Creating a textbook and its supplementary materials is a team effort involving many individuals and groups. We are especially grateful to thousands of Advanced Algebra Through Data Exploration and Discovering Algebra teachers and students, to teachers who participated in the summer institutes and workshops, and to manuscript readers, all of whom provided suggestions, reviewed material, located errors, and most of all, encouraged us to continue with the project.

Our students, their parents, and our administrators at Interlochen Arts Academy have played an important part in the development of this book. Most importantly, we wish to thank Carol Murdock, our parents, and our children for their love, encouragement, and support.

As authors we are grateful to the National Science Foundation for supporting our initial technology-and-writing project that led to the 1998 publication of Advanced Algebra Through Data Exploration. Discovering Advanced Algebra has been developed and shaped by what we learned during the writing and publication of both Advanced Algebra Through Data Exploration and Discovering Algebra, and our work with so many students, parents, and teachers who were searching for a more meaningful algebra curriculum.

Over the course of our careers, many individuals and groups have been instrumental in our development as teachers and authors. The Woodrow Wilson National Fellowship Foundation provided the initial impetus for involvement in leading workshops. Publications and conferences produced by the National Council of Teachers of Mathematics and Teachers Teaching with Technology have guided the development of this curriculum. Individuals such as Ron Carlson, Helen Compton, Frank Demana, Arne Engebretsen, Paul Foerster, Christian Hirsch, Glenda Lappan, Richard Odell, Heinz-Otto Peitgen, James Sandefur, James Schultz, Dan Teague, Charles VonderEmbsa, Bert Waits, and Mary Jean Winter have inspired us.

The development and production of Discovering Advanced Algebra has been a collaborative effort between the authors and the staff at Key Curriculum Press. We truly appreciate the cooperation and valuable contributions offered by the Editorial and Production Departments at Key Curriculum Press. Finally, a special thanks to Key’s president, Steven Rasmussen, for encouraging and publishing a technology-enhanced Discovering Mathematics series that offers groundbreaking content and learning opportunities.

Jerald Murdock
Ellen Kamischke
Eric Kamischke
The algebra you find in this book won’t look quite like the algebra you may have seen in older textbooks. The mathematics we learn and teach in school has to change continually to reflect changes in our world. Our workplaces are changing, and technology is present everywhere, fundamentally changing the work we do. There are some new topics that are now possible to explore with technology, and some standard topics that can be approached in new ways. As the National Council of Teachers of Mathematics (NCTM) Technology Principle says, “When technological tools are available, students can focus on decision making, reflection, reasoning, and problem solving.” This has been the focus of the authors and the Key Curriculum Press editorial team in the creation of Discovering Advanced Algebra: An Investigative Approach. As you progress through this book, you’ll see that graphing calculators and other technologies are used to explore patterns and to make, test, and generalize conjectures.

When Key Curriculum Press published the first version of this text, Advanced Algebra Through Data Exploration: A Graphing Calculator Approach, in 1998, few books were available that had a similar foundation in technology. In this revision, you’ll see that that foundation has been enriched with projects, explorations, and exercises that utilize not only graphing calculators, but also the powerful analysis tools The Geometer’s Sketchpad® and Fathom Dynamic Statistics™. Based on feedback from users and reviewers, this revision is reorganized and easier to read. Discovering Advanced Algebra also completes the fully updated Discovering Mathematics series. All of the features that make Discovering Algebra and Discovering Geometry innovative and exciting are now incorporated into this book as well, to make a coherent and streamlined series.

Investigations are at the heart of each book. Through the investigations, you’ll explore interesting problems and generalize concepts. And if you, as a student, forget a concept, formula, or procedure, you can always re-create it—because you developed it yourself the first time! You’ll find that this approach allows you to form a deep and conceptual understanding of advanced algebra topics.

As Glenda Lappan, mathematics professor at Michigan State University and former NCTM president, said about the first edition of this book, “Students coming out of a year with this text . . . will know the mathematics they know in deeper, more flexible ways. They will have developed a set of mathematical habits of mind that will serve them very well as students or users of mathematics. They will emerge with a sense of mathematics as a search for regularity that allows prediction.”

If you are a student, we hope that what you learn this year will serve you well in life. If you are a parent, we hope you will enjoy watching your student develop mathematical confidence. And if you are a teacher, we hope Discovering Advanced Algebra greatly enriches your classroom. The professional team at Key Curriculum Press wishes you success and joy in the lifetime of mathematics ahead of you. We look forward to hearing about your experiences.

Steven Rasmussen, President
Key Curriculum Press
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The goal of this stage of your mathematical journey is to develop advanced algebraic tools and the mathematical power that will help you participate fully as a productive citizen in a changing world. On this journey you will make connections between algebra and the world around you.

Important decision-making situations will confront you in life, and your ability to use mathematics and algebra can help you make informed decisions. You’ll need skills that can evolve and be adapted to new situations. You’ll need to interpret numerical information and use it as a basis for making decisions. And you’ll need to find ways to solve problems that arise in real life, not just in textbooks. Success in algebra is also a recognized gateway to many varied career opportunities.

You’ve already found out that learning algebra is more than memorizing facts, theories, and procedures. With your teacher as a guide, you’ll learn algebra by doing mathematics. You’ll make sense of important algebraic concepts, learn essential skills, and discover how to use algebra. This requires a far bigger commitment than just “waiting for the teacher to show you” or studying worked-out examples.

Your personal involvement is critical to successful group work during Investigations. Keep your measurements, data, and calculations neat and accurate to make your work easier and the concepts clearer in the long run. Talk about algebra, share ideas, and learn from and with your fellow group members. Work and communicate with your teammates to strengthen your understanding of the mathematical concepts. To enjoy and gain respect in your role as a team player, honor differences among group members, listen carefully when others are sharing, stay focused during the process, be responsible and respectful, and share your own ideas and suggestions.

The right technology can help you explore new ideas and answer questions that come up along the way. Using a graphing calculator, you will be able to manipulate large amounts of data quickly so that you can see the overall picture. Throughout the text you can refer to Calculator Notes for information that will help you use this tool. Technology is likely to play an important role in your life and future career. Learning to use your graphing calculator efficiently today, and being able to interpret its output, will prepare you to use other technologies successfully in situations to come.

The book itself will be a guide, leading you to explore ideas and ponder questions. Read it carefully—with paper, pencil, and calculator close at hand—and take good notes. Concepts and problems you have encountered before can help you solve new problems. Work through the Examples and answer the questions that are asked along the way. Some Exercises require a great deal of thought. Don’t give up. Make a solid attempt at each problem that is assigned. Sometimes you’ll make corrections and fill in details later, after you discuss a problem in class. Features called Project, Improving Your . . . Skills, and Take Another Look will challenge you to extend your learning and to apply it in creative ways.
Just as this book is your guide, your notebook can be a log of your travels through advanced algebra. In it you will record your notes and your work. You may also want to keep a journal of your personal impressions along the way. And just as every trip results in a photo album, you can place some of your especially notable accomplishments in a portfolio that highlights your trip. Collect pieces of work in your portfolio as you go, and refine the contents as you make progress on your journey. Each chapter ends with Asssessing What You’ve Learned. This feature suggests ways to review your progress and prepare for what comes next: organizing your notebook, writing in your journal, updating your portfolio, and other ways to reflect on what you have learned.

You should expect struggles, hard work, and occasional frustration. Yet, as you gain more algebra skills, you’ll overcome obstacles and be rewarded with a deeper understanding of mathematics, an increased confidence in your own problem-solving abilities, and the opportunity to be creative. From time to time, look back to reflect on where you have been. We hope that your journey through Discovering Advanced Algebra will be a meaningful and rewarding experience.

And now it is time to begin. You are about to discover some pretty fascinating things.
The Art of the Motorcycle, an exhibit at the Guggenheim Museum Las Vegas, required several very different problem-solving strategies. Architect Rem Koolhaas used diagrams and models to design a building that can house one large exhibit or several small galleries. Frank O. Gehry designed this installation as a visual representation of the materials and craftsmanship of the motorcycles. The organizers of the exhibit had to organize, schedule, and budget to bring the exhibit together as a whole.

In this chapter you will

- solve problems both on your own and as a member of a group
- use pictures and graphs as problem-solving tools
- learn a four-step process for solving problems with symbolic algebra
- practice strategies for organizing information before you solve a problem
A whole essay might be written on the danger of thinking without images.

SAMUEL COLERIDGE

Pictures, Graphs, and Diagrams

In this textbook there are many problems that ask you to look at situations in new and different ways. This chapter offers some strategies to approach these problems. Although some of the problems in this chapter are fictitious, they give you a chance to practice skills that you will use throughout the book and throughout life.

This first lesson focuses on using a sketch, graph, or diagram to help you find a solution.

EXAMPLE A

Allyndreth needs to mix some lawn fertilizer with 7 liters of water. She has two buckets that hold exactly 3 liters and 8 liters, respectively. Describe or illustrate a procedure that will give exactly 7 liters of water in the 8-liter bucket.

Solution

There is more than one solution to this problem. The picture sequence below shows one solution.

A written description of the solution to Example A might be complex and hard to understand. Yet the pictures help you keep track of the amount of water at each step of the solution. You also see how the water is poured into and out of the buckets. Can you think of a different solution? Does your solution take more or fewer steps?

Using pictures is one way to visualize a problem. Sometimes it helps if you actually use objects and act out the problem. For instance, in Example A you could use paper cups to represent the buckets and label each cup with the amount of water at each step of the solution. When you act out a problem, it helps to record positions and quantities on paper as you solve the problem so that you can recall your own steps.

Problem solving often requires a group effort. Different people have different approaches to solving problems, so working in a group gives you the opportunity to hear and see different strategies. Sometimes group members can divide the work based on each person's strengths and expertise, and other times it helps if everyone does the same task and then compares results. Each time you work in a group, decide how to share tasks so that each person has a productive role.
The following investigation will give you an opportunity to work in a group and an opportunity to practice some problem-solving strategies.

**Investigation**

**Camel Crossing the Desert**

A camel rests by a pile of 3000 bananas at the edge of a 1000-mile-wide desert. He plans to travel across the desert, transporting as many bananas as possible to the other side. He can carry up to 1000 bananas at any given time, but he must eat one banana at every mile.

What is the maximum number of bananas the camel can transport across the desert? How does he do it? Work as a group and prepare a written or visual solution.

Pictures are useful problem-solving tools in mathematics but are not limited to diagrams like those in Example A. Coordinate graphs are some of the most important problem-solving pictures in mathematics.

**EXAMPLE B**

A line passes through the point \((4, 7)\) and has slope \(\frac{3}{5}\). Find another point on the same line.

**Solution**

You *could* use a formula for slope and solve for an unknown point. But a graph may be a simpler way to find a solution.

Plot the point \((4, 7)\). Recall that slope is \(\frac{\text{change in } y}{\text{change in } x}\) and move from \((4, 7)\) according to the slope, \(\frac{3}{5}\). One possible point, \((9, 10)\), is shown.

**Mathematics CONNECTION**

Coordinate graphs are also called Cartesian graphs, named after the French mathematician and philosopher René Descartes (1596-1650). Descartes was not the first to use coordinate graphs, but he was the first to publish his work using two-dimensional graphs with a horizontal axis, a vertical axis, and an origin. Descartes's goal was to apply algebra to geometry, which is today called analytic geometry. Analytic geometry in turn laid the foundations of modern mathematics, including calculus.
Although pictures and diagrams have been the focus of this lesson, problem solving requires that you use a variety of strategies. As you work on the exercises, don't limit yourself. You are always welcome to use any and all of the strategies that you know.

George Pólya (1887-1985) was a Hungarian-American mathematician often recognized for his contribution to the study of problem solving. In his 1945 book, *How to Solve It*, he describes a four-step problem-solving process:

- Understand the problem
- Devise a plan
- Carry out the plan
- Look back

You might want to practice this four-step process as you work through the problems in this chapter.

You can learn more about Pólya and his contributions to mathematics and problem solving by using the Internet links at [www.keymath.com/CAA](http://www.keymath.com/CAA).

---

**EXERCISES**

**Practice Your Skills**

1. The pictures below show the first and last steps of bucket problems similar to Example A. Write a statement for each problem.

   a. 
   
   b. 

2. Find the slope of each line.

   a. 
   
   b. 

3. A line passes through the point (4, 7) and has slope $\frac{2}{3}$. Find two more points on the line other than (9, 10), which was found in Example B.
4. Explain how this graph helps you solve these proportion problems. Then solve each proportion.

   a. \( \frac{12}{15} = \frac{9}{c} \)
   b. \( \frac{12}{15} = \frac{6}{10} \)

5. The following problem appears in Problems for the Quickening of the Mind, a collection of problems compiled by the Anglo-Saxon scholar Alcuin of York (ca. 735-804 C.E.). Describe a strategy for solving this problem. Do not actually solve the problem.

   A wolf, a goat, and a cabbage must be moved across a river in a boat holding only one besides the ferryman. How can he carry them across so that the goat shall not eat the cabbage, nor the wolf the goat?

6. Find the slope of the line that passes through each pair of points.
   a. \((2, 5)\) and \((7, 10)\)
   b. \((3, -1)\) and \((8, 7)\)
   c. \((-3, 2)\) and \((2, -6)\)
   d. \((3, 3)\) and \((-5, -2)\)

7. For each scenario, draw and label a diagram. Do not actually solve the problem.
   a. A 25 ft ladder leans against the wall of a building with the foot of the ladder 10 ft from the wall.

   How high does the ladder reach?

   b. A cylindrical tank that has diameter 60 cm and length 150 cm rests on its side. The fluid in the tank leaks out from a valve on one base that is 20 cm off the ground. When no more fluid leaks out, what is the volume of the remaining fluid?

   c. Five sales representatives e-mail each of the others exactly once.

   How many e-mail messages do they send?

8. Use this graph to estimate these conversions between grams (g) and ounces (oz).
   a. \(11 \text{ oz} \approx \ ? \text{ g}\)
   b. \(350 \text{ g} \approx \ ? \text{ oz}\)
   c. \(15.5 \text{ oz} \approx \ ? \text{ g}\)
   d. \(180 \text{ g} \approx \ ? \text{ oz}\)
   e. What is the slope of this line? Explain the real-world meaning of the slope.

The unit of measurement today called the pound comes from the Roman unit *libra*, abbreviated "lb," a weight of around 5000 grains (a very small unit of weight). It was subdivided into ounces, or *onzas*, abbreviated "oz." In England the Saxon pound was based on a standard weight of 5400 grains kept in the Tower of London. This nonstandard system eventually became so confusing that many countries adopted the Système International (SI), or metric system, as the standard form of measurement.
9. **APPLICATION** Kyle has a summer job cleaning pools. He needs to measure exact amounts of chlorine but has only a 10-liter bucket and a 7-liter bucket. Assuming an unlimited supply of chlorine, describe or illustrate a procedure that will allow Kyle to:
   a. Give exactly 4 liters of chlorine in the 10-liter bucket.
   b. Give exactly 2 liters of chlorine in the 10-liter bucket.

10. You can use diagrams to represent algebraic expressions. Explain how this rectangle diagram demonstrates that $(x + 2)(x + 3)$ is equivalent to $x^2 + 2x + 3x + 6$.

   ![Rectangle Diagram](image).

11. Draw a rectangle diagram to represent each product. Use the diagrams to expand each product.
   a. $(x + 4)(x + 7)$
   b. $(x + 5)^2$
   c. $(x + 2)(y + 6)$
   d. $(x + 3)(x - 1)$

12. Solve the problem in Exercise 5. Explain your solution.

---

**Review**

13. Translate each verbal statement to a symbolic expression or an equation.
   a. Three more than a number
   b. Venus is 24.3 million miles farther from the Sun than Mercury.
   c. Seth owns twice as many CDs as his sister Erin.

14. Convert these fractions to decimal form.
   a. \(\frac{3}{8}\)
   b. \(\frac{13}{9}\)
   c. \(\frac{16}{25}\)
   d. \(\frac{5}{14}\)

15. Convert these decimal values to fraction form.
   a. 0.375
   b. 1.42
   c. 0.\(\overline{2}\)
   d. 0.\(\overline{3}\)

16. Use the Pythagorean Theorem to find each missing length.
   a. ![Triangle Diagram](image)
   b. ![Triangle Diagram](image)
   c. ![Triangle Diagram](image)
Symbolic Representation

The customer service team had planned to double the number of calls answered the second day, but they exceeded that by three dozen. Seventy-five dozen customer service calls in two days set a new record.

You can translate the paragraph above into an algebraic equation. Although you don't know how many calls were answered each day, an equation will help you figure it out.

For many years, problems like this were solved without writing equations. Then, around the 17th century, the development of symbolic algebra made writing equations and finding solutions much simpler. Verbal statements could then be translated into symbols by representing unknown quantities with letters, called variables, and converting the rest of the sentence into numbers and operations. (You will actually translate this telephone call problem into algebraic notation when you do the exercises.)

Language is very complex and subtle, designed for general descriptions and qualitative communication. For quantitative communication, translating words into symbols can be a very helpful problem-solving skill. The symbols stand for numbers that vary or remain constant, that are given in the problem, or that are unknown. In applications, the numbers quantify data such as time, weight, or position.

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**EXAMPLE A**

Twenty-nine pigs are to be placed into four pens arranged in a circle. As you move clockwise around the circle, each pen should have closer to 10 pigs than the previous pen. How should you divide the 29 pigs?

**Solution**

To represent this problem with symbolic algebra, you need to first determine which quantities are unknown. In this case, the number of pigs in each pen is unknown. Use the variables $a, b, c,$ and $d$ for these quantities.

How close each number is to 10 is expressed as:

$$|a - 10|, |b - 10|, |c - 10|, \text{ and } |d - 10|$$

If each pen has a number of pigs closer to 10 pigs than the previous pen, then this inequality must be true:

$$|a - 10| > |b - 10| > |c - 10| > |d - 10| > |a - 10|$$

However, this doesn't make sense! It says that $|a - 10|$ must be greater than $|a - 10|$, and that is impossible.

So you might be tempted to say there is no solution.

However, consider placing 7 pigs in the first pen, 12 pigs in the second pen, 10 pigs in the third pen, and no pigs in the last pen. Then, starting at the pen with 7 pigs, you see that 12 is closer to 10 than 7, 10 is closer to 10 than 12, "nothing" is closer to 10 than 10, and 7 is closer to 10 than 0.

You may claim, "Unfair!" after reading the answer to Example A, yet it is a solution. The solution lies in the multiple meanings of words. "Nothing" is used to mean something that does not exist and to mean the quantity of 0. Although symbolic algebra seems to have failed in Example A, it did help you recognize that no mathematical answer existed. If a problem does have a mathematical answer, there is usually a way to get it with symbolic algebra. Therefore, as you approach problems of a descriptive nature, it is often helpful to translate the problem into variables, expressions, and equations.

---

**Cultural CONNECTION**

Some cultures use systems of writing based on ideograms—symbols that represent an idea or a thing. The traditional Chinese system of writing, called 汉字, is one example. The complete system contains thousands of symbols, some of which look like the objects they represent. For example, the symbol for 树, or "tree," looks similar to a tree. 汉字 characters are also combined to create symbols with more complex meanings. The symbol for 森, or "forest," shows many trees. The symbol for 東, or "east," combines two symbols to show a sun rising behind a tree.
EXAMPLE B

Three friends went to the gym to work out. None of the friends would tell how much he or she could leg-press, but each hinted at their friends' leg-press amount. Chen said that Juanita and Lou averaged 87 pounds. Juanita said that Chen leg-pressed 6 pounds more than Lou. Lou said that eight times Juanita's amount equals seven times Chen's amount. Find how much each friend could leg-press.

Solution

First, list the unknown quantities and assign a variable to each.
Let \( C \) represent Chen's weight in pounds.
Let \( J \) represent Juanita's weight in pounds.
Let \( L \) represent Lou's weight in pounds.

Second, write equations from the problem.

Chen's statement translated into an algebraic equation. Call this Equation 1.

\[
\frac{J + L}{2} = 87
\]

Juanita's statement as Equation 2.

\[
C - L = 6
\]

Lou's statement as Equation 3.

\[
8J = 7C
\]

Third, solve the equations to find values for the variables.

Multiply both sides of Equation 1 by 2.
Equation 2.
Add the equations.

\[
7J + 7C = 1260
\]

Multiply both sides of the sum by 7.
Equation 3 allows you to substitute \( 8J \) for \( 7C \).
Add like terms.
Divide both sides by 15.

\[
J = 84
\]

Substitute 84 for \( J \) in Equation 3.
Solve for \( C \).

\[
C = 96
\]

Substitute 96 for \( C \) in Equation 2.
Solve for \( L \).

\[
96 - L = 6
\]

\[
L = 90
\]

Last, interpret your solution. Chen leg-presses 96 pounds, Juanita leg-presses 84 pounds, and Lou leg-presses 90 pounds.
You may notice that Example B used a four-step solution process. The investigation will give you a chance to try these four steps on your own or with a group.

**Investigation**

Problems, Problems, Problems

Select one or more of the problems below, and use these four steps to find a solution.

<table>
<thead>
<tr>
<th>Step 1</th>
<th>List the unknown quantities, and assign a variable to each.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 2</td>
<td>Write one or more equations that relate the unknown quantities to conditions of the problem.</td>
</tr>
<tr>
<td>Step 3</td>
<td>Solve the equations to find a value for each variable.</td>
</tr>
<tr>
<td>Step 4</td>
<td>Interpret your solution according to the context of the problem.</td>
</tr>
</tbody>
</table>

When you finish, write a paragraph answering this question: Which of the four problem-solving steps was hardest for you? Why?

**Problem 1**
When Adam and his sister Megan arrive at a party, they see that there is 1 adult chaperone for every 4 kids. Right behind them come 30 more boys, and Megan notices that the ratio is now 2 boys to 1 girl. However, behind the extra boys come 30 more girls, and Adam notices that there are now 4 girls for every 3 boys. What is the final ratio of adult chaperones to kids?

**Problem 2**
Abdul, Billy, and Celia agree to meet and can tomatoes from their neighborhood garden. Abdul picks 50 pounds of tomatoes from his plot of land. Billy picks 30 pounds of tomatoes from his plot. Unfortunately, Celia's plants did not get enough sun, and she cannot pick any tomatoes from her plot.

They spend the day canning, and each has 36 quarts of tomatoes to take home. Wanting to pay Abdul and Billy for the tomatoes they gave to her, Celia finds $8 in her wallet. How should Celia divide the money between her two friends?

**Problem 3**
A caterer claims that a birthday cake will serve either 20 children or 15 adults. Tina's party presently has 12 children and 7 adults. Is there enough cake?

The four problem-solving steps in the investigation help you organize information, work through an algebraic solution, and interpret the final answer. As you do the exercises in this lesson, refer back to these four steps and practice using them.
EXERCISES

Practice Your Skills

1. Explain what you would do to change the first equation to the second.

   a. \( a + 12 = 47 \)
   b. \( 5b = 24 \)
   c. \( -18 + c = 28 \)
   d. \( \frac{d}{15} = 4.5 \)

   a. \( a = 35 \)
   b. \( b = 4.8 \)
   c. \( c = 46 \)
   d. \( d = -67.5 \)

2. Which equation would help you solve the following problem?
Each member of the committee made three copies of the letter to the senator. Adding these to the 5 original letters, there are now a total of 32 letters. How large is the committee?

   A. \( 5 + c = 32 \)
   B. \( 3 + 5c = 32 \)
   C. \( 5 + 3c = 32 \)

3. Solve each equation.

   a. \( 5 + c = 32 \)
   b. \( 3 + 5c = 32 \)
   c. \( 5 + 3c = 32 \)

4. Which equation would help you solve this problem? What does \( x \) represent?
The customer service team had planned to double the number of calls answered the second day, but they exceeded that by three dozen. Seventy-five dozen customer service calls in two days set a new record.

   A. \( x + 3 = 75 \)
   B. \( x + 2x + 3 = 75 \)
   C. \( 2x + 3 = 75 \)

5. Solve each equation.

   a. \( x + 3 = 75 \)
   b. \( x + 2x + 3 = 75 \)
   c. \( 2x + 3 = 75 \)

Reason and Apply

6. Here is a problem and three related equations. Anita buys 6 large beads and 20 small beads to make a necklace. Ivan buys 4 large and 25 small beads for his necklace. Jill selects 8 large and 16 small beads to make an ankle bracelet. Without tax, Anita pays $2.70, and Ivan pays $2.85. How much will Jill pay?

\[
\begin{align*}
6L + 20S &= 270 \quad \text{(Equation 1)} \\
4L + 25S &= 285 \quad \text{(Equation 2)} \\
8L + 16S &= J \quad \text{(Equation 3)}
\end{align*}
\]

   a. What do the variables \( L, S, \) and \( J \) represent?
   b. What are the units of \( L, S, \) and \( J \)?
   c. What does Equation 1 represent?

This necklace was crafted around the 3rd to 2nd millennium B.C.E. in the southwest Asian country of Bactria (now in Afghanistan).
7. Follow these steps to solve Exercise 6.
   a. Multiply Equation 1 by negative two.
   b. Multiply Equation 2 by three.
   c. Add the resulting equations from 7a and b.
   d. Solve the equation in 7c for $S$. Interpret the real-world meaning of this solution.
   e. Use the value of $S$ to find the value of $L$. Interpret the real-world meaning of the value of $L$.
   f. Use the values of $S$ and $L$ to find the value of $J$. Interpret this solution.

8. The following problem appears in Liber abaci (1202), or Book of Calculations, by the Italian mathematician Leonardo Fibonacci (ca. 1170-1240).
   If $A$ gets 7 denarii from $B$, then $A$'s sum is fivefold $B$'s. If $B$ gets 5 denarii from $A$, then $B$'s sum is sevenfold $A$'s. How much has each?
   
   $\begin{align*}
   a + 7 &= 5(b - 7) & \text{(Equation 1)} \\
   b + 5 &= 7(a - 5) & \text{(Equation 2)}
   \end{align*}$

   a. What does $a$ represent?  
   b. What does $b$ represent?  
   c. Explain Equation 1 with words.  
   d. Explain Equation 2 with words.

9. Use Equations 1 and 2 from Exercise 8.
   a. Explain how to get $b + 5 = 7(5b - 42) - 5$
   b. Solve the equation in 9a for $b$.
   c. Use your answer from 9b to find the value of $a$.
   d. Use the context of Exercise 8 to interpret the values of $a$ and $b$.

10. According to Mrs. Randolph's will, each of her great-grandchildren living in Georgia received $700 more than each of her great-grandchildren living in Florida. In all, $206,100 was divided between 36 great-grandchildren.
    The Georgian great-grandchildren decide that the will wasn't really fair, so they each contribute $175 to be divided among the Floridian great-grandchildren. If all great-grandchildren now have an equal share, how many great-grandchildren live in Georgia?

    a. List the unknown quantities and assign a variable to each.
    b. Write one or more equations that relate the variables to conditions of the problem.
    c. Solve the equations to find a value for each variable.
    d. Interpret the value of each variable according to the context of the problem.

Review

11. **APPLICATION** A $30^\circ$-$60^\circ$-$90^\circ$ triangle and a $45^\circ$-$45^\circ$-$90^\circ$ triangle are two drafting tools used by people in careers such as engineering, architecture, and drafting. The angles in both triangle tools are combined to make a variety of angle measures in hand-drawn technical drawings, such as blueprints. Describe or illustrate a procedure that will give the following angle measures.

    a. $15^\circ$  
    b. $75^\circ$  
    c. $105^\circ$
12. Rewrite each expression without parentheses.
   a. \(3(x + 7)\)  
   b. \(-2(6 - n)\)  
   c. \(x(4 - x)\)

13. Substitute the given value of the variable(s) in each expression and evaluate.
   a. \(47 + 3x\) when \(x = 17\)  
   b. \(29 - 34n + 14m\) when \(n = -1\) and \(m = -24\)

14. Find the slope of the line that passes through each pair of points.
   a. \((4, 7)\) and \((8, 7)\)  
   b. \((2, 5)\) and \((-6, 3)\)

15. Rudy is renting a cabin on Lake Tahoe. He wants to make a recipe for hot chocolate that calls for 1 cup of milk. Unfortunately, the cabin is not well stocked, and he has only a 1-gallon container of milk, a 4-cup saucepan, and an empty 12-ounce soda can. Describe or illustrate a process that Rudy can use to measure exactly 1 cup of milk. (There are 8 ounces per cup and 16 cups per gallon.)

16. **Technology** Use geometry software to construct a line through two points. Measure the coordinates of the points and use them to calculate the slope. Observe how the value of the slope changes as you move one of the points. What is the slope when the line is horizontal? Vertical? What other observations can you make?

---

**Project**

**CREATE YOUR OWN COMPUTER ICON**

Many computers use an operating system with a desktop interface. A small picture, or icon, typically represents items on the desktop, such as folders, disk drives, and files. Many of these computer icons are specifically designed to be symbolic representations. A word processing file might look like a piece of paper. Other icons are designed to symbolically convey more abstract ideas. For example, a group of programs that work cooperatively might be represented by puzzle pieces that appear to connect to each other. Even within computer programs, symbolic icons are used for commands, such as a picture of a disk to represent the "save" command.

Try to create a computer icon of your own. Select a type of file, computer program, or command, and design an icon that helps the user symbolically understand the icon.

Your project should include

- A full-color drawing of your icon.
- A description of the item or command represented by your icon.
- An explanation of how and why you chose your design.

(What elements of your design help a computer user interpret your icon?)

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**Organizing Information**

If one and a half chickens lay one and a half eggs in one and a half days, then how long does it take six monkeys to make nine omelets?

What sort of problem-solving strategy can you apply to the silly problem above? You could draw a picture or make a diagram. You could assign variables to all sorts of unknown quantities. But do you really have enough information to solve the problem? Sometimes the best strategy is to begin by organizing what you know and what you want to know. With the information organized, you may then find a way to get to the solution.

There are different methods of organizing information. One valuable technique uses the units in the problem to help you see how the information fits together.

**EXAMPE A**

To qualify for the Interlochen 470 auto race, each driver must complete two laps of the track at an average speed of 100 miles per hour (mi/h). Due to some problems at the start, Naomi averages only 50 mi/h on her first lap. How fast must she go on the second lap to qualify for the race?

**Solution**

Sort the information into two categories: what you know and what you might need to know. Assign variables to the quantities that you don't know.

<table>
<thead>
<tr>
<th>Know</th>
<th>Need to know</th>
</tr>
</thead>
<tbody>
<tr>
<td>Speed of first lap: 50 mi/h</td>
<td>Speed of second lap (in mi/h): s</td>
</tr>
<tr>
<td>Average speed for both laps: 100 mi/h</td>
<td>Length of each lap (in mi): l</td>
</tr>
<tr>
<td></td>
<td>Time for first lap (in h): t₁</td>
</tr>
<tr>
<td></td>
<td>Time for second lap (in h): t₂</td>
</tr>
</tbody>
</table>
Next, look at the units to find connections between the pieces of information. Speed is measured in miles per hour and therefore calculated by dividing distance by time, so you can write these equations:

\[
\begin{align*}
50 \text{ mi/h} &= \frac{l \text{ mi}}{t_1 \text{ h}} \\
\text{speed, distance, and time for the first lap.} \\
3 \text{ mi/h} &= \frac{l \text{ mi}}{t_2 \text{ h}} \\
\text{the second lap.} \\
100 \text{ mi/h} &= \frac{2l \text{ mi}}{(t_1 + t_2) \text{ h}} \\
\text{the average speed needed for both laps to qualify.}
\end{align*}
\]

You can write the first and third equations as

\[t_1 \text{ h} = \frac{l \text{ mi}}{50 \text{ mi/h}} = \frac{l}{50} \text{ h} \quad (t_1 + t_2) \text{ h} = \frac{2l \text{ mi}}{100 \text{ mi/h}} = \frac{l}{50} \text{ h}
\]

This means that the time for the first lap, \(t_1\), and the time for both laps together \((t_1 + t_2)\), are the same and \(t_2 = 0\). There is no time at all to complete the second lap.

It is not possible for Naomi to qualify for the race.

In Example A, it was easy to find relationships between the known and unknown quantities because the units were miles, hours, or miles per hour. Sometimes you will need to use common knowledge to make the intermediate connections between the units.

**EXAMPLE B**

How many seconds are in a calendar year?

**Solution**

First, identify what you know and what you want to know.

<table>
<thead>
<tr>
<th>Know</th>
<th>Need to know</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>Number of seconds</td>
</tr>
</tbody>
</table>

It may seem like you don't have enough information, but consider these commonly known facts:

- 1 year = 365 days (non-leap-year)
- 1 day = 24 hours
- 1 hour = 60 minutes
- 1 minute = 60 seconds

You can write each equality as a fraction and multiply the chain of fractions such that the units reduce to leave seconds.

\[
\frac{1 \text{ year}}{1 \text{ year}} \cdot \frac{365 \text{ days}}{1 \text{ day}} \cdot \frac{24 \text{ hours}}{1 \text{ hour}} \cdot \frac{60 \text{ minutes}}{1 \text{ minute}} \cdot \frac{60 \text{ seconds}}{1 \text{ minute}} = 31,536,000 \text{ seconds}
\]

There are 31,536,000 seconds in a non-leap-year calendar year.
Some problems overwhelm you with lots of information. Identifying and categorizing what you know is always a good way to start organizing information.

**EXAMPLE C**

Lab assistant Jerry Anderson has just finished cleaning a messy lab table and is putting the equipment back on the table when he reads a note telling him *not* to disturb the positions of three water samples. Not knowing the correct order of the three samples, he finds these facts in the lab notes.

- The water that is highest in sulfur was on one end.
- The water that is highest in iron is in the Erlenmeyer flask.
- The water taken from the spring is not next to the water in the bottle.
- The water that is highest in calcium is left of the water taken from the lake.
- The water in the Erlenmeyer flask, the water taken from the well, and the water that is highest in sulfur are three distinct samples.
- The water in the round flask is not highest in calcium.

Organize the facts into categories. (This is the first step in actually determining which sample goes where. You will finish the problem when you do the exercises.)

**Solution**

Information is given about the types of containers, the sources of the water, the elements found in the samples, and the positions of the samples on the table. You can find three options for each category.

- **Containers:** round flask, Erlenmeyer flask, bottle
- **Sources:** spring, lake, well
- **Elements:** sulfur, iron, calcium
- **Positions:** left, center, right

Now that the information is organized and categorized, you need to see where it leads. You will finish this problem in Exercise 8.

**Science CONNECTION**

Piecing together clues in order to understand a bigger picture is a major problem-solving strategy of the Human Genome Project. Since 1990, researchers from around the world have been working cooperatively to map and sequence the human genome—the complete set of more than 3 billion human DNA base pairs. By 2000, despite an overwhelming amount of information to organize, researchers were able to sequence roughly 90% of the human genome. By understanding the organization and function of human DNA, the researchers hope to improve human health and create guidelines for the ethical use of genetic knowledge.

This is one of the computers used in the Human Genome Project.
Use this investigation as an opportunity to practice categorizing and organizing information as you did in Example C.

**Investigation**

**Who Owns the Zebra?**

There are five houses along one side of Birch Street, each of a unique color. The home-owners each drive a different car, and each has a different pet. The owners all read a different newspaper and plant only one thing in their garden.

- The family with the station wagon lives in the red house.
- The owner of the SUV has a dog.
- The family with the van reads the *Gazette*.
- The green house is immediately to the left of the white house.
- The *Chronicle* is delivered to the green house.
- The man who plants zucchini has birds.
- In the yellow house they plant corn.
- In the middle house they read the *Times*.
- The compact car parks at the first house.
- The family that plants eggplant lives in the house next to the house with cats.
- In the house next to the house where they have a horse, they plant corn.
- The woman who plants beets receives the *Daily News*.
- The owner of the sports car plants okra.
- The family with the compact car lives next to the blue house.
- They read the *Bulletin* in the house next to the house where they plant eggplant.

Who owns the zebra?

Organizing the known information and clarifying what you need to find out is a very useful strategy. Whether it involves simply keeping track of units or sorting out masses of information, organizing your data and making a plan are essential to finding a solution efficiently.
Malaysian-American architect and artist Daniel Castor (b 1966) created this pencil drawing of the Amsterdam Stock Exchange. Castor’s work—which he calls “jellyfish” drawings—organizes lots of information and several perspectives into one drawing. He describes his art as “[capturing], in two dimensions, the physical power of the spaces yielded by [the] design process, a power that cannot be adequately described by word or photograph.”

EXERCISES

Practice Your Skills

1. Use units to help you find the missing information.
   a. How many seconds would it take to travel 15 feet at 3.5 feet per second?
   b. How many centimeters are in 25 feet? (There are 2.54 centimeters per inch and 12 inches per foot.)
   c. How many miles could you drive with 15 gallons of gasoline at 32 miles per gallon?

2. Emily and Alejandro are part of a math marathon team on which they take turns solving math problems for 4 hours each day. On Monday Emily worked for 3 hours and Alejandro for 1 hour. Then on Tuesday Emily worked for 2 hours and Alejandro for 2 hours. On Monday they collectively solved 139 problems and on Tuesday they solved 130 problems. Find the average problem-solving rate for Emily and for Alejandro.
   a. Identify the unknown quantities and assign variables. What are the units for each variable?
   b. What does the equation $3e + 1a = 139$ represent?
   c. Write an equation for Tuesday.
   d. Which of these ordered pairs $(e, a)$ is a solution for the problem?
      i. (34, 37)  
      ii. (37, 28)  
      iii. (30, 35)  
      iv. (27, 23)
   e. Interpret the solution from 2d according to the context of the problem.
3. To qualify for the Interlochen 470 auto race, each driver must complete two laps of the track at an average speed of 100 mi/h. Benjamin averages only 75 mi/h on his first lap. How fast must he go on the second lap to qualify for the race?

4. Use the distributive property to expand and combine like terms when possible.
   a. $7.5(a - 3)$
   b. $12 + 4.7(b + 6)$
   c. $5c - 2(c - 12)$
   d. $8.4(35 - d) + 12.6d$

5. Solve each equation.
   a. $4.5(a - 7) = 26.1$
   b. $9 + 2.7(b + 3) = 20.7$
   c. $8c - 2(c - 5) = 70$
   d. $8.4(35 - d) + 12.6d = 327.6$

Reason and Apply

6. APPLICATION Alyse earns $15.40 per hour, and she earns time and a half for working past 8:00 P.M. Last week she worked 35 hours and earned $600.60. How many hours did she work past 8:00 P.M.?

7. The dimensions used to measure length, area, and volume are related by multiplication and division. Find the information about the following rectangular boxes. Include the units in your solution.
   a. A box has volume 486 in.$^3$ and height 9 in. Find the area of the base.
   b. A box has base area 3.60 m$^2$ and height 0.40 m. Find the volume.
   c. A box has base area 2.40 ft$^2$ and volume 2.88 ft$^3$. Find the height.
   d. A box has volume 12,960 cm$^3$, height 18 cm, and length 30 cm. Find the width.

8. Lab assistant Jerry Anderson has just finished cleaning a messy lab table and is putting the equipment back on the table when he reads a note telling him not to disturb the positions of three water samples. Not knowing the correct order of the three samples, he finds these facts in the lab notes.

   - The water that is highest in sulfur was on one end.
   - The water that is highest in iron is in the Erlenmeyer flask.
   - The water taken from the spring is not next to the water in the bottle.
   - The water that is highest in calcium is left of the water taken from the lake.
   - The water in the Erlenmeyer flask, the water taken from the well, and the water that is highest in sulfur are three distinct samples.
   - The water in the round flask is not highest in calcium.

Determine which water sample goes where. Identify each sample by its container, source, element, and position.
Event planners make a career organizing information. To plan a New Year's Eve celebration, for example, an event planner considers variables such as location, decorations, food and beverages, number of people, and staff. For the celebration of the year 2000, event planners around the world worked independently and cooperatively to organize unique events for each city or country and to coordinate recording and televising of the events, which occurred in every time zone. Problem-solving strategies that were taught in this lesson may have been used to coordinate these global events.

9. **APPLICATION**

Paul can paint the area of a 12-by-8 ft wall in 15 min. China can paint the same area in 20 min.

a. In which equation does \( t \) represent how long it would take Paul and China to paint the area of a 12-by-8 ft wall together? Explain your choice.

\[
\text{i. } \frac{15t}{96} + \frac{20t}{96} = \frac{1}{96} \quad \text{ii. } (96)15t + (96)20t = 96 \quad \text{iii. } \frac{96t}{13} + \frac{96t}{20} = 96
\]

b. Solve all three equations in 9a.

c. How long would it take Paul and China to paint the wall together?

10. Kiane has a photograph of her four cats sitting in a row. The cats are different ages, and each cat has its own favorite toy and favorite sleeping spot.

- Rocky and the 10-year-old cat would never sit next to each other for a photo.
- The cat that sleeps in the blue chair and the cat that plays with the rubber mouse are the two oldest cats.
- The cat that plays with the silk rose is the third cat in the photo.
- Sadie and the cat on Sadie's left in the photo don't sleep on the furniture.
- The 8-year-old cat sleeps on the floor.
- The cat that sleeps on the sofa eats the same food as the 13-year-old cat.
- Pascal likes to chase the 5-year-old cat.
- If you add the ages of the cat that sleeps in a box and the one that plays with a stuffed toy, you get the age of Winks.
- The cat that sleeps on the blue chair likes to hide the catnip ball, which belongs to one of the cats sitting next to it in the photo.

Who plays with the catnip-filled ball?
Review

11. Rewrite each expression using the properties of exponents so that the variable appears only once.
   a. \((r^5)(r^7)\)
   b. \(\frac{2a^6}{6a^2}\)
   c. \(\left(\frac{c}{d}\right)^3\)
   d. \(3(2a^2)^4\)

12. Joel is 16 years old. His cousin Rachel is 12.
   a. What is the difference in their ages?
   b. What is the ratio of Joel's age to Rachel's age?
   c. In eight years, what will be the difference in their ages?
   d. In eight years, what will be the ratio of Joel's age to Rachel's age?

13. Draw a rectangle diagram to represent each product. Use the diagrams to expand each product.
   a. \((x + 1)(x + 5)\)
   b. \((x + 3)^2\)
   c. \((x + 3)(x - 3)\)

14. APPLICATION Iwanda sells African bead necklaces through a consignment shop. At the end of May, the shop paid her $100 from the sale of 8 necklaces. At the end of June, she was paid $187.50 from the sale of 15 necklaces. Assume that the consignment shop pays Iwanda the same amount of money for each necklace sold.
   a. Make a linear graph showing the relationship between the number of necklaces sold and the amount of money Iwanda gets.
   b. Use your graph to estimate how much money Iwanda will get if the shop sells only 6 of her necklaces in July.
   c. How much does the consignment shop pay Iwanda from the sale of each necklace?
There is no single way to solve a problem. Different people prefer to use different problem-solving strategies, yet not all strategies can be applied to all problems. In this chapter you practiced only a few specific strategies. You may have also used some strategies that you remember from other courses. The next paragraph gives you a longer list of problem-solving strategies to choose from.

**Organize the information** that is given, or that you figure out in the course of your work, in a list or table. **Draw a picture**, graph, or diagram, and label it to illustrate information you are given and what you are trying to find. There are also special types of diagrams that help you organize information, such as tree diagrams and Venn diagrams. **Make a physical representation** of the problem. That is, act it out, make a model, or use manipulatives. **Look for a pattern** in numbers or units of measure. Be sure all your measures use the same system of units and that you compare quantities with the same unit. **Eliminate some possibilities.** If you know what the answer cannot be, you are partway there. **Solve subproblems** that present themselves as part of the problem context, or **solve a simpler problem** by substituting easier numbers or looking at a special case. Don't forget to **use algebra**! Assign variables to unknown quantities and write expressions for related quantities. Translate verbal statements into equations, and solve the equations. **Work backward** from the solution to the problem. For instance, you can solve equations by undoing operations. **Use guess-and-check**, adjusting each successive guess by the result of your previous guess. Finally, but most importantly, **read the problem**! Be sure you know what is being asked.

**EXERCISES**

1. You are given a 3-liter bucket, a 5-liter bucket, and an unlimited supply of water. Describe or illustrate a procedure that will give exactly 4 liters of water in the 5-liter bucket.

2. Draw a rectangle diagram to represent each product. Use the diagrams to expand each product.
   a. \((x + 3)(x + 4)\)
   b. \((2x)(x + 3)\)
   c. \((x + 6)(x - 2)\)
   d. \((x - 4)(2x - 1)\)

3. Use the Pythagorean Theorem to find each missing length.
   a. [Diagram of a right triangle with sides 3 cm and hypotenuse x cm]
   b. [Diagram of a right triangle with sides 13 in. and hypotenuse 12 in.]

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4. This graph shows the relationship between distance driven and gasoline consumed for two cars going 60 mi/h.
   a. How far can Car A drive on 7 gallons (gal) of gasoline?
   b. How much gasoline is needed for Car B to drive 342 mi?
   c. Which car can drive farther on 8.5 gal of gas? How much farther?
   d. What is the slope of each line? Explain the real-world meaning of each slope.

5. Solve each equation. Check each answer by substituting into the original equation.
   a. $3(x - 5) + 2 = 26$
   b. $3.75 - 1.5(y + 4.5) = 0.75$

6. In 6a and b, translate each verbal statement to a symbolic expression. Combine the expressions to solve 6c.
   a. Six more than twice a number
   b. Five times three less than a number
   c. Six more than twice a number is five times three less than the number. Find the number.

7. APPLICATION Keisha and her family are moving to a new apartment 12 miles from their old one. You-Do-It Truck Rental rents a small truck for $19.95 per day plus $0.35 per mile. Keisha's family hopes they can complete the move with five loads all on the same day. She estimates that she will drive the truck another 10 miles for pickup and return.
   a. Write an expression that represents the cost of a one-day rental with any number of miles.
   b. How much will Keisha pay if she does complete the move with five loads?
   c. How much can Keisha save if she completes the move with four loads?
8. Toby has only a balance scale, a single 40 g mass, and a stack of both white and red blocks. (Assume that all white blocks have the same mass and all red blocks have the same mass.) Toby discovers that four white blocks and one red block balance two white blocks, two red blocks, and the 40 g mass. He also finds that five white blocks and two red blocks balance one white block and five red blocks.

a. List the unknown quantities and assign a variable to each.

b. Translate Toby's discoveries into equations.

c. Solve the equations to find a value for each variable.

d. Interpret the solution according to the context of the problem.

9. Amy says that it is her birthday and that six times her age five years ago is twice as much as twice her age next year. How old is Amy?

10. Scott has 47 coins totaling $5.02. He notices that the number of pennies is the same as the number of quarters and that the sum of the number of pennies and quarters is one more than the sum of the number of nickels and dimes. How many of each coin does Scott have?

11. The height of a golf ball in flight is given by the equation \( h = -16t^2 + 48t \), where \( h \) represents the height in feet above the ground and \( t \) represents the time in seconds since the ball was hit. Find \( h \) and interpret the real-world meaning of the result when

   a. \( t = 0 \)  
   b. \( t = 2 \)  
   c. \( t = 3 \)

12. Rewrite each expression using the properties of exponents so that the variable appears only once.

   a. \((4x^{-2})(x)\)  
   b. \(\frac{4x^2}{8x^3}\)  
   c. \((x^3)^5\)

13. Consider the equation \( y = 2^x \).

   a. Find \( y \) when \( x = 0 \).  
   b. Find \( y \) when \( x = 3 \).  
   c. Find \( y \) when \( x = -2 \).  
   d. Find \( x \) when \( y = 32 \).

14. Use units to help you find the missing information.

   a. How many ounces are in 5 gallons? (There are 8 ounces per cup, 4 cups per quart, and 4 quarts per gallon.)
   b. How many meters are in 1 mile? (There are 2.54 centimeters per inch, 12 inches per foot, 5280 feet per mile, and 100 centimeters per meter.)
15. Bethany Rogers temps as a sales assistant. She learns that all three sales representatives have important meetings with clients, but she only uncovers these clues.

- Mr. Bell is a sales representative although he is not meeting with Mr. Green.
- Ms. Hunt is the client who will be meeting in the lunch room.
- Mr. Green is the client with a 9:00 A.M. appointment.
- Mr. Mendoza is the sales representative meeting in the conference room.
- Ms. Phoung is a client, but she will not be in the 3:00 P.M. meeting.
- Mrs. Plum is a sales representative, but not the one meeting at 12:00 noon.
- The client with the 9:00 A.M. appointment is not meeting in the convention hall.

Help Bethany figure out which sales representative is meeting with which client, where, and when.

**TAKE ANOTHER LOOK**

1. You have seen that the multiplication expression \((x + 2)(x + 3)\) can be represented with a rectangle diagram in which the length and width of the rectangle represent the factors and the area represents the product. Find a way to represent the multiplication of three factors, such as \((x + 2)(x + 3)(x + 4)\). Explain how the geometry of the diagram represents the product.

2. Recall Jerry Anderson's problem with the water samples (Exercise 8 in Lesson 0.3). Notice that the table below has six subsections that each compare two of the characteristics. Each statement given in the problem translates into yeses (Y) or noes (N) in the cells of the table. When you have two noes in the same row or column of one subsection, the third cell must be a yes. And when you have a yes in any cell of a subsection, the other four cells in the same row and column must be noes. The table will eventually show you how the characteristics match up. Use this table to solve Jerry's problem.

<table>
<thead>
<tr>
<th>Erlenmeyer flask</th>
<th>Round flask</th>
<th>Bottle</th>
<th>Calcium</th>
<th>Sulfur</th>
<th>Iron</th>
<th>Lake</th>
<th>Well</th>
<th>Spring</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left</td>
<td>Center</td>
<td>Right</td>
<td>Lake</td>
<td>Well</td>
<td>Spring</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Calcium</td>
<td>Sulfur</td>
<td>Iron</td>
<td>Lake</td>
<td>Well</td>
<td>Spring</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sulfur</td>
<td>Iron</td>
<td>Lake</td>
<td>Well</td>
<td>Spring</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

[Table with Yes/No values filled in]
3. Find a way to use the distributive property to rewrite \((x + 2)(x + 3)\) without parentheses. Compare and contrast this method to the rectangle diagram method. Look back at Exercise 11 in Lesson 0.1, and explain how you could use the distributive property for each multiplication.

4. Use your graphing calculator or geometry software to explore the slopes of lines. What is the slope of a horizontal line? Of a vertical line? What slopes create a diagonal line at a 45° angle from the x-axis? For which slopes does the line increase from left to right? For which slopes does it decrease? Can you estimate a line's slope simply by looking at a graph? Write a short paper summarizing your findings.

Assessing What You've Learned

WRITE IN YOUR JOURNAL  Recording your thoughts about the mathematics you are learning, as well as about areas of confusion or frustration, help point out when you should seek assistance from your teacher and what questions you could ask. Keeping a journal is a good way to collect these informal notes, and if you write in it regularly, you'll track the progress of your understanding throughout the course. Your journal can also remind you of interesting contributions to make in class or prompt questions to ask during a review period.

Here are some questions you might start writing about.

- How has your idea about what algebra is changed since you finished your first-year course in algebra? Do you have particular expectations about what you will learn in advanced algebra? If so, what are they?
- What are your strengths and weaknesses as a problem-solver? Do you consider yourself well organized? Do you have a systematic approach? Give an everyday example of problem solving that reminds you of work you did in this chapter.

GIVE A PRESENTATION  In the working world, most people will need to give a presentation once in a while or contribute thoughts and opinions at meetings. Making a presentation to your class gives you practice in planning, conveying your ideas clearly, and adapting to the needs of an audience.

Choose an investigation or a problem from this chapter, and describe the problem-solving strategies you and your group used to solve it. Here are some suggestions to plan your presentation. (Even planning a presentation requires problem solving!)

- Work with a partner or team. Divide tasks equally. Your role should use skills that are well established, as well as stretch your abilities in new areas.
- Discuss the topic thoroughly. Connect the work you did on the problem with the objectives of the chapter.
- Outline your talk, and decide what details to mention for each point and what charts, graphs, or pictures would clarify the presentation.
- Speak clearly and loudly. Reveal your interest in the topic you chose by making eye contact with listeners.