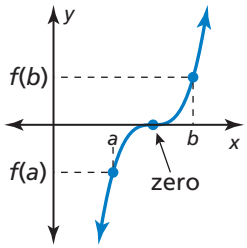
You can use the *Location Principle* to help you find real zeros of polynomial functions.

Core Concept

The Location Principle

If f is a polynomial function, and a and b are two real numbers such that $f(a) < 0$ and $f(b) > 0$, then f has at least one real zero between a and b.

To use this principle to locate real zeros of a polynomial function, find a value a at which the polynomial function is negative and another value b at which the function is positive. You can conclude that the function has *at least* one real zero between a and b.



EXAMPLE 2 Locating Real Zeros of a Polynomial Function

Find all real zeros of

$$f(x) = 6x^3 + 5x^2 - 17x - 6.$$

SOLUTION

Step 1 Use a graphing calculator to make a table.

X	Y1
0	−6
1	−12
2	28
3	150
4	390
5	784
6	1368

X=1

Step 2 Use the Location Principle. From the table shown, you can see that $f(1) < 0$ and $f(2) > 0$. So, by the Location Principle, f has a zero between 1 and 2. Because f is a polynomial function of degree 3, it has three zeros. The only possible *rational* zero between 1 and 2 is $\frac{3}{2}$. Using synthetic division, you can confirm that $\frac{3}{2}$ is a zero.

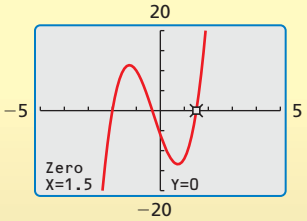
Step 3 Write $f(x)$ in factored form. Dividing $f(x)$ by its known factor $x - \frac{3}{2}$ gives a quotient of $6x^2 + 14x + 4$. So, you can factor $f(x)$ as

$$\begin{aligned} f(x) &= \left(x - \tfrac{3}{2}\right)(6x^2 + 14x + 4) \\ &= 2\left(x - \tfrac{3}{2}\right)(3x^2 + 7x + 2) \\ &= 2\left(x - \tfrac{3}{2}\right)(3x + 1)(x + 2). \end{aligned}$$

▶ From the factorization, there are three zeros. The zeros of f are $\frac{3}{2}$, $-\frac{1}{3}$, and -2.

Check this by graphing f.

Check



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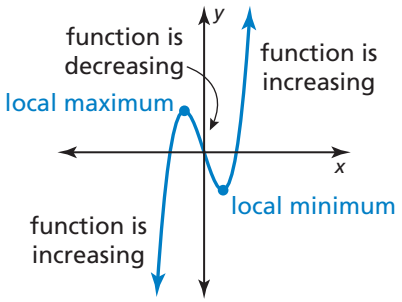
3. Find all real zeros of $f(x) = 18x^3 + 21x^2 - 13x - 6$.

Turning Points

Another important characteristic of graphs of polynomial functions is that they have *turning points* corresponding to local maximum and minimum values.

- The y-coordinate of a turning point is a **local maximum** of the function when the point is higher than all nearby points.
- The y-coordinate of a turning point is a **local minimum** of the function when the point is lower than all nearby points.

The turning points of a graph help determine the intervals for which a function is increasing or decreasing.



READING

Local maximum and local minimum are sometimes referred to as *relative maximum* and *relative minimum*.

Core Concept

Turning Points of Polynomial Functions

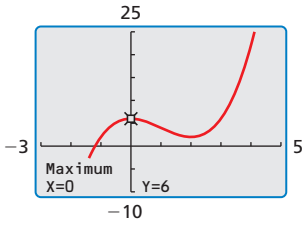
1. The graph of every polynomial function of degree n has *at most* $n - 1$ turning points.
2. If a polynomial function of degree n has n distinct real zeros, then its graph has *exactly* $n - 1$ turning points.

EXAMPLE 3 Finding Turning Points

Graph each function. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which each function is increasing or decreasing.

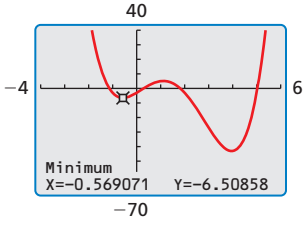
a. $f(x) = x^3 - 3x^2 + 6$ **b.** $g(x) = x^4 - 6x^3 + 3x^2 + 10x - 3$

SOLUTION



a. Use a graphing calculator to graph the function. The graph of f has one x-intercept and two turning points. Use the graphing calculator's *zero*, *maximum*, and *minimum* features to approximate the coordinates of the points.

▶ The x-intercept of the graph is $x \approx -1.20$. The function has a local maximum at (0, 6) and a local minimum at (2, 2). The function is increasing when $x < 0$ and $x > 2$ and decreasing when $0 < x < 2$.



b. Use a graphing calculator to graph the function. The graph of g has four x-intercepts and three turning points. Use the graphing calculator's *zero*, *maximum*, and *minimum* features to approximate the coordinates of the points.

▶ The x-intercepts of the graph are $x \approx -1.14$, $x \approx 0.29$, $x \approx 1.82$, and $x \approx 5.03$. The function has a local maximum at (1.11, 5.11) and local minimums at (−0.57, −6.51) and (3.96, −43.04). The function is increasing when $-0.57 < x < 1.11$ and $x > 3.96$ and decreasing when $x < -0.57$ and $1.11 < x < 3.96$.

Monitoring Progress

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4. Graph $f(x) = 0.5x^3 + x^2 - x + 2$. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing.

Even and Odd Functions

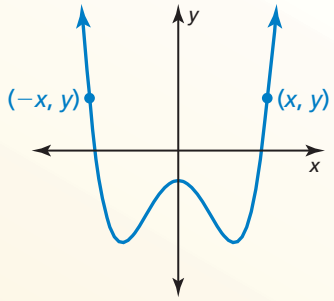
Core Concept

Even and Odd Functions

A function f is an **even function** when $f(-x) = f(x)$ for all x in its domain. The graph of an even function is *symmetric about the y-axis*.

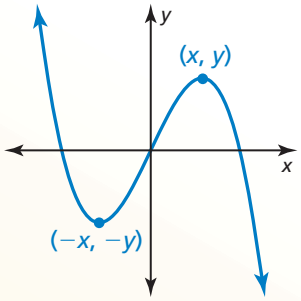
A function f is an **odd function** when $f(-x) = -f(x)$ for all x in its domain. The graph of an odd function is *symmetric about the origin*. One way to recognize a graph that is symmetric about the origin is that it looks the same after a 180° rotation about the origin.

Even Function



For an even function, if (x, y) is on the graph, then $(-x, y)$ is also on the graph.

Odd Function



For an odd function, if (x, y) is on the graph, then $(-x, -y)$ is also on the graph.

EXAMPLE 4 Identifying Even and Odd Functions

Determine whether each function is *even*, *odd*, or *neither*.

a. $f(x) = x^3 - 7x$ **b.** $g(x) = x^4 + x^2 - 1$ **c.** $h(x) = x^3 + 2$

SOLUTION

a. Replace x with $-x$ in the equation for f, and then simplify.

$$f(-x) = (-x)^3 - 7(-x) = -x^3 + 7x = -(x^3 - 7x) = -f(x)$$

▶ Because $f(-x) = -f(x)$, the function is odd.

b. Replace x with $-x$ in the equation for g, and then simplify.

$$g(-x) = (-x)^4 + (-x)^2 - 1 = x^4 + x^2 - 1 = g(x)$$

▶ Because $g(-x) = g(x)$, the function is even.

c. Replacing x with $-x$ in the equation for h produces

$$h(-x) = (-x)^3 + 2 = -x^3 + 2.$$

▶ Because $h(x) = x^3 + 2$ and $-h(x) = -x^3 - 2$, you can conclude that $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$. So, the function is neither even nor odd.

Monitoring Progress

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Determine whether the function is *even*, *odd*, or *neither*.

5. $f(x) = -x^2 + 5$ **6.** $f(x) = x^4 - 5x^3$ **7.** $f(x) = 2x^5$

4.8 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** A local maximum or local minimum of a polynomial function occurs at a _____ point of the graph of the function.

2. **WRITING** Explain what a local maximum of a function is and how it may be different from the maximum value of the function.

Monitoring Progress and Modeling with Mathematics

ANALYZING RELATIONSHIPS **In Exercises 3–6, match the function with its graph.**

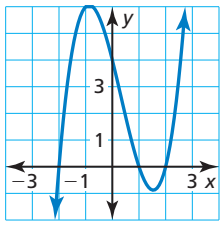
3. $f(x) = (x - 1)(x - 2)(x + 2)$

4. $h(x) = (x + 2)^2(x + 1)$

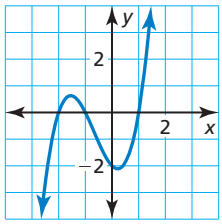
5. $g(x) = (x + 1)(x - 1)(x + 2)$

6. $f(x) = (x - 1)^2(x + 2)$

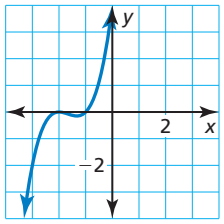
A.



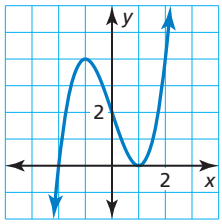
B.



C.



D.



In Exercises 7–14, graph the function. *(See Example 1.)*

7. $f(x) = (x - 2)^2(x + 1)$

8. $f(x) = (x + 2)^2(x + 4)^2$

9. $h(x) = (x + 1)^2(x - 1)(x - 3)$

10. $g(x) = 4(x + 1)(x + 2)(x - 1)$

11. $h(x) = \frac{1}{3}(x - 5)(x + 2)(x - 3)$

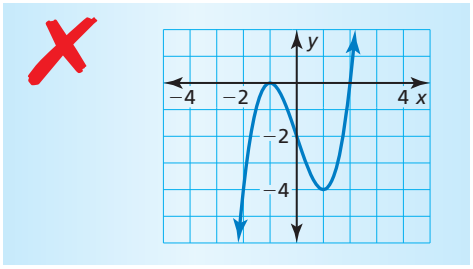
12. $g(x) = \frac{1}{12}(x + 4)(x + 8)(x - 1)$

13. $h(x) = (x - 3)(x^2 + x + 1)$

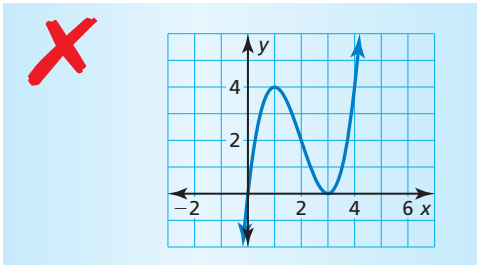
14. $f(x) = (x - 4)(2x^2 - 2x + 1)$

ERROR ANALYSIS **In Exercises 15 and 16, describe and correct the error in using factors to graph *f*.**

15. $f(x) = (x + 2)(x - 1)^2$



16. $f(x) = x^2(x - 3)^3$



In Exercises 17–22, find all real zeros of the function. *(See Example 2.)*

17. $f(x) = x^3 - 4x^2 - x + 4$

18. $f(x) = x^3 - 3x^2 - 4x + 12$

19. $h(x) = 2x^3 + 7x^2 - 5x - 4$

20. $h(x) = 4x^3 - 2x^2 - 24x - 18$

21. $g(x) = 4x^3 + x^2 - 51x + 36$

22. $f(x) = 2x^3 - 3x^2 - 32x - 15$

In Exercises 23–30, graph the function. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing. *(See Example 3.)*

23. $g(x) = 2x^3 + 8x^2 - 3$

24. $g(x) = -x^4 + 3x$

25. $h(x) = x^4 - 3x^2 + x$

26. $f(x) = x^5 - 4x^3 + x^2 + 2$

27. $f(x) = 0.5x^3 - 2x + 2.5$

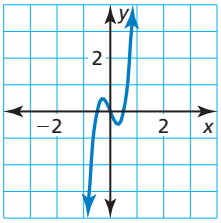
28. $f(x) = 0.7x^4 - 3x^3 + 5x$

29. $h(x) = x^5 + 2x^2 - 17x - 4$

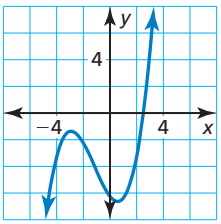
30. $g(x) = x^4 - 5x^3 + 2x^2 + x - 3$

In Exercises 31–36, estimate the coordinates of each turning point. State whether each corresponds to a local maximum or a local minimum. Then estimate the real zeros and find the least possible degree of the function.

31.



32.



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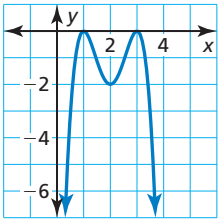

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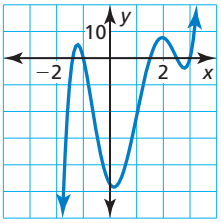

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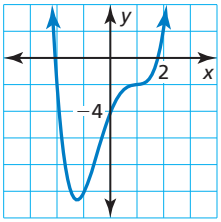

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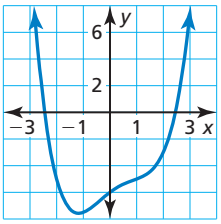


OPEN-ENDED **In Exercises 37 and 38, sketch a graph of a polynomial function f having the given characteristics.**

37.
- The graph of f has x-intercepts at $x = -4$, $x = 0$, and $x = 2$.
- f has a local maximum value when $x = 1$.
- f has a local minimum value when $x = -2$.

38.
- The graph of f has x-intercepts at $x = -3$, $x = 1$, and $x = 5$.
- f has a local maximum value when $x = 1$.
- f has a local minimum value when $x = -2$ and when $x = 4$.

In Exercises 39–46, determine whether the function is *even, odd,* or *neither*. *(See Example 4.)*

39. $h(x) = 4x^7$

40. $g(x) = -2x^6 + x^2$

41. $f(x) = x^4 + 3x^2 - 2$

42. $f(x) = x^5 + 3x^3 - x$

43. $g(x) = x^2 + 5x + 1$

44. $f(x) = -x^3 + 2x - 9$

45. $f(x) = x^4 - 12x^2$

46. $h(x) = x^5 + 3x^4$

47. **USING TOOLS** When a swimmer does the breaststroke, the function

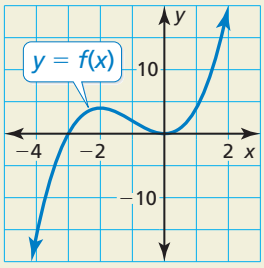
$$S = -241t^7 + 1060t^6 - 1870t^5 + 1650t^4 - 737t^3 + 144t^2 - 2.43t$$

models the speed S (in meters per second) of the swimmer during one complete stroke, where t is the number of seconds since the start of the stroke and $0 \le t \le 1.22$. Use a graphing calculator to graph the function. At what time during the stroke is the swimmer traveling the fastest?

48. **USING TOOLS** During a recent period of time, the number S (in thousands) of students enrolled in public schools in a certain country can be modeled by $S = 1.64x^3 - 102x^2 + 1710x + 36{,}300$, where x is time (in years). Use a graphing calculator to graph the function for the interval $0 \le x \le 41$. Then describe how the public school enrollment changes over this period of time.

49. **WRITING** Why is the adjective *local*, used to describe the maximums and minimums of cubic functions, sometimes not required for quadratic functions?

50. HOW DO YOU SEE IT? The graph of a polynomial function is shown.



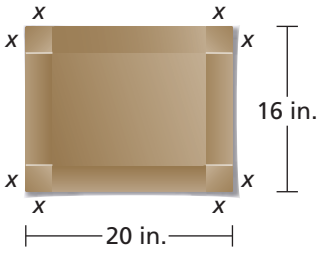
a. Find the zeros, local maximum, and local minimum values of the function.

b. Compare the x-intercepts of the graphs of $y = f(x)$ and $y = -f(x)$.

c. Compare the maximum and minimum values of the functions $y = f(x)$ and $y = -f(x)$.

51. MAKING AN ARGUMENT Your friend claims that the product of two odd functions is an odd function. Is your friend correct? Explain your reasoning.

52. MODELING WITH MATHEMATICS You are making a rectangular box out of a 16-inch-by-20-inch piece of cardboard. The box will be formed by making the cuts shown in the diagram and folding up the sides. You want the box to have the greatest volume possible.



a. How long should you make the cuts?

b. What is the maximum volume?

c. What are the dimensions of the finished box?

53. PROBLEM SOLVING Quonset huts are temporary, all-purpose structures shaped like half-cylinders. You have 1100 square feet of material to build a quonset hut.

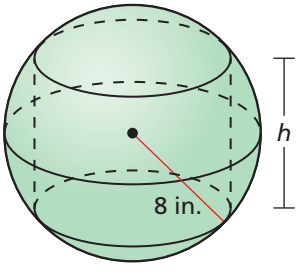
a. The surface area S of a quonset hut is given by $S = \pi r^2 + \pi r\ell$. Substitute 1100 for S and then write an expression for ℓ in terms of r.

b. The volume V of a quonset hut is given by $V = \frac{1}{2}\pi r^2\ell$. Write an equation that gives V as a function in terms of r only.

c. Find the value of r that maximizes the volume of the hut.

54. THOUGHT PROVOKING Write and graph a polynomial function that has one real zero in each of the intervals $-2 < x < -1$, $0 < x < 1$, and $4 < x < 5$. Is there a maximum degree that such a polynomial function can have? Justify your answer.

55. MATHEMATICAL CONNECTIONS A cylinder is inscribed in a sphere of radius 8 inches. Write an equation for the volume of the cylinder as a function of h. Find the value of h that maximizes the volume of the inscribed cylinder. What is the maximum volume of the cylinder?



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

State whether the table displays *linear data*, *quadratic data*, or *neither*. Explain. *(Section 2.4)*

56.

Months, x	0	1	2	3
Savings (dollars), y	100	150	200	250

57.

Time (seconds), x	0	1	2	3
Height (feet), y	300	284	236	156

4.9 Modeling with Polynomial Functions

Essential Question How can you find a polynomial model for real-life data?

EXPLORATION 1 Modeling Real-Life Data

Work with a partner. The distance a baseball travels after it is hit depends on the angle at which it was hit and the initial speed. The table shows the distances a baseball hit at an angle of 35° travels at various initial speeds.

Initial speed, x (miles per hour)	80	85	90	95	100	105	110	115
Distance, y (feet)	194	220	247	275	304	334	365	397

a. Recall that when data have equally-spaced x-values, you can analyze patterns in the differences of the y-values to determine what type of function can be used to model the data. If the first differences are constant, then the set of data fits a linear model. If the second differences are constant, then the set of data fits a quadratic model.

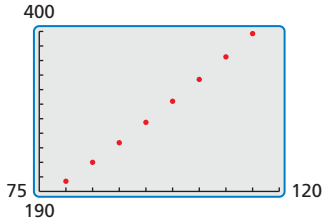
Find the first and second differences of the data. Are the data linear or quadratic? Explain your reasoning.

194 220 247 275 304 334 365 397

USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technological tools to explore and deepen your understanding of concepts.

b. Use a graphing calculator to draw a scatter plot of the data. Do the data appear linear or quadratic? Use the *regression* feature of the graphing calculator to find a linear or quadratic model that best fits the data.



c. Use the model you found in part (b) to find the distance a baseball travels when it is hit at an angle of 35° and travels at an initial speed of 120 miles per hour.

d. According to the *Baseball Almanac*, "Any drive over 400 feet is noteworthy. A blow of 450 feet shows exceptional power, as the majority of major league players are unable to hit a ball that far. Anything in the 500-foot range is genuinely historic." Estimate the initial speed of a baseball that travels a distance of 500 feet.

Communicate Your Answer

2. How can you find a polynomial model for real-life data?

3. How well does the model you found in Exploration 1(b) fit the data? Do you think the model is valid for any initial speed? Explain your reasoning.

4.9 Lesson

Core Vocabulary

finite differences, *p. 220*

Previous

scatter plot

What You Will Learn

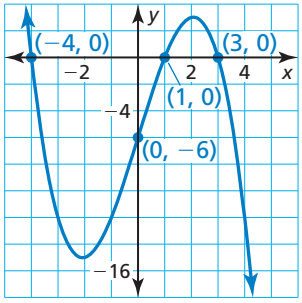
- Write polynomial functions for sets of points.
- Write polynomial functions using finite differences.
- Use technology to find models for data sets.

Writing Polynomial Functions for a Set of Points

You know that two points determine a line and three points not on a line determine a parabola. In Example 1, you will see that four points not on a line or a parabola determine the graph of a cubic function.

EXAMPLE 1 Writing a Cubic Function

Write the cubic function whose graph is shown.



SOLUTION

Step 1 Use the three x-intercepts to write the function in factored form.

$$f(x) = a(x + 4)(x - 1)(x - 3)$$

Step 2 Find the value of a by substituting the coordinates of the point $(0, -6)$.

$$-6 = a(0 + 4)(0 - 1)(0 - 3)$$

$$-6 = 12a$$

$$-\tfrac{1}{2} = a$$

▶ The function is $f(x) = -\frac{1}{2}(x + 4)(x - 1)(x - 3)$.

Check

Check the end behavior of f. The degree of f is odd and $a < 0$. So, $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$, which matches the graph. ✓

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Write a cubic function whose graph passes through the given points.

1. $(-4, 0), (0, 10), (2, 0), (5, 0)$

2. $(-1, 0), (0, -12), (2, 0), (3, 0)$

Finite Differences

When the x-values in a data set are equally spaced, the differences of consecutive y-values are called **finite differences**. Recall from Section 2.4 that the first and second differences of $y = x^2$ are:

equally-spaced x-values

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9

first differences: −5 −3 −1 1 3 5

second differences: 2 2 2 2 2

Notice that $y = x^2$ has degree *two* and that the *second* differences are constant and nonzero. This illustrates the first of the two properties of finite differences shown on the next page.

Core Concept

Properties of Finite Differences

1. If a polynomial function $y = f(x)$ has degree n, then the nth differences of function values for equally-spaced x-values are nonzero and constant.
2. Conversely, if the nth differences of equally-spaced data are nonzero and constant, then the data can be represented by a polynomial function of degree n.

The second property of finite differences allows you to write a polynomial function that models a set of equally-spaced data.

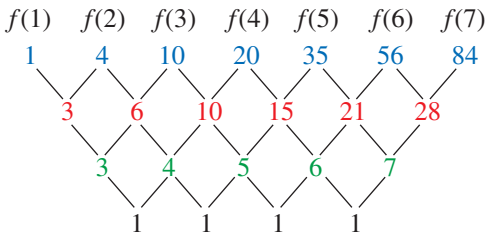
EXAMPLE 2 Writing a Function Using Finite Differences

Use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function.

x	1	2	3	4	5	6	7
$f(x)$	1	4	10	20	35	56	84

SOLUTION

Step 1 Write the function values. Find the first differences by subtracting consecutive values. Then find the second differences by subtracting consecutive first differences. Continue until you obtain differences that are nonzero and constant.



Write function values for equally-spaced x-values.

First differences

Second differences

Third differences

Because the third differences are nonzero and constant, you can model the data *exactly* with a cubic function.

Step 2 Enter the data into a graphing calculator and use cubic regression to obtain a polynomial function.

```
CubicReg
y=ax³+bx²+cx+d
a=.1666666667
b=.5
c=.3333333333
d=0
R²=1
```

Because $\frac{1}{6} \approx 0.1666666667$, $\frac{1}{2} = 0.5$, and $\frac{1}{3} \approx 0.333333333$, a polynomial function that fits the data exactly is

$$f(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x.$$

Monitoring Progress

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3. Use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function.

x	-3	-2	-1	0	1	2
$f(x)$	6	15	22	21	6	-29

Finding Models Using Technology

In Examples 1 and 2, you found a cubic model that *exactly* fits a set of data. In many real-life situations, you cannot find models to fit data exactly. Despite this limitation, you can still use technology to approximate the data with a polynomial model, as shown in the next example.

EXAMPLE 3 Real-Life Application

According to the U.S. Department of Energy, *biomass* includes "agricultural and forestry residues, municipal solid wastes, industrial wastes, and terrestrial and aquatic crops grown solely for energy purposes." Among the uses for biomass is production of electricity and liquid fuels such as ethanol.

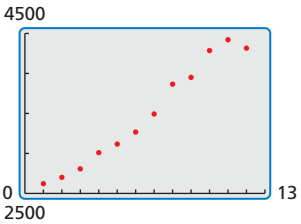
The table shows the total U.S. biomass energy consumptions y (in trillions of British thermal units, or Btus) in the year t, where $t = 1$ corresponds to 2001. Find a polynomial model for the data. Use the model to estimate the total U.S. biomass energy consumption in 2013.

t	1	2	3	4	5	6
y	2622	2701	2807	3010	3117	3267

t	7	8	9	10	11	12
y	3493	3866	3951	4286	4421	4316

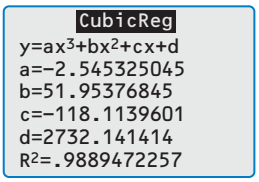
SOLUTION

Step 1 Enter the data into a graphing calculator and make a scatter plot. The data suggest a cubic model.

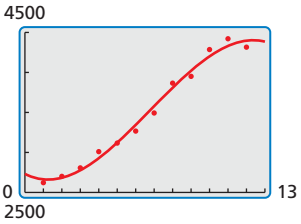


Step 2 Use the *cubic regression* feature. The polynomial model is

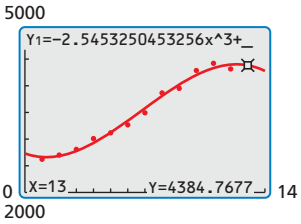
$$y = -2.545t^3 + 51.95t^2 - 118.1t + 2732.$$



Step 3 Check the model by graphing it and the data in the same viewing window.



Step 4 Use the *trace* feature to estimate the value of the model when $t = 13$.



▶ The approximate total U.S. biomass energy consumption in 2013 was about 4385 trillion Btus.

Monitoring Progress

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Use a graphing calculator to find a polynomial function that fits the data.

4.

x	1	2	3	4	5	6
y	5	13	17	11	11	56

5.

x	0	2	4	6	8	10
y	8	0	15	69	98	87

4.9 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

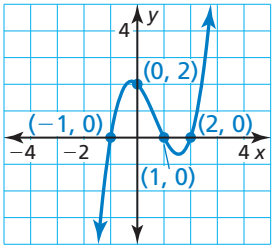
1. **COMPLETE THE SENTENCE** When the x-values in a set of data are equally spaced, the differences of consecutive y-values are called _____.

2. **WRITING** Explain how you know when a set of data could be modeled by a cubic function.

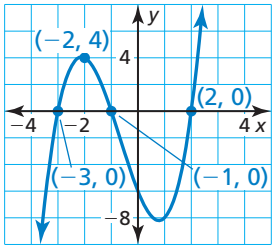
Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, write a cubic function whose graph is shown. *(See Example 1.)*

3.



4.



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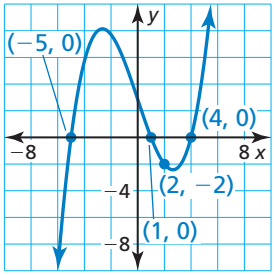

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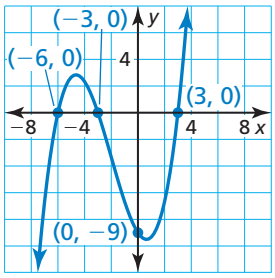


In Exercises 7–12, use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function. *(See Example 2.)*

7.

x	−6	−3	0	3	6	9
$f(x)$	−2	15	−4	49	282	803

8.

x	−1	0	1	2	3	4
$f(x)$	−14	−5	−2	7	34	91

9. (−4, −317), (−3, −37), (−2, 21), (−1, 7), (0, −1), (1, 3), (2, −47), (3, −289), (4, −933)

10. (−6, 744), (−4, 154), (−2, 4), (0, −6), (2, 16), (4, 154), (6, 684), (8, 2074), (10, 4984)

11. (−2, 968), (−1, 422), (0, 142), (1, 26), (2, −4), (3, −2), (4, 2), (5, 2), (6, 16)

12. (1, 0), (2, 6), (3, 2), (4, 6), (5, 12), (6, −10), (7, −114), (8, −378), (9, −904)

13. **ERROR ANALYSIS** Describe and correct the error in writing a cubic function whose graph passes through the given points.

$(-6, 0), (1, 0), (3, 0), (0, 54)$

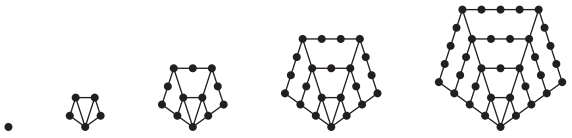
$54 = a(0 - 6)(0 + 1)(0 + 3)$

$54 = -18a$

$a = -3$

$f(x) = -3(x - 6)(x + 1)(x + 3)$

14. **MODELING WITH MATHEMATICS** The dot patterns show pentagonal numbers. The number of dots in the nth pentagonal number is given by $f(n) = \frac{1}{2}n(3n - 1)$. Show that this function has constant second-order differences.



15. **OPEN-ENDED** Write three different cubic functions that pass through the points (3, 0), (4, 0), and (2, 6). Justify your answers.

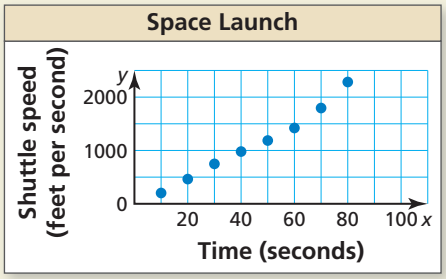
16. **MODELING WITH MATHEMATICS** The table shows the ages of cats and their corresponding ages in human years. Find a polynomial model for the data for the first 8 years of a cat's life. Use the model to estimate the age (in human years) of a cat that is 3 years old. *(See Example 3.)*

Age of cat, x	1	2	4	6	7	8
Human years, y	15	24	32	40	44	48

17. MODELING WITH MATHEMATICS The data in the table show the average speeds y (in miles per hour) of a pontoon boat for several different engine speeds x (in hundreds of revolutions per minute, or RPMs). Find a polynomial model for the data. Estimate the average speed of the pontoon boat when the engine speed is 2800 RPMs.

x	10	20	25	30	45	55
y	4.5	8.9	13.8	18.9	29.9	37.7

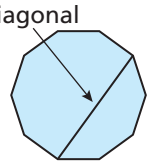
18. HOW DO YOU SEE IT? The graph shows typical speeds y (in feet per second) of a space shuttle x seconds after it is launched.



a. What type of polynomial function models the data? Explain.

b. Which nth-order finite difference should be constant for the function in part (a)? Explain.

19. MATHEMATICAL CONNECTIONS The table shows the number of diagonals for polygons with n sides. Find a polynomial function that fits the data. Determine the total number of diagonals in the decagon shown.



Number of sides, n	3	4	5	6	7	8
Number of diagonals, d	0	2	5	9	14	20

20. MAKING AN ARGUMENT Your friend states that it is not possible to determine the degree of a function given the first-order differences. Is your friend correct? Explain your reasoning.

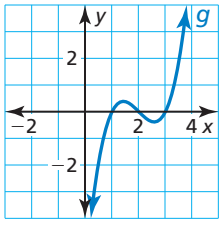
21. WRITING Explain why you cannot always use finite differences to find a model for real-life data sets.

22. THOUGHT PROVOKING A, B, and C are zeros of a cubic polynomial function. Choose values for A, B, and C such that the distance from A to B is less than or equal to the distance from A to C. Then write the function using the A, B, and C values you chose.

23. MULTIPLE REPRESENTATIONS Order the polynomial functions according to their degree, from least to greatest.

A. $f(x) = -3x + 2x^2 + 1$

B.



C.

x	−2	−1	0	1	2	3
$h(x)$	8	6	4	2	0	−2

D.

x	−2	−1	0	1	2	3
$k(x)$	25	6	7	4	−3	10

24. ABSTRACT REASONING Substitute the expressions $z, z + 1, z + 2, \ldots, z + 5$ for x in the function $f(x) = ax^3 + bx^2 + cx + d$ to generate six equally-spaced ordered pairs. Then show that the third-order differences are constant.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation using square roots. *(Section 3.1)*

25. $x^2 - 6 = 30$

26. $5x^2 - 38 = 187$

27. $2(x - 3)^2 = 24$

28. $\frac{4}{3}(x + 5)^2 = 4$

Solve the equation using the Quadratic Formula. *(Section 3.4)*

29. $2x^2 + 3x = 5$

30. $2x^2 + \frac{1}{2} = 2x$

31. $2x^2 + 3x = -3x^2 + 1$

32. $4x - 20 = x^2$

4.5–4.9 What Did You Learn?

Core Vocabulary

repeated solution, *p. 190*
complex conjugates, *p. 199*
local maximum, *p. 214*
local minimum, *p. 214*
even function, *p. 215*
odd function, *p. 215*
finite differences, *p. 220*

Core Concepts

Section 4.5

The Rational Root Theorem, *p. 191*
The Irrational Conjugates Theorem, *p. 193*

Section 4.6

The Fundamental Theorem of Algebra, *p. 198*
The Complex Conjugates Theorem, *p. 199*
Descartes's Rule of Signs, *p. 200*

Section 4.7

Transformations of Polynomial Functions, *p. 206*
Writing Transformed Polynomial Functions, *p. 207*

Section 4.8

Zeros, Factors, Solutions, and Intercepts of Polynomials, *p. 212*
The Location Principle, *p. 213*
Turning Points of Polynomial Functions, *p. 214*
Even and Odd Functions, *p. 215*

Section 4.9

Writing Polynomial Functions for Data Sets, *p. 220*
Properties of Finite Differences, *p. 221*

Mathematical Practices

1. Explain how understanding the Complex Conjugates Theorem allows you to construct your argument in Exercise 46 on page 203.
2. Describe how you use structure to accurately match each graph with its transformation in Exercises 7–10 on page 209.

Performance Task

For the Birds – Wildlife Management

How does the presence of humans affect the population of sparrows in a park? Do more humans mean fewer sparrows? Or does the presence of humans increase the number of sparrows up to a point? Are there a minimum number of sparrows that can be found in a park, regardless of how many humans there are? What can a mathematical model tell you?

To explore the answers to these questions and more, go to ***BigIdeasMath.com***.

4 Chapter Review

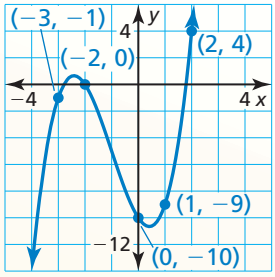
Dynamic Solutions available at *BigIdeasMath.com*

4.1 Graphing Polynomial Functions *(pp. 157–164)*

Graph $f(x) = x^3 + 3x^2 - 3x - 10$.

To graph the function, make a table of values and plot the corresponding points. Connect the points with a smooth curve and check the end behavior.

x	-3	-2	-1	0	1	2	3
$f(x)$	-1	0	-5	-10	-9	4	35



The degree is odd and the leading coefficient is positive.
So, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

Decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

1. $h(x) = -x^3 + 2x^2 - 15x^7$

2. $p(x) = x^3 - 5x^{0.5} + 13x^2 + 8$

Graph the polynomial function.

3. $h(x) = x^2 + 6x^5 - 5$

4. $f(x) = 3x^4 - 5x^2 + 1$

5. $g(x) = -x^4 + x + 2$

4.2 Adding, Subtracting, and Multiplying Polynomials *(pp. 165–172)*

a. Multiply $(x - 2)$, $(x - 1)$, and $(x + 3)$ in a horizontal format.

$$\begin{aligned}(x - 2)(x - 1)(x + 3) &= (x^2 - 3x + 2)(x + 3)\\ &= (x^2 - 3x + 2)x + (x^2 - 3x + 2)3\\ &= x^3 - 3x^2 + 2x + 3x^2 - 9x + 6\\ &= x^3 - 7x + 6\end{aligned}$$

b. Use Pascal's Triangle to expand $(4x + 2)^4$.

The coefficients from the fourth row of Pascal's Triangle are 1, 4, 6, 4, and 1.

$$\begin{aligned}(4x + 2)^4 &= 1(4x)^4 + 4(4x)^3(2) + 6(4x)^2(2)^2 + 4(4x)(2)^3 + 1(2)^4\\ &= 256x^4 + 512x^3 + 384x^2 + 128x + 16\end{aligned}$$

Find the sum or difference.

6. $(4x^3 - 12x^2 - 5) - (-8x^2 + 4x + 3)$

7. $(x^4 + 3x^3 - x^2 + 6) + (2x^4 - 3x + 9)$

8. $(3x^2 + 9x + 13) - (x^2 - 2x + 12)$

Find the product.

9. $(2y^2 + 4y - 7)(y + 3)$

10. $(2m + n)^3$

11. $(s + 2)(s + 4)(s - 3)$

Use Pascal's Triangle to expand the binomial.

12. $(m + 4)^4$

13. $(3s + 2)^5$

14. $(z + 1)^6$

4.3 Dividing Polynomials *(pp. 173–178)*

Use synthetic division to evaluate $f(x) = -2x^3 + 4x^2 + 8x + 10$ when $x = -3$.

$$\begin{array}{r|cccc} -3 & -2 & 4 & 8 & 10 \\ & & 6 & -30 & 66 \\ \hline & -2 & 10 & -22 & 76 \end{array}$$

▶ The remainder is 76. So, you can conclude from the Remainder Theorem that $f(-3) = 76$. You can check this by substituting $x = -3$ in the original function.

Check

$$\begin{aligned} f(-3) &= -2(-3)^3 + 4(-3)^2 + 8(-3) + 10 \\ &= 54 + 36 - 24 + 10 \\ &= 76 \checkmark \end{aligned}$$

Divide using polynomial long division or synthetic division.

15. $(x^3 + x^2 + 3x - 4) \div (x^2 + 2x + 1)$

16. $(x^4 + 3x^3 - 4x^2 + 5x + 3) \div (x^2 + x + 4)$

17. $(x^4 - x^2 - 7) \div (x + 4)$

18. Use synthetic division to evaluate $g(x) = 4x^3 + 2x^2 - 4$ when $x = 5$.

4.4 Factoring Polynomials *(pp. 179–186)*

a. Factor $x^4 + 8x$ completely.

$$\begin{aligned} x^4 + 8x &= x(x^3 + 8) && \text{Factor common monomial.} \\ &= x(x^3 + 2^3) && \text{Write } x^3 + 8 \text{ as } a^3 + b^3. \\ &= x(x + 2)(x^2 - 2x + 4) && \text{Sum of Two Cubes Pattern} \end{aligned}$$

b. Determine whether $x + 4$ is a factor of $f(x) = x^5 + 4x^4 + 2x + 8$.

Find $f(-4)$ by synthetic division.

$$\begin{array}{r|cccccc} -4 & 1 & 4 & 0 & 0 & 2 & 8 \\ & & -4 & 0 & 0 & 0 & -8 \\ \hline & 1 & 0 & 0 & 0 & 2 & 0 \end{array}$$

▶ Because $f(-4) = 0$, the binomial $x + 4$ is a factor of $f(x) = x^5 + 4x^4 + 2x + 8$.

Factor the polynomial completely.

19. $64x^3 - 8$ **20.** $2z^5 - 12z^3 + 10z$ **21.** $2a^3 - 7a^2 - 8a + 28$

22. Show that $x + 2$ is a factor of $f(x) = x^4 + 2x^3 - 27x - 54$. Then factor $f(x)$ completely.

4.5 Solving Polynomial Equations *(pp. 189–196)*

a. Find all real solutions of $x^3 + x^2 - 8x - 12 = 0$.

Step 1 List the possible rational solutions. The leading coefficient of the polynomial $f(x) = x^3 + x^2 - 8x - 12$ is 1, and the constant term is -12. So, the possible rational solutions of $f(x) = 0$ are

$$x = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{6}{1}, \pm\frac{12}{1}.$$

Step 2 Test possible solutions using synthetic division until a solution is found.

$$\begin{array}{r|rrrr} 2 & 1 & 1 & -8 & -12 \\ & & 2 & 6 & -4 \\ \hline & 1 & 3 & -2 & -16 \end{array} \qquad \begin{array}{r|rrrr} -2 & 1 & 1 & -8 & -12 \\ & & -2 & 2 & 12 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$f(2) \neq 0$, so $x - 2$ is not a factor of $f(x)$. $f(-2) = 0$, so $x + 2$ is a factor of $f(x)$.

Step 3 Factor completely using the result of synthetic division.

$(x + 2)(x^2 - x - 6) = 0$ Write as a product of factors.

$(x + 2)(x + 2)(x - 3) = 0$ Factor the trinomial.

▶ So, the solutions are $x = -2$ and $x = 3$.

b. Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros -4 and $1 + \sqrt{2}$.

By the Irrational Conjugates Theorem, $1 - \sqrt{2}$ must also be a zero of f.

$f(x) = (x + 4)\left[x - (1 + \sqrt{2})\right]\left[x - (1 - \sqrt{2})\right]$ Write $f(x)$ in factored form.

$= (x + 4)\left[(x - 1) - \sqrt{2}\right]\left[(x - 1) + \sqrt{2}\right]$ Regroup terms.

$= (x + 4)\left[(x - 1)^2 - 2\right]$ Multiply.

$= (x + 4)\left[(x^2 - 2x + 1) - 2\right]$ Expand binomial.

$= (x + 4)(x^2 - 2x - 1)$ Simplify.

$= x^3 - 2x^2 - x + 4x^2 - 8x - 4$ Multiply.

$= x^3 + 2x^2 - 9x - 4$ Combine like terms.

Find all real solutions of the equation.

23. $x^3 + 3x^2 - 10x - 24 = 0$ **24.** $x^3 + 5x^2 - 2x - 24 = 0$

Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

25. $1, 2 - \sqrt{3}$ **26.** $2, 3, \sqrt{5}$ **27.** $-2, 5, 3 + \sqrt{6}$

28. You use 240 cubic inches of clay to make a sculpture shaped as a rectangular prism. The width is 4 inches less than the length and the height is 2 inches more than three times the length. What are the dimensions of the sculpture? Justify your answer.

4.6 The Fundamental Theorem of Algebra *(pp. 197–204)*

Find all zeros of $f(x) = x^4 + 2x^3 + 6x^2 + 18x - 27$.

Step 1 Find the rational zeros of f. Because f is a polynomial function of degree 4, it has four zeros. The possible rational zeros are ± 1, ± 3, ± 9, and ± 27. Using synthetic division, you can determine that 1 is a zero and -3 is also a zero.

Step 2 Write $f(x)$ in factored form. Dividing $f(x)$ by its known factors $x - 1$ and $x + 3$ gives a quotient of $x^2 + 9$. So,

$$f(x) = (x - 1)(x + 3)(x^2 + 9).$$

Step 3 Find the complex zeros of f. Solving $x^2 + 9 = 0$, you get $x = \pm 3i$. This means $x^2 + 9 = (x + 3i)(x - 3i)$.

$$f(x) = (x - 1)(x + 3)(x + 3i)(x - 3i)$$

▶ From the factorization, there are four zeros. The zeros of f are 1, -3, $-3i$, and $3i$.

Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

29. $3, 1 + 2i$ **30.** $-1, 2, 4i$ **31.** $-5, -4, 1 - i\sqrt{3}$

Determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for the function.

32. $f(x) = x^4 - 10x + 8$ **33.** $f(x) = -6x^4 - x^3 + 3x^2 + 2x + 18$

4.7 Transformations of Polynomial Functions *(pp. 205–210)*

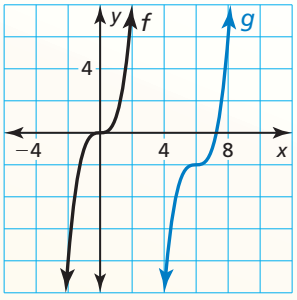
Describe the transformation of $f(x) = x^3$ represented by $g(x) = (x - 6)^3 - 2$. Then graph each function.

Notice that the function is of the form $g(x) = (x - h)^3 + k$. Rewrite the function to identify h and k.

$$g(x) = (x - 6)^3 + (-2)$$

(h = 6, k = -2)

▶ Because $h = 6$ and $k = -2$, the graph of g is a translation 6 units right and 2 units down of the graph of f.



Describe the transformation of f represented by g. Then graph each function.

34. $f(x) = x^3$, $g(x) = (-x)^3 + 2$ **35.** $f(x) = x^4$, $g(x) = -(x + 9)^4$

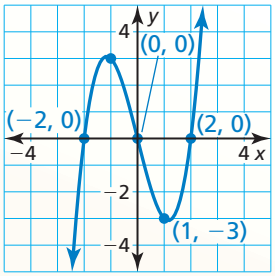
Write a rule for g.

36. Let the graph of g be a horizontal stretch by a factor of 4, followed by a translation 3 units right and 5 units down of the graph of $f(x) = x^5 + 3x$.

37. Let the graph of g be a translation 5 units up, followed by a reflection in the y-axis of the graph of $f(x) = x^4 - 2x^3 - 12$.

4.8 Analyzing Graphs of Polynomial Functions *(pp. 211–218)*

Graph the function $f(x) = x(x + 2)(x - 2)$. Then estimate the points where the local maximums and local minimums occur.

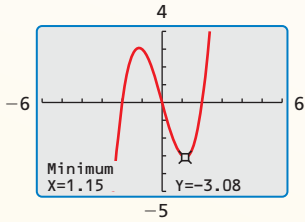


Step 1 Plot the x-intercepts. Because -2, 0, and 2 are zeros of f, plot $(-2, 0)$, $(0, 0)$, and $(2, 0)$.

Step 2 Plot points between and beyond the x-intercepts.

x	-3	-2	-1	0	1	2	3
y	-15	0	3	0	-3	0	15

Step 3 Determine end behavior. Because $f(x)$ has three factors of the form $x - k$ and a constant factor of 1, f is a cubic function with a positive leading coefficient. So $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.



Step 4 Draw the graph so it passes through the plotted points and has the appropriate end behavior.

▶ The function has a local maximum at $(-1.15, 3.08)$ and a local minimum at $(1.15, -3.08)$.

Graph the function. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing.

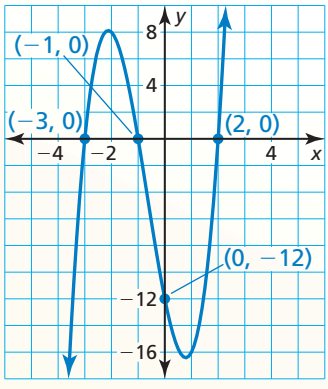
38. $f(x) = -2x^3 - 3x^2 - 1$ **39.** $f(x) = x^4 + 3x^3 - x^2 - 8x + 2$

Determine whether the function is *even*, *odd*, or *neither*.

40. $f(x) = 2x^3 + 3x$ **41.** $g(x) = 3x^2 - 7$ **42.** $h(x) = x^6 + 3x^5$

4.9 Modeling with Polynomial Functions *(pp. 219–224)*

Write the cubic function whose graph is shown.



Step 1 Use the three x-intercepts to write the function in factored form.

$$f(x) = a(x + 3)(x + 1)(x - 2)$$

Step 2 Find the value of a by substituting the coordinates of the point $(0, -12)$.

$$-12 = a(0 + 3)(0 + 1)(0 - 2)$$
$$-12 = -6a$$
$$2 = a$$

▶ The function is $f(x) = 2(x + 3)(x + 1)(x - 2)$.

43. Write a cubic function whose graph passes through the points $(-4, 0)$, $(4, 0)$, $(0, 6)$, and $(2, 0)$.

44. Use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function.

x	1	2	3	4	5	6	7
$f(x)$	-11	-24	-27	-8	45	144	301

4 Chapter Test

Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

1. $3, 1 - \sqrt{2}$

2. $-2, 4, 3i$

Find the product or quotient.

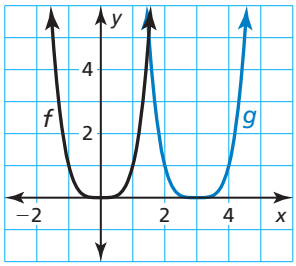
3. $(x^6 - 4)(x^2 - 7x + 5)$

4. $(3x^4 - 2x^3 - x - 1) \div (x^2 - 2x + 1)$

5. $(2x^3 - 3x^2 + 5x - 1) \div (x + 2)$

6. $(2x + 3)^3$

7. The graphs of $f(x) = x^4$ and $g(x) = (x - 3)^4$ are shown.



a. How many zeros does each function have? Explain.

b. Describe the transformation of f represented by g.

c. Determine the intervals for which the function g is increasing or decreasing.

8. The volume V (in cubic feet) of an aquarium is modeled by the polynomial function $V(x) = x^3 + 2x^2 - 13x + 10$, where x is the length of the tank.

Volume = 3 ft^3

a. Explain how you know $x = 4$ is *not* a possible rational zero.

b. Show that $x - 1$ is a factor of $V(x)$. Then factor $V(x)$ completely.

c. Find the dimensions of the aquarium shown.

9. One special product pattern is $(a - b)^2 = a^2 - 2ab + b^2$. Using Pascal's Triangle to expand $(a - b)^2$ gives $1a^2 + 2a(-b) + 1(-b)^2$. Are the two expressions equivalent? Explain.

10. Can you use the synthetic division procedure that you learned in this chapter to divide *any* two polynomials? Explain.

11. Let T be the number (in thousands) of new truck sales. Let C be the number (in thousands) of new car sales. During a 10-year period, T and C can be modeled by the following equations where t is time (in years).

$$T = 23t^4 - 330t^3 + 3500t^2 - 7500t + 9000$$

$$C = 14t^4 - 330t^3 + 2400t^2 - 5900t + 8900$$

a. Find a new model S for the total number of new vehicle sales.

b. Is the function S *even*, *odd*, or *neither*? Explain your reasoning.

12. Your friend has started a golf caddy business. The table shows the profits p (in dollars) of the business in the first 5 months. Use finite differences to find a polynomial model for the data. Then use the model to predict the profit after 7 months.

Month, t	1	2	3	4	5
Profit, p	4	2	6	22	56

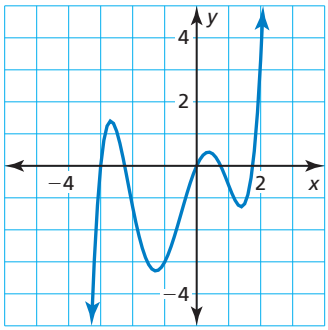
4 Cumulative Assessment

1. The synthetic division below represents $f(x) \div (x - 3)$. Choose a value for m so that $x - 3$ is a factor of $f(x)$. Justify your answer.

3	1	−3	m	3
		3	0	
	1	0		

−3	3
−2	2
−1	1

2. Analyze the graph of the polynomial function to determine the sign of the leading coefficient, the degree of the function, and the number of real zeros. Explain.



3. Which statement about the graph of the equation $12(x - 6) = -(y + 4)^2$ is *not* true?

Ⓐ The vertex is $(6, -4)$.

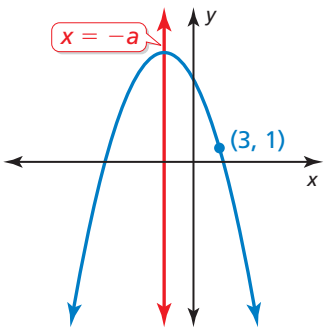
Ⓑ The axis of symmetry is $y = -4$.

Ⓒ The focus is $(3, -4)$.

Ⓓ The graph represents a function.

4. A parabola passes through the point shown in the graph. The equation of the axis of symmetry is $x = -a$. Which of the given points could lie on the parabola? If the axis of symmetry was $x = a$, then which points could lie on the parabola? Explain your reasoning.

(1, 1)	(0, 1)	(−2, 1)
(−3, 1)	(−4, 1)	(−5, 1)



5. Select values for the function to model each transformation of the graph of $f(x) = x$.

$$g(x) = \square\left(x - \square\right) + \square$$

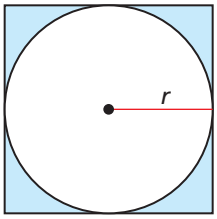
a. The graph is a translation 2 units up and 3 units left.

b. The graph is a translation 2 units right and 3 units down.

c. The graph is a vertical stretch by a factor of 2, followed by a translation 2 units up.

d. The graph is a translation 3 units right and a vertical shrink by a factor of $\frac{1}{2}$, followed by a translation 4 units down.

6. The diagram shows a circle inscribed in a square. The area of the shaded region is 21.5 square meters. To the nearest tenth of a meter, how long is each side of the square?



Ⓐ 4.6 meters Ⓑ 8.7 meters Ⓒ 9.7 meters Ⓓ 10.0 meters

7. Classify each function as *even*, *odd*, or *neither*. Justify your answer.

a. $f(x) = 3x^5$

b. $f(x) = 4x^3 + 8x$

c. $f(x) = 3x^3 + 12x^2 + 1$

d. $f(x) = 2x^4$

e. $f(x) = x^{11} - x^7$

f. $f(x) = 2x^8 + 4x^4 + x^2 - 5$

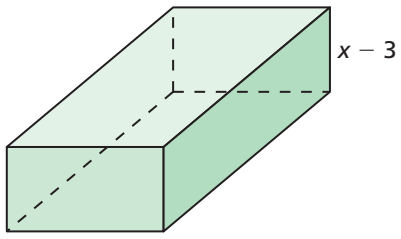
8. The volume of the rectangular prism shown is given by $V = 2x^3 + 7x^2 - 18x - 63$. Which polynomial represents the area of the base of the prism?

Ⓐ $2x^2 + x - 21$

Ⓑ $2x^2 + 21 - x$

Ⓒ $13x + 21 + 2x^2$

Ⓓ $2x^2 - 21 - 13x$



9. The number R (in tens of thousands) of retirees receiving Social Security benefits is represented by the function

$$R = 0.286t^3 - 4.68t^2 + 8.8t + 403, \quad 0 \le t \le 10$$

where t represents the number of years since 2000. Identify any turning points on the given interval. What does a turning point represent in this situation?